Cochains

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Introduction

The goal of this essay is to show how certain algebras of differential forms may be used to describe the rational homotopy type of a topological space or of a continuous map.

For the sake of technical simplicity, we shall restrict our attention to simply-connected topological spaces; however, the methods we present generalize to *nilpotent spaces* (for a general discussion of nilpotent spaces, see [McC01, chap. 8*bis*]; for a treatment of rational homotopy theory in this setting see [BG76]), and similar tools may be used to study the nilpotent part of the fundamental group of any topological space (see [GM81, chap. XII]).

We first need a suitable notion of rational homotopy type. Given a topological space X, one can for instance try and compute $H_*(X; \mathbf{Q}) = H_*(X; \mathbf{Z}) \otimes \mathbf{Q}$; in his 1951 thesis [Ser51], Jean-Pierre Serre, applying this formula to Eilenberg-MacLane spaces and using what is now known as the Leray-Serre spectral sequence, showed how, in some cases, one could also compute the *rational homotopy groups* $\pi_* \otimes \mathbf{Q}$ of a topological space. This all amounts to extracting rational homotopy-theoretic information about a space.

However, Serre's methods also allowed him, in [Ser51] and in his subsequent paper [Ser53], to prove a rational version of "Whitehead's theorem", namely: a continuous map between simply-connected spaces induces isomorphisms on rational homotopy groups if and only if it induces isomorphisms on rational homology groups. Thus one can define a *rational homotopy equivalence* to be a map satisfying these conditions, and say that two simply-connected spaces have the *same rational homotopy type* if there is a chain of rational homotopy equivalences

$$X \to Z_1 \leftarrow Z_2 \leftarrow \ldots \to Y$$

connecting them. On the other hand, it is not clear what the rational homotopy type of a map should be.

To phrase this question in very general terms, what we need is a *localization* of the category of simply-connected topological spaces with respect to the rational homotopy equivalences.

Definition. Let $\mathcal{C}$ be a category and let S be a class of arrows of $\mathcal{C}$. A *localization* of $\mathcal{C}$ with respect to S is a functor $\gamma : \mathcal{C} \to S^{-1}C$ such that

(1) for every $f \in S$, $\gamma(f)$ is an isomorphism;

(2) for every other functor $F : C \to D$ sending the arrows of S to isomorphisms, there is a unique $G : S^{-1}C \to D$ such that $F = G\gamma$.

If it exists, such a localization is unique up to isomorphism.

Example. Let $\mathcal{A}$ be the category of abelian groups, and let $\mathcal{C}$ be the category with the same objects as $\mathcal{A}$ but with maps

$$\mathcal{C}(G, H) = \mathcal{A}(G \otimes_{\mathbf{Z}} \mathbf{Q}, H \otimes_{\mathbf{Z}} \mathbf{Q}).$$

Then the functor $\gamma : \mathcal{A} \to \mathcal{C}$ which is the identity on objects and which sends a map f to $f \otimes_{\mathbf{Z}} \mathbf{Q}$ is a localization of $\mathcal{A}$ with respect to the maps whose kernel and cokernel are torsion. Notice that $\mathcal{C}$ is equivalent to the full subcategory of $\mathcal{A}$ whose objects are the groups G such that $G \cong G \otimes_{\mathbf{Z}} \mathbf{Q}$ (by abuse of language, we shall frequently refer to such groups as *rational vector spaces*).

A *rational homotopy category* can be constructed formally by inverting the rational homotopy equivalences; using his theory of model categories, Quillen [Qui69, II.6] showed it to be equivalent to the category of simply-connected CW complexes all of whose homotopy groups are rational vector spaces. His method, however, is not as straightforward as one might hope, since the absence of coproducts in the category of simply-connected spaces means that it cannot be naively endowed with a model category structure.

We shall follow a more direct approach (due to Sullivan [Sul70, Sul74]) and construct, for any simply-connected space X, a rational homotopy equivalence $X \to X_{\mathbf{Q}}$, where $X_{\mathbf{Q}}$ is a CW complex all of whose homotopy groups are rational vector spaces. This map will behave very much like localization in the category of $\mathbf{Z}$-modules, and will allow us to associate to every homotopy class of maps $X \to Y$ a unique homotopy class of maps $X_{\mathbf{Q}} \to Y_{\mathbf{Q}}$, thereby defining a notion of rational homotopy type for maps. Along the way, we obtain a characterization of the rational homotopy type of a space by its *rational Postnikov tower*.

The correspondence between algebra and topology which is at the core of Sullivan's theory then works as follows. One starts with a simply-connected topological space X *of finite type* (i.e. all of whose Betti numbers are finite — the cohomology of X need not be finite-dimensional), and associates to X a natural *commutative* differential graded algebra $A^*(X)$ over the rationals which computes the rational singular cohomology of X. (Note that the singular cochain algebra $C^*(X; \mathbf{Q})$ is not commutative, although the induced cohomology algebra is.) From $A^*(X)$ one constructs a certain free algebra $(\Lambda V, d)$ satisfying some minimality condition, such that there is a map $(\Lambda V, d) \to A^*(X)$ inducing an isomorphism on cohomology. This *minimal model* for $A^*(X)$, which is a kind of algebraic analogue of the Postnikov tower, is shown to be unique up to isomorphism. Moreover, there is an exact structural correspondence between this model and the rational Postnikov tower of X: the model may be built "in stages" from the tower, and the tower may be recovered from the model. Therefore, the minimal model contains all the rational homotopy-theoretic information about X; for instance, the dimension of $\pi_n(X) \otimes \mathbf{Q}$ is equal to the number of generators of $(\Lambda V, d)$ in degree n. Going the other way, any minimal algebra over $\mathbf{Q}$ of finite type arises as the minimal model of $A^*(X)$ for some space X.

Similarly, one can represent any map between simply-connected spaces X and Y by a map between the minimal models for $A^*(X)$ and $A^*(Y)$, and there is a converse. But to make it precise, as well as to prove the uniqueness of the minimal model (which is essential to establish the structural correspondence described above), it is necessary to develop a homotopy theory for commutative differential algebras.

Sullivan used his construction of rational differential forms to transfer to algebra the topological notion of homotopy, and developed an algebraic obstruction theory, strongly analogous to that of topological spaces (his original methods may be found in the Friedlander-Griffiths-Morgan lecture notes [GM81], dating back to 1970–1971). We shall follow a slightly different path: we found Quillen's axiomatic approach to homotopy theory (first exposed in [Qui67]), through model categories, to be a compelling tool both conceptually and technically; it makes the analogies between topological spaces and algebras precise and allows one to streamline the proofs in a very efficient way. Therefore, and though we by no means use model categories to their full power in this essay, we decided to develop the homotopy theory of commutative differential graded algebras in this framework. In doing so we heavily relied on [BG76].

We now give a rough outline of the essay. In section 1, we first review some classical results about Postnikov towers and introduce the notions of principal fibration and of k-invariant. We then present the Leray-Serre spectral sequence and apply it to prove a rational form of the Hurewicz theorem. In section 2, we discuss rational homotopy type; using the tools of section 1, we introduce the notion of rational space and of localization, we prove the existence of a localization in the simply-connected case, and show how the rational homotopy type of a simply-connected CW complex is entirely characterized by its rational Postnikov tower of principal fibrations. In section 3 we develop the homotopy theory of commutative differential graded algebras, using Quillen's model categories. We prove the existence and uniqueness of the minimal model of a simply-connected CDGA, and briefly discuss homotopy categories. In section 4 we present Sullivan's structural correspondence between minimal models and rational Postnikov towers; this requires a rather lengthy discussion of the transgression map of a fibration. Our end result may be phrased in the following way: the rational homotopy category of simply-connected topological spaces of finite type is equivalent to the homotopy category of simply-connected CDGAs over $\mathbf{Q}$ of finite type. We briefly describe Sullivan's "PL forms" but do not construct them in any detail. We explain how to relate these to the algebra of smooth forms on a manifold and discuss the notion of real homotopy type.

Finally, we would like to thank our supervisor Prof. Grojnowski for guiding us to this beautiful theory.

Contents

Section 1

Postnikov towers and the Leray-Serre spectral sequence

We first define weak homotopy type and recall how the weak homotopy type of a space is completely described by its Postnikov tower. Next, we review some material about Eilenberg-MacLane spaces and the links between homotopy and cohomology; this allows us to define the k-invariant of a principal fibration — an essential tool throughout the essay — and to analyze further the Postnikov tower of a simply-connected CW complex.

In the second half of this section we introduce the Leray-Serre spectral sequence and use it inductively on the Postnikov tower of a space to prove a rational Hurewicz theorem, which is the foundation stone of all our later study of rational homotopy type. An easy adaptation of these methods would allow one to compute the rational homotopy groups of spheres; we do not carry out this computation, but it is easily found in the literature (see for instance [BT82] or [Hat]).

1.1 Weak homotopy type

Definition 1.1.1. A continuous map $f : X \to Y$ is called a *weak (homotopy) equivalence* if

$$\pi_i(f) : \pi_i(X) \to \pi_i(Y) \text{ is an isomorphism for all } i \geq 0;$$

two spaces are said to have the *same weak homotopy type* if there is a chain of weak equivalences

$$X \to Z_1 \leftarrow Z_2 \leftarrow \ldots \to Y$$

connecting them.

Remark 1.1.2. It is a theorem of Whitehead that two weakly equivalent CW complexes are in fact homotopy equivalent; see [Hat02, pp. 346-347]. Since we are mostly interested in CW complexes, this notion might therefore seem superfluous. It is however an exact prototype for the rational homotopy type to be defined in section 2, and is also an essential part of the simplest example of a model category structure (see section 3); this is why we are discussing it here.

We first quote the following classical result from homotopy theory. See for instance [Hat02, pp. 352–353].

Proposition 1.1.3. *Let X be a topological space. There is a CW complex X_0 and a weak homotopy equivalence $X_0 \to X$; moreover X_0 is unique up to homotopy equivalence.*

An easy consequence of the Hurewicz theorem is that a weak equivalence between simply-connected spaces also induces isomorphisms on homology and cohomology groups.

Proposition 1.1.4. *Let $f : X \to Y$ be a weak homotopy equivalence between simply-connected spaces. Then*

$$H_i(f) : H_i(X) \to H_i(Y) \text{ and } H^i(f) : H^i(Y) \to H^i(X)$$

are isomorphisms for all $i \geq 0$.

Proof. Make f an inclusion by replacing Y by the homotopy equivalent space given by the mapping cylinder of f. The homotopy long exact sequence for the pair (Y, X) gives $\pi_i(Y, X, x_0) = 0$ for all $i > 0$; the (relative) Hurewicz theorem then gives $H_i(Y, X) = 0$ for all i. The conclusion follows from the homology long exact sequence and (for cohomology) from the universal coefficient theorem. $\square$

Recall that a *fibration* $E \xrightarrow{p} B$ is a map satisfying the homotopy lifting property for any space; if the base B is path-connected, the fibres over every point are all homotopy equivalent, so in this setting, the fibre of the fibration is well-defined up to homotopy. Additionally, the homotopy groups of the fibre, of the total space and of the base fit in a long exact sequence. See for example [BT82, p. 199 sqq.] for details. We shall often use the notation $F \hookrightarrow E \xrightarrow{p} B$.

Let X be a connected topological space. We can construct a commutative diagram

$$\begin{tikzcd} & \vdots \\ & X_3 \arrow[d, "p_2"] \\ & X_2 \arrow[d, "p_1"] \\ X \arrow[r, "i_1"'] \arrow[ur, "i_2"] \arrow[uur, "i_3"] & X_1 \end{tikzcd}$$

where:

(1) The maps $X \xrightarrow{i_n} X_n$ induce isomorphisms on π_i, $i \leq n$;

(2) $\pi_i(X_n) = 0$ for $i > n$;

(3) the maps $X_{n+1} \xrightarrow{p_n} X_n$ are fibrations.

Such a system of fibrations is called a *Postnikov tower* for X.

Remark 1.1.5. The easiest way to build a suitable X_n is to attach cells of dimension $n + 2$ and higher to X so as to kill all homotopy groups of dimension greater than n; see [Hat02, pp. 354–355].

We shall make repeated use of such systems, in the remainder of this section, to study X through inductive arguments over the X_n. The following lemma will be useful. It is easily proved, through a cell-by-cell Mayer-Vietoris argument, for a Postnikov tower $\{X_n,\ p_n,\ i_n\}$ constructed as suggested in remark 1.1.5 on the preceding page; it may be extended to any other Postnikov tower $\{X'_n,\ p'_n,\ i'_n\}$, since we can prove that there always exists weak homotopy equivalences $X_n \to X'_n$.

Lemma 1.1.6. *Let $\{X_n,\ p_n,\ i_n\}$ be a Postnikov tower for X. Then*

$$H_k(i_n) : H_k(X) \to H_k(X_n)$$

is an isomorphism for $1 \leq k \leq n$ and a surjection for $k = n+1$.

We now state an important fact that prepares the ground for our study of rational homotopy type in section 2. See [Hat02, pp. 410-411] for a proof.

Proposition 1.1.7. *Let $\{X_n,\ p_n,\ i_n\}$ be a Postnikov tower for X. The map*

$$X \to \varprojlim X_n.$$

is a weak homotopy equivalence.

In other words, a Postnikov tower for X characterizes X entirely up to weak homotopy equivalence.

1.2 Eilenberg-MacLane spaces, principal fibrations and k-invariants

We are now going to analyse further the Postnikov towers of CW-complexes; this will lead us to a complete and economic classification of weak homotopy types, which we shall use in section 2.

From now on, we shall be working with CW-complexes; we assume that basepoints have been chosen, but usually omit mentioning them in order to keep the notations as simple as possible. We shall write $[X, Y]$ for the set of homotopy classes of maps $X \to Y$, and $\langle X, Y\rangle$ for the set of homotopy classes of basepoint preserving maps $X \to Y$.

We first discuss Eilenberg-MacLane spaces and principal fibrations.

Definition 1.2.1. An *Eilenberg-MacLane space of type* $(G,\ n)$ is a topological space K such that

$$\pi_i(K) \cong \begin{cases} G & \text{if } i = n, \\ 0 & \text{otherwise.} \end{cases}$$

We have the following. For a proof see [Hat02, pp. 365–366].

Proposition 1.2.2. *For every integer $n \geq 0$ and every group G, with G abelian if $n > 1$, there exists a CW complex which is an Eilenberg-MacLane space of type (G, n). It is unique up to homotopy equivalence. We shall use the notation $K(G, n)$ to denote any such CW complex equipped with an identification*

$$\chi_{K(G,n)} : G \xrightarrow{\cong} \pi_n(K(G, n), *).$$

Moreover, consider two Eilenberg-MacLane spaces $K(G,n)$ and $K(H,n)$ and a homomorphism $\phi : G \to H$. Then there is a basepoint preserving map $f : K(G,n) \to K(H,n)$ such that the diagram

$$\begin{tikzcd} G \arrow[rr, "\phi"] \arrow[d, "\chi_{K(G,n)}"'] & & H \arrow[d, "\chi_{K(H,n)}"] \\ \pi_n(K(G,n),*) \arrow[rr, "\pi_n(f)"'] & & \pi_n(K(H,n),*) \end{tikzcd}$$

commutes.

The following fundamental theorem may be proved in two ways: either through very abstract and general methods, as in [Hat02, section 4.3], or through a rather transparent application of the so-called "obstruction theory", which unfortunately requires cumbersome prerequisites. A readable and concise sketch of the latter approach may be found in [GM81, chapters V–VI].

Theorem 1.2.3. *Let X be a CW complex and let π be an abelian group. For every $n \geq 1$, there is a natural bijection*

$$T : \langle X, K(\pi,n)\rangle \to H^n(X;\pi).$$

Moreover, this bijection is of the form

$$T : \phi \mapsto H^n(\phi)(\alpha)$$

for some class $\alpha \in H^n(K(\pi,n);\pi)$.

Here, naturality means that for any $f : X \to K(\pi,n)$ and $g : Y \to X$, we have $T([gf]) = H^n(g)(T([f]))$.

Remark 1.2.4. The $\alpha \in H^n(K(\pi,n);\pi)$ in the statement of the theorem may be taken to be the class corresponding to the morphism $\kappa : H_n(K(\pi,n);\mathbf{Z}) \to \pi$ such that the composite

$$\pi \xrightarrow{\chi_{K(\pi,n)}} \pi_n(K(\pi,n),*) \xrightarrow{h} H_n(K(\pi,n);\mathbf{Z}) \xrightarrow{\kappa} \pi,$$

where h is the Hurewicz isomorphism, is the identity. We shall call this α the *fundamental class* of $K(\pi,n)$.

Remark 1.2.5. If $n > 1$, then $K(\pi,n)$ is simply-connected, and therefore the natural map $\langle X, K(\pi,n)\rangle \to [X, K(\pi,n)]$ is a bijection (see [Hat02, section 4.A]). Thus if $n > 1$ there is a bijection $[X, K(\pi,n)] \to H^n(X;\pi)$; it is natural with respect to *simply-connected* spaces X.

Let us now revert to the study of Postnikov towers. Let $\{X_n,\ p_n,\ i_n\}$ be a Postnikov tower for the CW complex X; we may choose the X_n to be CW complexes (see remark 1.1.5 on page 6). The fibre of $F \hookrightarrow X_{n+1} \to X_n$ is an Eilenberg-MacLane space $K(\pi_{n+1}(X), n+1)$, as can be seen from the homotopy long exact sequence; in general, we call a fibration with fibre some $K(\pi,n)$ a (π,n)-*fibration.*

Definition 1.2.6. Let B be a connected CW complex. We say that $K(\pi,n) \hookrightarrow E \xrightarrow{p} B$ is a *principal* (π,n)-fibration if it is a pullback of the pathspace fibration over $K(\pi,n+1)$.

$$\begin{tikzcd} K(\pi,n) \arrow[r] & E \arrow[r] \arrow[d, "p"'] & PK(\pi,n+1) \arrow[d] \\ & B \arrow[r, "k"'] & K(\pi,n+1) \end{tikzcd}$$

The map $k : B \to K(\pi,n+1)$ in the previous definition corresponds to a class $H^{n+1}(B;\pi)$ according to theorem 1.2.3 on the preceding page (and to remark 1.2.5); this class is *a well-defined invariant of the fibration*, as may be proved by identifying it to a class intrinsically defined in terms of the map $E \to B$ (see [GM81, chap. VI]). It is called the *k-invariant* of the principal fibration.

Remark 1.2.7. Using the explicit form of the correspondence between maps into $K(\pi,n+1)$ and classes in $H^{n+1}(B;\pi)$ together with remark 1.2.4, we can describe the k-invariant of p as the pullback by k of the fundamental class of $K(\pi,n+1)$.

The main result of this section is the following. For a proof see for instance [Spa66, p. 440 sqq.].

Theorem 1.2.8. *Let X be a simply-connected CW complex. Then X has a Postnikov tower $\{X_n,\ p_n,\ i_n\}$ where the X_n are CW complexes and the p_n are principal fibrations, as in the diagram below.*

$$\begin{tikzcd} & \vdots & \\ & X_4 \arrow[r, "k_4"'] \arrow[d, "p_3"] & K(\pi_5(X),6) \\ & X_3 \arrow[r, "k_3"'] \arrow[d, "p_2"] & K(\pi_4(X),5) \\ X \arrow[r, "i_2"'] \arrow[ur, "i_3"'] \arrow[uur, "i_4"] & X_2 = K(\pi_2(X),2) \arrow[r, "k_2"'] & K(\pi_3(X),4) \end{tikzcd}$$

The k-invariant of the fibration $K(\pi_{n+1}(X),n+1) \hookrightarrow X_{n+1} \to X_n$, which is a cohomology class in $H^{n+2}(X_n;\pi_{n+1}(X))$, is called the n-th k-invariant of X.

So for a simply-connected CW complex X, the homotopy groups and the k-invariants of X characterize the (weak) homotopy type of X entirely.

1.3 The Leray-Serre spectral sequences

We are now going to state the two main theorems about the Leray-Serre spectral sequence. We refer the reader to [BT82] for a nice construction of the spectral sequence, and to [McC01] for reference.

Theorem 1.3.1 (Leray-Serre for homology)**.** *Let $F \hookrightarrow E \xrightarrow{p} B$ be a fibration with B simply-connected and F path-connected; let G be any abelian group. There is a first quadrant spectral sequence $\{E^r_{*,*},\ d^r\}$, $d^r : E^r_{*,*} \to E^r_{*-r,*+r-1}$, converging to $H_*(E;G)$ with*

$$E^2_{p,q} \cong H_p(B; H_q(F;G)).$$

Theorem 1.3.2 (Leray-Serre for cohomology)**.** *Let $F \hookrightarrow E \xrightarrow{p} B$ be a fibration with B simply-connected and F path-connected; let G be any abelian group. There is a first quadrant spectral sequence of algebras $\{E_r^{*,*},\ d_r\}$, $d_r : E_r^{*,*} \to E_r^{*+r,*-r+1}$, converging to $H^*(E;G)$ as an algebra, with*

$$E_2^{p,q} \cong H^p(B; H^q(F;G)).$$

If additionally $G = \boldsymbol{k}$ is a field, we have

$$E_2^{p,q} \cong H^p(B;\boldsymbol{k}) \otimes_{\boldsymbol{k}} H^q(F;\boldsymbol{k})$$

as algebras.

Remark 1.3.3. Under the hypotheses of theorem 1.3.1, the map $H_i(f;G) : H_i(E;G) \to H_i(B;G)$ may be recovered as the composite

$$H_i(E;G) \to E^\infty_{i,0} \cong E^{i+1}_{i,0} \to \ldots \to E^2_{i,0} \cong H_i(B;G).$$

There is a dual statement for cohomology.

The reader may assume that we are working with $\mathbf{Z}$ coefficients wherever no further precision is given.

As a first application, we sketch a proof of the following lemma, which will be of repeated use.

Lemma 1.3.4. *For any $n > 0$, $H^*(K(\boldsymbol{Z}, n); \boldsymbol{Q})$ is the free algebra (in the graded commutative sense) over $\boldsymbol{Q}$ on one generator of degree n; that is, a polynomial algebra on one generator when n is even, and an exterior algebra on one generator when n is odd.*

Proof. The proof is inductive over n. The base case is immediate since $K(\mathbf{Z}, 1) \simeq S^1$. Now suppose the result has been proved for some odd n, and consider the rational cohomology spectral sequence associated to the pathspace fibration $K(\mathbf{Z}, n) \hookrightarrow PK(\mathbf{Z}, n+1) \to K(\mathbf{Z}, n+1)$. We show that the E_2 term, which is also the E_{n+1} term, looks like:

$$\begin{array}{c|ccccccccccc}
 & \vdots & & & & & & & & & & \\
 & 0 & & & & & & & & & & \\
n & \mathbf{Q}a & & & & \mathbf{Q}ab & & & & \mathbf{Q}ab^2 & & \\
 & 0 & & & & & & & & & & \\
 & \vdots & & & & & & & & & & \\
 & 0 & & & & & & & & & & \\
 & \mathbf{Q} & 0 & \cdots & 0 & \mathbf{Q}b & 0 & \cdots & 0 & \mathbf{Q}b^2 & & \cdots \\
\hline
 & & & & & n+1 & & & & 2(n+1) & &
\end{array}$$

Indeed, the maps $d_{n+1} : E_{n+1}^{i,n} \to E_{n+1}^{i+n+1,0}$ have to be isomorphisms since the total space of the fibration is contractible. This immediately gives us the additive structure of $H^*(K(\mathbf{Z}, n+1); \mathbf{Q})$. Now let a be a generator of $H^n(K(\mathbf{Z}, n); \mathbf{Q}) \cong E_2^{0,n}$, and let b be its image under d_{n+1}; it is a generator of $H^{n+1}(K(\mathbf{Z}, n+1); \mathbf{Q}) \cong E_2^{n+1,0}$. From the fact that

$$E_2^{p,q} \cong H^p(K(\mathbf{Z}, n+1); \mathbf{Q}) \otimes H^q(K(\mathbf{Z}, n); \mathbf{Q})$$

we deduce that $E_{n+1}^{n+1,n}$ is generated by ab; then the graded Leibniz rule for d_{n+1} implies that $H^{2(n+1)}(K(\mathbf{Z}, n+1); \mathbf{Q})$ is generated by b^2. In the same way $E_{n+1}^{2(n+1),n}$ is generated by ab^2 and $H^{3(n+1)}(K(\mathbf{Z}, n+1); \mathbf{Q})$ by b^3; and so on. The case when n is even is similar. $\square$

1.4 The Hurewicz-Serre theorem

Our next goal is to prove the following *rational* version of the Hurewicz theorem. Recall that an abelian group G is said to be *torsion* if $G \otimes_{\mathbf{Z}} \mathbf{Q} = 0$.

Theorem 1.4.1 (Rational Hurewicz)**.** *Let X be a simply-connected space. If $\pi_i(X)$ is torsion for $1 \leq i < n$, then $H_i(X)$ is torsion for $1 \leq i < n$ and there is an isomorphism*

$$\pi_n(X) \otimes_{\mathbf{Z}} \mathbf{Q} \xrightarrow{\cong} H_n(X) \otimes_{\mathbf{Z}} \mathbf{Q}.$$

This theorem only relies on the following three properties of the collection $\mathcal{C}$ of torsion abelian groups:

(1) for any short exact sequence $0 \to L \to M \to N \to 0$ of abelian groups,

$$M \in \mathcal{C} \iff L \text{ and } N \in \mathcal{C};$$

(2) for any $A, B \in \mathcal{C}$, $A \otimes_{\mathbf{Z}} B \in \mathcal{C}$ and $\mathrm{Tor}(A, B) \in \mathcal{C}$;

(3) for any $G \in \mathcal{C}$, $H_i(K(G, 1)) \in \mathcal{C}$ for all $i > 0$ (in other words, the positive dimensional homology groups of G are in $\mathcal{C}$).

Let us briefly check these. Property (1) and the first part of (2) are immediate. The second part of (2) and (3) are first proved for sums of cyclic groups, using the explicit construction of $K(\mathbf{Z}_m, 1)$ (as an infinite-dimensional lens space) and the Künneth theorem. Then, one may write G as a direct limit of sums of cyclic groups, and use the second part of proposition 1.2.2 on page 7 to get a direct system of Eilenberg-MacLane spaces converging to an Eilenberg-MacLane space of type (G, n). To conclude, recall that H_* and Tor commute with direct limits (see [Hat02, p. 244 and p. 267]).

For any collection of abelian groups satisfying these three properties (in the language of [Ser53], for any *class* of abelian groups satisfying Serre's properties (II_A) and (III)), a similar generalized Hurewicz theorem holds, as follows. Note that property (1) implies that the trivial group is in $\mathcal{C}$.

Theorem 1.4.2 (Hurewicz-Serre)**.** *Let X be a simply-connected space, and let $\mathcal{C}$ be a collection of abelian groups satisfying the properties above. If $\pi_i(X) \in \mathcal{C}$*

for $1 \le i < n$*, then* $H_i(X) \in \mathcal{C}$ *for* $1 \le i < n$ *and there is an isomorphism modulo* $\mathcal{C}$ *between* $\pi_n(X)$ *and* $H_n(X)$*: a map*

$$\pi_n(X) \to H_n(X)$$

whose kernel and cokernel are in $\mathcal{C}$*.*

Example 1.4.3. This theorem may be applied to the collection of finitely generated abelian groups; it easily follows that any space all of whose homology groups are finitely generated also has finitely generated homotopy groups. In particular, the homotopy groups of spheres are finitely generated.

The proofs being identical, we shall prove theorem 1.4.1 on the previous page in this generalized form. We start with a lemma.

Lemma 1.4.4. *Let* $\mathcal{C}$ *be a collection of abelian groups satisfying the properties above, and* $G \in \mathcal{C}$*. Then*

$$H_i(K(G,n)) \in \mathcal{C}$$

for every $i \ge 1, n \ge 1$*.*

Proof. The case $n = 1$ is property (3) above. From there, use the homology Leray-Serre spectral sequence inductively on the pathspace fibration below.

$$\begin{tikzcd} \Omega K(G,n+1) \simeq K(G,n) \arrow[r, hook] & PK(G,n+1) \simeq * \arrow[d] \\ & K(G,n+1) \end{tikzcd}$$

At every step, prove that $H_i(K(G,n+1)) \in \mathcal{C}$ ($i > 0$) inductively on i, using the fact that $H_i(K(G,n+1)) \cong E^2_{i,0}$ ($i > 0$) has to be killed by some terms $E^r_{p,q}$ for $q < i$. Now from (1) we see that $E^r_{p,q}$ is in $\mathcal{C}$ whenever $E^2_{p,q}$ is, and from (2) and the universal coefficient theorem we see that $E^2_{p,q}$ is in $\mathcal{C}$ if $E^2_{p,0}$ and $E^2_{0,q}$ are. $\square$

Proof of theorem 1.4.2 on the preceding page. Consider the Postnikov tower for X. We shall show inductively on $2 \le k < n$ that $H_i(X_k) \in \mathcal{C}$ for all $i > 0$. Since $H_k(X) = H_k(X_k)$ (cf. lemma 1.1.6 on page 7), this will in particular prove the first half of the proposition. As long as $\pi_k(X) = 0$, the result is obvious, X_k then being contractible; for $k = k_0$, the dimension of the first non-trivial homotopy group, it is our lemma, since $X_{k_0} = K(\pi_{k_0}(X), k_0)$. Now consider the Leray-Serre homology spectral sequence for the fibration $F = K(\pi_k(X), k) \hookrightarrow X_k \to X_{k-1}$. Since the positive degree homology groups of the base space and of the fibre are in $\mathcal{C}$, the universal coefficient theorem and property (2) together give

$$E^2_{p,q} = H_p(X_{k-1}) \otimes H_q(F) \oplus \mathrm{Tor}(X_{k-1}, F) \in \mathcal{C} \quad \text{for } (p,q) \neq (0,0).$$

Using property (1) we then get $H_i(X_k) \in \mathcal{C}$ for $i > 0$.

Finally, consider the fibration $K(\pi_n(X), n) \hookrightarrow X_n \to X_{n-1}$. The E^2 term of the homology Leray-Serre spectral sequence looks like:

$$
\begin{array}{l|ll}
 & \pi_n(X) & \\
 & 0 & \\
 & \vdots & \\
 & 0 & \\
 & \mathbf{Z} & \qquad H_i(X_{n-1}) \in \mathcal{C} \\
\hline
 & &
\end{array}
$$

and so we may write $H_n(X) = H_n(X_n)$ as an extension

$$0 \to E^{\infty}_{0,n} \to H_n(X) \to E^{\infty}_{n,0} \to 0;$$

but since $d^n : H_{n+1}(X_{n-1}) \to \pi_n(X)$ is the only differential affecting $E^{*}_{0,n}$, we also have a short exact sequence

$$0 \to H_{n+1}(X_{n-1}) \xrightarrow{d^n} \pi_n(X) \to E^{\infty}_{0,n} \to 0.$$

Combining these two, we get a map $\pi_n(X) \to H_n(X)$ whose kernel and cokernel are in $\mathcal{C}$. □

Remark 1.4.5. This result may be generalized to the non simply-connected case, with $\pi_1(X) \in \mathcal{C}$, provided X is path-connected, $\pi_1(X)$ is abelian and the action of $\pi_1(X)$ on all higher homotopy groups $\pi_n(X)$ is trivial (such a space is occasionally called *simple* or *abelian*). See for instance [Hat, chap. I, pp. 14–17].

We shall gather some important corollaries of this result in the next section, when studying rational homotopy type.

Section 2

Rational homotopy type

The aim of this section is double: to define a suitable notion of rational homotopy type for spaces and maps, and to give a description of the rational homotopy type of a space through a rational analogue of the Postnikov tower of principal fibrations of section 1. This description will form the topological basis of the correspondence exposed in section 4.

2.1 Rational homotopy equivalences

We first introduce the rational homotopy type of a space, along the lines sketched in the introduction; our definition is mirrored on the weak homotopy type defined in section 1.

Definition 2.1.1. A map $f : X \to Y$ is called a rational (homotopy) equivalence if the localized map

$$\pi_i(f) \otimes \mathbf{Q} : \pi_i(X) \otimes \mathbf{Q} \to \pi_i(Y) \otimes \mathbf{Q}$$

is an isomorphism for all $i \geq 0$. Two spaces are said to have the *same rational homotopy type* if there is a chain of rational equivalences

$$X \to Z_1 \leftarrow Z_2 \leftarrow \ldots \to Y$$

connecting them.

There is an analogue of proposition 1.1.4 for rational homotopy equivalences, which relies on the rational Hurewicz theorem. First, note that the universal coefficient theorem gives us an isomorphism $H_*(X; \mathbf{Q}) \cong H_*(X) \otimes \mathbf{Q}$ under which $H_*(f, \mathbf{Q})$ is identified with the localized map $H_*(f) \otimes \mathbf{Q}$. Similarly, we have an isomorphism $H^*(X; \mathbf{Q}) \cong \mathrm{Hom}(H_*(X); \mathbf{Q})$ identifying $H^*(f; \mathbf{Q})$ with the dual over $\mathbf{Q}$ of $H_*(f)$; this map, in turn, is readily identified with the dual of $H_*(f; \mathbf{Q})$.

Proposition 2.1.2. *Let $f : X \to Y$ be a rational homotopy equivalence between simply-connected spaces. Then*

$$H_i(f; \boldsymbol{Q}) : H_i(X; \boldsymbol{Q}) \to H_i(Y; \boldsymbol{Q}) \text{ and } H^i(f; \boldsymbol{Q}) : H^i(Y; \boldsymbol{Q}) \to H^i(X; \boldsymbol{Q})$$

are isomorphisms for all $i \geq 0$.

Proof. In light of our discussion above, it suffices to show the result for homology.

We shall first make the additional hypothesis that $\pi_2(f) : \pi_2(X) \to \pi_2(Y)$ is onto. Up to homotopy equivalence, we may convert f into a fibration $F \hookrightarrow X \to Y$; since localization preserves exactness, the homotopy long exact sequence of the fibration gives

$$\pi_i(F) \otimes \mathbf{Q} = 0 \text{ for } i \geq 1.$$

Our additional hypothesis guarantees $\pi_1(F) = 0$. We may therefore apply the rational Hurewicz theorem to F: we obtain

$$H_i(F; \mathbf{Q}) = 0 \text{ for } i \geq 1.$$

Hence the homology Leray-Serre spectral sequence with rational coefficients for this fibration collapses at the E^2 term, and the conclusion immediately follows from remark 1.3.3 on page 10.

In the general case, $\pi_1(F)$ is torsion, but F need not be simply-connected. However, one can prove that F is an H-space, and therefore simple; see [Spa66, p. 511] for details. One may then invoke the strengthened Hurewicz-Serre theorem mentioned in remark 1.4.5 on page 13 and proceed as above. □

Remark 2.1.3. It is also possible to prove this statement using a relative version of the rational Hurewicz theorem, thus mirroring the proof of proposition 1.1.4 on page 6; this is the approach of Serre's original paper [Ser53]. It requires the (reasonably straightforward) development of a relative version of the Leray-Serre spectral sequence. In any case, some additional arguments are needed when $\pi_2(f)$ is not onto; this point is easily overlooked, as in [GM81, p. 89].

Example 2.1.4. If $n \geq 2$, there is a rational homotopy equivalence $K(\mathbf{Z}, n) \to K(\mathbf{Q}, n)$ (take a map that realizes the inclusion $\mathbf{Z} \to \mathbf{Q}$; it induces isomorphisms on rational homotopy groups). We may therefore immediately deduce from lemma 1.3.4 on page 10 that $H^*(K(\mathbf{Q}, n); \mathbf{Q})$ is a free (graded commutative) algebra on one generator of degree n.

More generally, let V be a rational vector space. Then $H^*(K(V, n); \mathbf{Q})$ is a free algebra generated in dimension n; hence it is the free algebra on $H^n(K(V, n); \mathbf{Q}) \cong \mathrm{Hom}(H_n(K(V, n); \mathbf{Z}), \mathbf{Q}) \cong \mathrm{Hom}(V, \mathbf{Q}) = V^*$.

2.2 Rational spaces

As explained in the introduction, we shall show below that any *simply-connected* CW complex X is rationally equivalent to a certain *rational* CW complex $X_{\mathbf{Q}}$. We first need to make sense of that notion of rational space; this is done through the following result.

Theorem 2.2.1. *Let X be a simply-connected space. The following conditions are equivalent:*

(1) $\pi_i(X)$ is a rational vector space for all $i \geq 1$;

(2) $H_i(X)$ is a rational vector space for all $i \geq 1$.

A simply-connected space satisfying these conditions is said to be rational.

Note that this is not a special case of the Hurewicz-Serre theorem, since the collection of rational vector spaces does not satisfy property (1) of 1.4; however, it does satisfy the weaker property that any extension of two rational vector spaces is a rational vector space, as well as property (2). This allows us to prove the following lemma, an analogue of which we used in the proof of the Hurewicz-Serre theorem.

Lemma 2.2.2. *Let $F \hookrightarrow E \xrightarrow{p} B$ be a fibration. If the integral homology groups $H_{>0}(F; \boldsymbol{Z})$ and of $H_{>0}(B; \boldsymbol{Z})$ are rational vector spaces, then so are the $H_{>0}(E; \boldsymbol{Z})$.*

Proof. For the homology Leray-Serre spectral sequence associated to the fibration, we have

$$E^2_{p,q} = H_p(E; \mathbf{Z}) \otimes H_q(F; \mathbf{Z})$$

as there is no torsion term; the $E^2_{p,q}$, $(p, q) \neq (0, 0)$, are therefore rational vector spaces. It is an easy exercice that any homomorphism of abelian groups between rational vector spaces is necessarily $\mathbf{Q}$-linear; in particular its kernel and image are rational vector spaces. This implies that the $E^\infty_{p,q}$, $(p, q) \neq (0, 0)$, are rational vector spaces. Thus for $i > 0$, $H_i(E; \mathbf{Z})$ is obtained as the result of successive extensions of rational vector spaces, and the result follows. □

Property (3) also holds for rational vector spaces. We now establish it, together with an analogue of lemma 1.4.4.

Lemma 2.2.3. *The integral homology groups $H_{>0}(K(\boldsymbol{Q}, n); \boldsymbol{Z})$ are rational vector spaces for every $n \geq 1$.*

Proof. For $n = 1$, the result follows from the explicit construction of a $K(\mathbf{Q}, 1)$: namely, take the direct limit of the sequence of maps $f_k : K(\mathbf{Z}, 1) \to K(\mathbf{Z}, 1)$ where f_k realizes the group homomorphism $z \mapsto kz$ on π_1. Since such a map is of degree k on homology, and homology commutes with direct limits, one sees that $H_1(K(\mathbf{Q}, 1)) = \mathbf{Q}$ and that all higher homology groups vanish.

We then extend the result inductively to any n. Consider the homology spectral sequences with integral and rational coefficients associated to the fibration $K(\mathbf{Q}, n) \hookrightarrow PK(\mathbf{Q}, n+1) \to K(\mathbf{Q}, n+1)$; the map between these two spectral sequences induces an isomorphism on $E^2_{0,q}$, $q > 0$, and $E^\infty_{p,q}$, $(p, q) \neq (0, 0)$. We then invoke Zeeman's comparison theorem to conclude that it also induces an isomorphism on $E^2_{p,0}$, $p > 0$ (see for instance [McC01, pp. 82–85] — the theorem is stated and proved for cohomology, but ignoring the additional hypothesis and the complications in the proof required to get a conclusion on the product structure, it is easily transposed). Thus since the groups $H_{>0}(K(\mathbf{Q}, n+1), \mathbf{Q})$ are rational vector spaces, so are the $H_{>0}(K(\mathbf{Q}, n+1), \mathbf{Z})$. □

With these two lemmas, we can proceed to the proof of the theorem.

Proof of theorem 2.2.1 on the preceding page. (1) $\implies$ (2). Let $\{X_n,\ p_n,\ i_n\}$ be a Postnikov tower for X. We prove by induction on $n \geq 2$ that $H_i(X_n)$ is a rational vector space for $i > 0$. The result then follows from lemma 1.1.6 on page 7. The base case (when n is the dimension of the first non-trivial homotopy group) follows from the lemma above. Now as usual consider the fibration $K(\pi_n(X), n) \hookrightarrow X_n \to X_{n-1}$. From the lemma 2.2.3 and the induction hypothesis, it follows that the integral homology groups (of positive degree) of the base space and of the fibre are rational vector spaces; hence so are the $H_{>0}(X_n; \mathbf{Z})$ (lemma 2.2.2 on the previous page).

(2) $\implies$ (1). Here, we use a slightly different construction instead of the Postnikov tower. A *Whitehead tower*, or *upside-down Postnikov tower*, of X is a diagram of the form

$$\begin{tikzcd} & \vdots \\ K(\pi_3(X), 2) \arrow[r] & X_3 \arrow[d, "p_3"'] \\ K(\pi_2(X), 1) \arrow[r] & X_2 \arrow[d, "p_2"'] \\ K(\pi_1(X), 0) \arrow[r] & X_1 \arrow[d, "p_1"'] \\ & X = X_0 \end{tikzcd}$$

where:

(1) $\pi_i(X_n) = 0$ for $i \leq n$;

(2) the maps $X_n \xrightarrow{p_n} X_{n-1}$ are fibrations with fibre $K(\pi_n(X), n-1)$;

(3) the composite maps $X_n \to X$ induce isomorphisms on π_i, $i > n$.

Any topological space admits such a tower (see [BT82, pp. 253–254] or [Hat02, p. 356]). In this setting, the Hurewicz theorem yields

$$\pi_{n+1}(X) = \pi_{n+1}(X_n) = H_{n+1}(X_n)$$

so in our case it suffices to show that for every n, $H_{n+1}(X_n)$ is a rational vector space. This follows by inductive application of lemma 2.2.2 on the preceding page. □

A rational homotopy equivalence between rational CW complexes is a weak homotopy equivalence and thus a homotopy equivalence. Therefore, for rational spaces, rational homotopy theory reduces to ordinary homotopy theory.

2.3 Localization and rational homotopy type

We now introduce the notion of *localization at (0)* (or *rationalization*) of a simply-connected topological space; such a localization can be described both as a rational homotopy equivalence to a rational space and through a certain universel property.

Proposition 2.3.1. *Let X and X' be simply-connected CW complexes, with X' rational, and let $f : X \to X'$ be a map. The following conditions on f are equivalent:*

(1) f is a rational homotopy equivalence;

(2) f has the following universal property: for any simply-connected rational CW complex Y and any map $g : X \to Y$, there is a map $h : X' \to Y$ unique up to homotopy such that

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g"'] & X' \arrow[d, "h"] \\ & Y \end{tikzcd}$$

commutes.

We call a map f satisfying these conditions a rationalization*, or* localization at (0)*, of X. When such a map exists, the rational CW complex X' is unique up to homotopy equivalence; we shall denote it by $X_{\mathbf{Q}}$.*

Proof. (1) $\implies$ (2). Deform f to be the inclusion of a subcomplex into a CW complex. It is an easy application of obstruction theory that in this setting, a map $X \to Y$ may be extended to a map $X' \to Y$ if all $H^{i+1}(X, X'; \pi_i(Y))$, $i \geq 1$ vanish; and that a homotopy over X of two maps $X' \to Y$ may be extended over X' if all $H^{i+1}(X, X'; \pi_{i+1}(Y))$, $i \geq 1$ vanish (see for instance [Hu59, chap. VI] or [GM81, chap. V]). Now f is a rational equivalence, so $H^i(X, X'; \mathbf{Q}) = 0$, $i \geq 1$; since the homotopy groups of Y are rational vector spaces, the conditions above follow from the universal coefficient theorem.

(2) $\implies$ (1). The universal property of f for $Y = K(\mathbf{Q}, n)$, $n \geq 2$, shows that composition with f induces a bijection between $[X, K(\mathbf{Q}, n)]$ and $[X', K(\mathbf{Q}, n)]$. It follows from the naturality of the correspondence of theorem 1.2.3 on page 8 (and of remark 1.2.5 just underneath) that f induces an isomorphism on rational cohomology. □

Example 2.3.2. In remark 2.1.4 on page 15, we mentioned the rational homotopy equivalence $K(\mathbf{Z}, n) \to K(\mathbf{Q}, n)$. More generally, there is a rationalization $K(\pi, n) \to K(\pi \otimes \mathbf{Q}, n)$ constructed by realizing the map $\pi \to \pi \otimes \mathbf{Q}$.

We shall prove below that every simply-connected CW complex admits a localization; let us admit this result for a moment. Let X and Y be two simply-connected CW complexes. It is clear that if X and Y are of the same rational homotopy type, then $X_{\mathbf{Q}} = Y_{\mathbf{Q}}$; we may therefore identify $X_{\mathbf{Q}}$ with the rational homotopy type of X. Now consider a map $f : X \to Y$. The universal property of rationalization gives us some $f_{\mathbf{Q}} : X_{\mathbf{Q}} \to Y_{\mathbf{Q}}$, unique up to homotopy, such that

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X_{\mathbf{Q}} \arrow[r, "f_{\mathbf{Q}}"'] & Y_{\mathbf{Q}} \end{tikzcd}$$

commutes; this defines a map $[X, Y] \to [X_{\mathbf{Q}}, Y_{\mathbf{Q}}]$. The homotopy class of $f_{\mathbf{Q}}$ is called the *rational homotopy class* of f.

From there we can construct a localization of the category $\mathcal{T}_1$ of simply-connected topological spaces with respect to the rational homotopy equivalences, as promised in the introduction. Indeed, let $\mathrm{Ho}_{\mathbf{Q}}\,\mathcal{T}_1$ be the category with the same objects as $\mathcal{T}_1$ but with maps

$$\mathrm{Ho}_{\mathbf{Q}}\,\mathcal{T}_1\ (X,Y) = [X_{\mathbf{Q}}, Y_{\mathbf{Q}}].$$

It is easily seen that the functor $\gamma : \mathcal{T}_1 \to \mathrm{Ho}_{\mathbf{Q}}\,\mathcal{T}_1$ which is the identity on objects and sends f to $[f_{\mathbf{Q}}]$ is a localization of $\mathcal{T}_1$ with respect to the rational homotopy equivalences. Notice that the category $\mathrm{Ho}_{\mathbf{Q}}\,\mathcal{T}_1$ is equivalent to the homotopy category (in the usual sense, that is with homotopy classes of maps as arrows) of (simply-connected) rational CW complexes.

2.4 The rational Postnikov tower

We now introduce the rational analogue of the Postnikov tower of principal fibrations described in section 1; we show how to build such a tower and then use it to construct the rationalization of a space.

Definition 2.4.1. Let X be a simply-connected space. A *rational Postnikov tower* for X is a commutative diagram

$$\begin{tikzcd}
 & \vdots \\
 & X_{4,\mathbf{Q}} \arrow[d, "q_3"] \\
 & X_{3,\mathbf{Q}} \arrow[d, "q_2"] \\
X \arrow[r, "j_2"'] \arrow[ur, "j_3"] \arrow[uur, "j_4"] & X_{2,\mathbf{Q}} = K(\pi_2(X) \otimes \mathbf{Q}, 2)
\end{tikzcd}$$

where:

(1) the spaces $X_{n,\mathbf{Q}}$, $n \geq 2$, are rational;

(2) $\pi_i(X_{n,\mathbf{Q}}) = 0$ for $i > n$;

(3) the maps $X \xrightarrow{j_n} X_{n,\mathbf{Q}}$ induce isomorphisms on the rational homotopy groups $\pi_i \otimes \mathbf{Q}$ for $i \leq n$;

(4) the maps $X_{n+1,\mathbf{Q}} \xrightarrow{q_n} X_{n,\mathbf{Q}}$ are fibrations.

Remark 2.4.2. Let X be a simply-connected CW complex, and let $\{X_n, p_n, i_n\}$ be a Postnikov tower for X constructed as suggested in remark 1.1.5 on page 6. If a rational Postnikov tower as above exists, then it is easily seen that $j_n : X \to X_{n,\mathbf{Q}}$ can be extended to a map $X_n \to X_{n,\mathbf{Q}}$ inducing isomorphisms on rational homotopy groups; thus necessarily $X_{n,\mathbf{Q}} \simeq (X_n)_{\mathbf{Q}}$, which justifies our notation.

For CW complexes, it is indeed possible to construct a rational Postnikov tower through an inductive rationalization of the integral one. This is how we prove the following theorem.

Theorem 2.4.3. *Let X be a simply-connected CW complex. Then X has a rational Postnikov tower $\{X_{n,\boldsymbol{Q}},\ q_n,\ j_n\}$ with the $X_{n,\boldsymbol{Q}}$ rational CW complexes and the q_n principal fibrations, as in the diagram below.*

$$\begin{tikzcd}
 & \vdots & \\
 & X_{4,\boldsymbol{Q}} \arrow[r, "k_{4,\boldsymbol{Q}}"'] \arrow[d, "q_3"] & K(\pi_5(X)\otimes \boldsymbol{Q}, 6) \\
 & X_{3,\boldsymbol{Q}} \arrow[r, "k_{3,\boldsymbol{Q}}"'] \arrow[d, "q_2"] & K(\pi_4(X)\otimes \boldsymbol{Q}, 5) \\
X \arrow[r, "j_2"'] \arrow[ur, "j_3"'] \arrow[uur, "j_4"] & X_{2,\boldsymbol{Q}} = K(\pi_2(X)\otimes \boldsymbol{Q}, 2) \arrow[r, "k_{2,\boldsymbol{Q}}"'] & K(\pi_3(X)\otimes \boldsymbol{Q}, 4)
\end{tikzcd}$$

The k-invariant of the principal fibration q_n, which is a cohomology class in $H^{n+2}(X_{n,\boldsymbol{Q}};\pi_{n+1}(X)\otimes \boldsymbol{Q})$, is called the n-th rational k-invariant of X.

Proof. Let $\{X_n,\ p_n,\ i_n\}$ be an integral Postnikov tower for X. We inductively construct a rational Postnikov tower for X such that $X_{n,\mathbf{Q}} = (X_n)_{\mathbf{Q}}$. For the sake of brevity, we write π_n for $\pi_n(X)$.

For $n = 2$, there is a rationalization $K(\pi_2, 2) \to K(\pi_2 \otimes \mathbf{Q}, 2)$ (remark 2.3.2). (Note that if π_2 is torsion this is just a map to $\{*\}$.) Now suppose we have a rationalization $X_n \to X_{n,\mathbf{Q}}$. As usual there is a rational equivalence $K(\pi_{n+1}, n+2) \to K(\pi_{n+1} \otimes \mathbf{Q}, n+2)$; it induces a commutative square

$$\begin{tikzcd}
PK(\pi_{n+1}, n+2) \arrow[r] \arrow[d] & PK(\pi_{n+1}\otimes \mathbf{Q}, n+2) \arrow[d] \\
K(\pi_{n+1}, n+2) \arrow[r] & K(\pi_{n+1}\otimes \mathbf{Q}, n+2)
\end{tikzcd}$$

the upper map sending a path to its image under the lower map. Now let $k_n : X_n \to K(\pi_{n+1}, n+2)$ be the map inducing the fibration $X_{n+1} \to X_n$; the universal property of $X_n \to X_{n,\mathbf{Q}}$ gives us a map $k_{n,\mathbf{Q}} : X_{n,\mathbf{Q}} \to K(\pi_{n+1} \otimes \mathbf{Q}, n+2)$ such that

$$\begin{tikzcd}
X_n \arrow[r] \arrow[d, "k_n"'] & X_{n,\mathbf{Q}} \arrow[d, "k_{n,\mathbf{Q}}"] \\
K(\pi_{n+1}, n+2) \arrow[r] & K(\pi_{n+1}\otimes \mathbf{Q}, n+2)
\end{tikzcd}$$

commutes. This map induces a fibration $X_{n+1,\mathbf{Q}} \to X_{n,\mathbf{Q}}$. Then the following solid line diagram commutes.

$$\begin{tikzcd}
 & PK(\pi_{n+1}, n+2) \arrow[rr] \arrow[dd] & & PK(\pi_{n+1}\otimes \mathbf{Q}, n+2) \arrow[dd] \\
X_{n+1} \arrow[ur] \arrow[rr, dashed] \arrow[dd] & & X_{n+1,\mathbf{Q}} \arrow[ur] & \\
 & K(\pi_{n+1}, n+2) \arrow[rr] & & K(\pi_{n+1}\otimes \mathbf{Q}, n+2) \\
X_n \arrow[ur, "k_n"] \arrow[rr] & & X_{n,\mathbf{Q}} \arrow[ur, "k_{n,\mathbf{Q}}"'] \arrow[from=uu, crossing over] &
\end{tikzcd}$$

Hence one gets the same map when going from X_{n+1} to $K(\pi_{n+1} \otimes \mathbf{Q}, n+2)$ through $PK(\pi_{n+1} \otimes \mathbf{Q}, n+2)$ or through $X_{n,\mathbf{Q}}$; the universal property of the pullback therefore gives a map $X_{n+1} \to X_{n+1,\mathbf{Q}}$, the dashed arrow in the diagram. It is a rational equivalence and thus a rationalization. $\square$

Corollary 2.4.4. *Let X be a simply-connected CW complex. Then X admits a rationalization $f : X \to X_{\mathbf{Q}}$.*

Proof. Consider the rational Postnikov tower for X constructed in the theorem above. The map $X \to \varprojlim X_{n,\mathbf{Q}}$ is a rational homotopy equivalence, as can be seen from the same arguments as in the integral case (proposition 1.1.7 on page 7). Proposition 1.1.3 on page 6 gives us a weak (thus rational) homotopy equivalence $Z \to \varprojlim X_n$, where Z is a (necessarily rational) CW complex. Turning this weak homotopy equivalence into an inclusion, we get a pair $(Z, \varprojlim X_n)$ all of whose relative homotopy groups vanish. In this setting, it is easy to prove (by induction over the skeleta) that any map from a CW complex to $\varprojlim X_n$ may be retracted into a map with image in Z (see [Hat02, p. 347]). In other words, there exists a map $f : X \to Z$ such that

$$\begin{tikzcd} & Z \arrow[d] \\ X \arrow[ur, "f"] \arrow[r] & \varprojlim X_n \end{tikzcd}$$

commutes up to homotopy; then f has to be a rational homotopy equivalence, and $Z = X_{\mathbf{Q}}$. $\square$

Note that a rational Postnikov tower for X is just a Postnikov tower for $X_{\mathbf{Q}}$. Such a tower characterizes the rational homotopy type of X entirely.

Section 3

A homotopy theory for CDGAs

In this section $\mathbf{k}$ is any field of characteristic zero.

Definition 3.0.5. A *differential graded algebra* (or DGA) over $\mathbf{k}$ is a non-negatively graded $\mathbf{k}$-vector space

$$C^* = \bigoplus_{i \geq 0} C^i$$

equipped with a product $C^p \otimes C^q \to C^{p+q}$ and with a *differential* of degree 1, i.e. a map

$$d : C^p \to C^{p+1} \quad \text{with} \quad d^2 = 0,$$

which is also a *derivation* with respect to the product, i.e. satisfies the graded Leibniz rule

$$d(\alpha\beta) = (d\alpha)\beta + (-1)^p \alpha(d\beta)$$

for $\alpha \in C^p$. If $C \neq 0$, *we require the product to have a unit* (which has to be in C^0).

Additionally, we say that a differential graded algebra (C^*, d) is *commutative* (is a *commutative differential graded algebra*, CDGA) if it satisfies the relation of graded commutativity

$$\alpha\beta = (-1)^{pq}\beta\alpha \quad \text{for} \quad \alpha \in C^p,\ \beta \in C^q.$$

A *map* of differential graded algebras between (C^*, d) and (D^*, d) is a family $(f^n)_{n \geq 0}$ of maps $C^n \to D^n$ commuting with the differential and compatible with the product.

We shall denote the category of CDGAs over $\mathbf{k}$ by $\mathcal{G}$.

Remark 3.0.6. This terminology is not entirely standard: some authors do not include any condition on the grading in the definition and would refer to non-negatively graded (C)DGAs as (commutative) *cochain algebras*.

For any DGA (C^*, d) we can form the graded algebra $H^*(C)$ in the usual way; a map of DGAs $f : (C^*, d) \to (D^*, d)$ then induces a map $H^*(f) : H^*(C) \to H^*(D)$.

Definition 3.0.7. We say that a DGA (C^*, d) is *connected* if $H^0(C) = \mathbf{k}$, and *simply-connected* if additionally $H^1(C) = 0$.

For any graded object C^*, we shall freely use transparent notations like $C^{\leq n}$, $C^{\geq n}$, or C^+ to denote the elements of degree $\leq n$ (resp. $\geq n$, > 0).

We introduce one more piece of notation. Let $V = (V^i)_{i\geq 0}$ be a graded $\mathbf{k}$-vector space. We write ΛV for the free graded commutative algebra on V; more explicitly, if $\mathcal{B}$ is a homogeneous basis of V, then ΛV is the polynomial algebra on the elements of $\mathcal{B}$ of even degree tensor the exterior algebra on the elements of $\mathcal{B}$ of odd degree.

Pedantic remark 3.0.8. Remember that our algebras have units: the $\mathbf{k}$-vector space ΛV contains a generator $1 \in (\Lambda V)^0$ even if $V^0 = 0$. More generally, note that in any CDGA we have $d(1) = 0$.

3.1 Model categories

This subsection is a general introduction to model categories. None of the individual results are particularly hard on their own; accordingly, we only included a few proofs, giving what we hope is a representative sample of the techniques. Details and proofs may be found in Quillen's original memoir [Qui67] or in the introductory paper by Dwyer and Spalinski [DS95]. We followed the terminology and notation of the latter (which is mostly identical with that of [Qui69]).

Let $\mathcal{C}$ be a category with three distinguished classes of arrows, the *weak equivalences* ($\xrightarrow{\sim}$), the *fibrations* ($\twoheadrightarrow$) and the *cofibrations* ($\hookrightarrow$). An *acyclic fibration* ($\overset{\sim}{\twoheadrightarrow}$) is an arrow which is both a fibration and a weak equivalence; an *acyclic cofibration* ($\overset{\sim}{\hookrightarrow}$) is an arrow which is both a cofibration and a weak equivalence.

Definition 3.1.1. We say that these choices define a *model category structure* on $\mathcal{C}$ if the following hold.

M1. All finite limits and colimits exist in $\mathcal{C}$.

M2. Let $A \xrightarrow{f} A'$ and $A' \xrightarrow{g} A''$ be two arrows. If any two of f, g and gf are weak equivalences, then so is the third.

M3. If g is a *retract* of f, that is if there is a commutative diagram

$$\begin{tikzcd} Y \arrow[r, "i"] \arrow[d, "g"'] & X \arrow[r, "r"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ Y' \arrow[r, "i'"'] & X' \arrow[r, "r'"'] & Y' \end{tikzcd}$$

such that $ri = \mathrm{Id}_Y$, $r'i' = \mathrm{Id}_{Y'}$ and if f is a weak equivalence, a fibration or a cofibration, then so is g.

M4. Given the following solid line diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

an arrow as shown dashed exists in the following two situations:

(i) i is a cofibration and p an acyclic fibration;

(ii) i is an acyclic cofibration and p a fibration.

M5. Any arrow $f : A \to B$ can be factored as:

(i) $A \hookrightarrow A' \overset{\sim}{\twoheadrightarrow} B$;

(ii) $A \overset{\sim}{\hookrightarrow} A' \twoheadrightarrow B$.

Remark 3.1.2. These axioms are self-dual, that is, a model category structure on $\mathcal{C}$ gives rise to a model category structure on the opposite category $\mathcal{C}^{\mathrm{op}}$ in which the fibrations and the cofibrations have been swapped. Therefore, if some statement is true in any model category, then so is the dual statement obtained by exchanging the words *fibration* and *cofibration*.

Before proceeding to prove the first properties of model category structures, let us give some terminology.

Definition 3.1.3. If there is an arrow like the one shown dashed making the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

commute, we say that p has the *right lifting property* with respect to i and that i has the *left lifting property* with respect to p.

Then one way to rephrase M4 above is: cofibrations have the left lifting property with respect to all acyclic fibrations, and fibrations have the right lifting property with respect to all acyclic cofibrations.

The category $\mathcal{C}$ has all finite limits and colimits, so in particular has an initial object $\varnothing$ and a terminal object $*$. Using these we give the following definition.

Definition 3.1.4. We say that an object X of $\mathcal{C}$ is *fibrant* (resp. *cofibrant*) if $X \to *$ is a fibration (resp. if $\varnothing \to X$ is a cofibration).

The first property we prove is that in a model category, the classes of fibrations and of cofibrations determine each other. This is a kind of converse of M4.

Proposition 3.1.5. *Let $\mathcal{C}$ be a model category. Then an arrow f is a cofibration (resp. an acyclic cofibration) if and only if it has the left lifting property with respect to all acyclic fibrations (resp. to all fibrations); and dually for fibrations.*

Proof. As an example, suppose $f : A \to B$ has the left lifting property with respect to all fibrations. Use M5 to factor f as

$$A \underset{\sim}{\overset{i}{\hookrightarrow}} A' \overset{p}{\twoheadrightarrow} B$$

and notice that by hypothesis there exists an arrow $B \xrightarrow{r} A'$ such that the diagram

$$\begin{tikzcd} A \arrow[r, hook, "\sim", "i"'] \arrow[d, "f"'] & A' \arrow[d, "p"] \\ B \arrow[r, "\mathrm{Id}_B"'] \arrow[ur, dashed, "r"'] & B \end{tikzcd}$$

commutes. Then f is a retract of i

$$\begin{tikzcd} A \arrow[r, "\mathrm{Id}_A"] \arrow[d, "f"'] & A \arrow[r, "\mathrm{Id}_A"] \arrow[d, hook, "\wr"', "i"] & A \arrow[d, "f"] \\ B \arrow[r, "r"'] & B' \arrow[r, "p"'] & B \end{tikzcd}$$

and so is an acyclic cofibration according to M3. $\square$

Remark 3.1.6. In particular, every isomorphism is a weak equivalence, a fibration and a cofibration.

Examples 3.1.7.
1. There is a model category structure on the category $\mathcal{T}$ of topological spaces in which:
   - the weak equivalences are the weak homotopy equivalences;
   - the fibrations are the Serre fibrations, i.e. the maps that have the homotopy lifting property with respect to CW complexes;
   - the cofibrations are the maps that have the left lifting property with respect to every acyclic fibration.

   Notice that every object is fibrant. An important class of cofibrations is constituted by the inclusions $X \to Y$ where Y is obtained from X by attaching cells (not necessarily in order of their dimension); in particular, CW complexes are cofibrant. In fact, one can prove that every cofibration is a retract of such a map (see [DS95, p. 108]).
2. We would like to define a model category structure with the rational homotopy equivalences as weak equivalences. Unfortunately, the category of simply-connected spaces is not closed under finite colimits (the same objection would apply to the category of nilpotent spaces). This is more than an insignificant nuisance, but a fair amount of the theory is still applicable; see [Qui69, II.6] for a full discussion.

An essential property of model categories is that they carry intrinsic notions of homotopy between maps. We now describe these.

Definition 3.1.8. A *cylinder object* for A is an object $A \wedge I$ of $\mathcal{C}$ together with a factorization

$$A \coprod A \xrightarrow{i} A \wedge I \xrightarrow{\sim} A$$

of the map $\mathrm{Id}_A \coprod \mathrm{Id}_A : A \coprod A \to A$; a cylinder object for A is called *good* if i is a cofibration. We write $i_0, i_1 : A \to A \wedge I$ for the two maps represented by i.

Dually, a *path object* for X is an object X^I of $\mathcal{C}$ together with a factorization

$$X \xrightarrow{\sim} X^I \xrightarrow{p} X \times X$$

of the diagonal map $X \to X \times X$; a path object for X is called *good* if p is a fibration. We write $p_0, p_1 : X^I \to X \times X$ for the two maps represented by p.

Examples 3.1.9. In the case of topological spaces (example 3.1.7 above) one choice of a cylinder object for A is $A \times [0,1]$, and one choice of a path object for X is $X^{[0,1]}$ (with the compact-open topology).

Definition 3.1.10. Two maps $f, g : A \to X$ are said to be *left homotopic* (written $f \overset{l}{\sim} g$) if there is a cylinder object $A \wedge I$ for A and a map $H : A \wedge I \to X$ such that the composite

$$A \coprod A \xrightarrow{i} A \wedge I \xrightarrow{H} X$$

is the map $f \coprod g : A \coprod A \to X$.

Dually, two maps $f, g : A \to X$ are said to be *right homotopic* (written $f \overset{r}{\sim} g$) if there is a path object X^I for X and a map $H : A \to X^I$ such that the composite

$$A \xrightarrow{H} X^I \xrightarrow{p} X \times X$$

is the map $(f, g) : A \to X \times X$.

Left and right homotopy are not equivalence relations in general; however, we have the following.

Proposition 3.1.11. *If A is cofibrant, then $\overset{l}{\sim}$ is an equivalence relation on $\mathcal{C}(A, X)$; dually, if X is fibrant, then $\overset{r}{\sim}$ is an equivalence relation on $\mathcal{C}(A, X)$.*

As an example of how these notions may be used, we prove the following fact. When A is cofibrant, we write $[A, X]_l$ for the set of left homotopy classes in $\mathcal{C}(A, X)$; similarly, when X is fibrant, we write $[A, X]_r$ for the set of right homotopy classes in $\mathcal{C}(A, X)$.

Proposition 3.1.12. *Let A be cofibrant and let $p : Y \overset{\sim}{\twoheadrightarrow} X$ be an acylic fibration. Then composition with p induces a bijection $[A, Y]_l \to [A, X]_l$.*

$$\begin{tikzcd}
 & Y \arrow[d, "p", "\wr"'] \\
A \arrow[r] \arrow[ur, dashed] & X
\end{tikzcd}$$

Dually, if X is fibrant and $i : A \overset{\sim}{\hookrightarrow} B$ is an acylic cofibration, then composition with i induces a bijection $[B, X]_r \to [A, X]_r$.

Proof. We prove the first part. To show the surjectivity, apply M4(i) to the diagram

$$\begin{tikzcd}
\varnothing \arrow[r] \arrow[d, hook] & Y \arrow[d, "p", "\wr"'] \\
A \arrow[r, "f"'] \arrow[ur, dashed] & X
\end{tikzcd}$$

for any $f : A \to X$.

Now consider $f, g : A \to Y$ such that $pf \overset{l}{\sim} pg$; this left homotopy is given through some cylinder object $A \wedge I$ for A. We may replace it by a good cylinder object $A \wedge I'$ (indeed, apply M5(i) to factor the map $A \coprod A \to A \wedge I$ as a cofibration followed by an acyclic fibration); so we have a factorization

$$A \coprod A \hookrightarrow A \wedge I' \xrightarrow{H} X$$

of $pf \coprod pg$. Applying M4(i) we get a lift as shown dashed in the diagram

$$\begin{tikzcd} A \coprod A \arrow[r, "f \coprod g"] \arrow[d, hook] & Y \arrow[d, two heads, "p", "\wr"'] \\ A \wedge I' \arrow[r, "H"'] \arrow[ur, dashed] & X \end{tikzcd}$$

and this gives us the desired homotopy $f \overset{l}{\sim} g$. ☐

The next proposition relates left and right homotopy; in particular, it implies that if A is cofibrant and X is fibrant, then the two homotopy relations agree and can be described through the use of just one good path or cylinder object.

Proposition 3.1.13. *Let $f, g : A \to X$ be two maps. If A is cofibrant and $f \overset{l}{\sim} g$, then $f \overset{r}{\sim} g$ through any good path object for X; dually, if X is fibrant and $f \overset{r}{\sim} g$, then $f \overset{l}{\sim} g$ through any good cylinder object for A.*

Accordingly, if A is cofibrant and X is fibrant, we say that two maps $f, g : A \to X$ are *homotopic* if they are left or right homotopic. We write $[A, X]$ for the set of homotopy classes of maps $A \to X$. With the additional assumption that A is fibrant, we have a generalization of proposition 3.1.12 on the preceding page.

Proposition 3.1.14. *Let A be fibrant-cofibrant, let X and Y be fibrant and let $p : Y \xrightarrow{\sim} X$ be a weak equivalence (not necessarily a fibration). Then composition with p induces a bijection $[A, Y] \to [A, X]$.*

$$\begin{tikzcd} & Y \arrow[d, "p", "\wr"'] \\ A \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

Proof. Apply M5(ii) to factor p as

$$Y \underset{\sim}{\overset{i}{\hookrightarrow}} Y' \underset{\sim}{\overset{j}{\twoheadrightarrow}} X$$

the second map being acyclic because of M2. Then since A is fibrant M4(ii) yields a left inverse $r : Y' \to Y$ for i. From the second part of proposition 3.1.12 on the previous page, one sees that r is actually a two-sided *homotopy inverse* for i; therefore composition with i induces a bijection $[A, Y] \to [A, Y']$. To conclude, apply the first part of proposition 3.1.12 to j. ☐

Remark 3.1.15. In the context of topological spaces, a result akin to this one sometimes goes under the name of "Whitehead's lifting lemma".

Similarly, one proves the following useful fact.

Proposition 3.1.16. *A map $f : A \to X$ between fibrant-cofibrant objects is a weak equivalence if and only if it admits a homotopy inverse.*

We conclude this presentation with a discussion of the *homotopy category* $\mathrm{Ho}\,\mathcal{C}$ associated to a model category $\mathcal{C}$. It is the localization of $\mathcal{C}$ with respect to the weak equivalences; of course, it does not depend on the fibrations or cofibrations. Indeed, one of the motivations of Quillen's theory is that endowing a category $\mathcal{C}$ one wants to localize with an additional homotopic structure (the fibrations and the cofibrations) allows one to give a more explicit description of $\mathrm{Ho}\,\mathcal{C}$. This description is the following.

Proposition 3.1.17. *The homotopy category $Ho\,\mathcal{C}$ associated to the model category $\mathcal{C}$ is equivalent to the category $\pi\mathcal{C}_{cf}$ whose objects are the fibrant-cofibrant objects of $\mathcal{C}$ and whose maps are the homotopy classes of maps determined by the model category structure.*

3.2 The model category of CDGAs

We are now going to describe a model category structure on the category $\mathcal{G}$ of CDGAs over $\mathbf{k}$. Before we start, note that $\mathcal{G}$ is closed under finite limits and colimits. In particular, two CDGAs (C, d) and (D, d) admit both a coproduct which is just their tensor product as graded algebras

$$(C \otimes D)^n = \bigoplus_{p+q=n} C^p \otimes D^q$$

with the obvious differential, and a product which is the underlying product of graded vector spaces

$$(C \times D)^n = C^n \times D^n$$

with componentwise multiplication and differentiation.

We now give the following definitions. Note that we shall keep the notations of the presentation of model categories above: do not mistake our weak equivalences for isomorphisms! We say that $f : (C, d) \to (D, d)$ is:

- a weak equivalence if $H^*(f) : H^*(C) \to H^*(D)$ is an isomorphism (f is a *quasi-isomorphism*);
- a fibration if f is *surjective*;
- a cofibration if f has the left lifting property with respect to all acyclic fibrations.

From proposition 3.1.5 on page 24 we know that this choice of the cofibrations is the only possible one once the fibrations and weak equivalences have been fixed; but we still have to check that the model category axioms hold, which amounts to checking that there indeed exists a model category with the weak equivalences and fibrations we specified.

Remark 3.2.1. As in the case of topological spaces, every object is fibrant.

Proposition 3.2.2. *The classes of weak equivalences, fibrations and cofibrations given above define a model category structure on $\mathcal{G}$.*

Before proceeding to the proof, we describe a large class of cofibrations, for both immediate and later use.

Definition 3.2.3. We shall call *relative Sullivan algebra with respect to* (B, d) any CDGA of the form $(B \otimes \Lambda V, d)$ where:

- $(B, d) \cong (B \otimes 1, d)$ is a connected sub-CDGA;
- V admits an exhaustive filtration $V(0) \subseteq V(1) \subseteq V(2) \subseteq \dots$ such that $d : V(0) \to B$ and for $k \geq 1$, $d : V(k) \to B \otimes \Lambda V(k-1)$.

A *Sullivan algebra* is a relative Sullivan algebra with $B = 0$.

Remark 3.2.4. We warn the reader that the authors of [FHT01] give a slightly different definition, in that they require V to have no elements of degree 0.

In other words, a relative Sullivan algebra $(B \otimes \Lambda V, d)$ is an algebra obtained from B by progressively adding new "free generators", with the condition that their differential has to be in the previously constructed part of the algebra.

Example 3.2.5. To shed some light on this definition, we give the following counter-example: $(\Lambda(x_1, x_2, x_3), d)$ with $\deg x_i = 1$, $dx_1 = x_2 x_3$, $dx_2 = x_1 x_3$, $dx_3 = x_1 x_2$ is free, but not Sullivan.

Lemma 3.2.6. *Let $(B \otimes \Lambda V, d)$ be a relative Sullivan algebra. Then the inclusion $(B, d) \to (B \otimes \Lambda V, d)$ is a cofibration. In particular, Sullivan algebras are cofibrant.*

Proof. Suppose we are given an acyclic fibration (i.e. a surjection inducing an isomorphism on homology) $(C, d) \xrightarrow[\sim]{\eta} (D, d)$ and the solid line diagram below.

$$\begin{tikzcd}
(B,d) \arrow[r] \arrow[d] & (C,d) \arrow[d, two heads, "\eta", "\wr"'] \\
(B \otimes \Lambda V, d) \arrow[r, "\psi"'] \arrow[ur, dashed, "\phi"] & (D,d)
\end{tikzcd}$$

We need to construct ϕ as shown dashed. We already have ϕ on (B, d); we extend it to $(B \otimes \Lambda V(k), d)$ by induction on k. Assume ϕ has been constructed on $(B \otimes \Lambda V(k-1), d)$ (or if $k = 0$, on (B, d)), and write $V(k) = V(k-1) \oplus W$ where $d : W \to B \otimes \Lambda V(k-1)$. Let (w_i) be a basis of W. For each w_i, we need to construct $c_i \in C$ such that $\eta(c_i) = \psi(w_i)$ and $dc_i = \phi(dw_i)$.

We have $d(\phi(dw_i)) = \phi(d^2 w_i) = 0$ and $\eta(\phi(dw_i)) = \psi(dw_i) = d\psi(w_i)$, so $\phi(dw_i)$ is a cocycle sent by the quasi-isomorphism η to a coboundary: hence $\phi(dw_i) = dt$ for some $t \in C$. Now $d(\eta(t) - \psi(w_i)) = 0$, so there is some $u \in C$ with $du = 0$ such that

$$\eta(u) = \eta(t) - \psi(w_i) + dv$$

for some $v \in D$. Finally, the surjectivity of η gives us $v' \in C$ such that $\eta(v') = v$. Set

$$c_i = t - u + dv'.$$

Then

$$\eta(c_i) = \eta(t) - \eta(u) + d\eta(v') = \psi(w_i)$$
$$dc_i = dt - du + d^2v' = \phi(dw_i)$$

and c_i meets the required conditions. $\square$

Proof of proposition 3.2.2 on the previous page. Axiom M1 is discussed above; M2, M3 and M4(i) are immediate.

We first show M5(ii). Let $f : (A, d) \to (B, d)$ be a map of CDGAs. Let $(b_i)_{i\in I}$ be a basis of the $\mathbf{k}$-vector space B. Define $(E(B), \delta) = (\Lambda(c_i, c'_i)_{i\in I}, \delta)$ where $\deg c_i = \deg b_i$, $\deg c'_i = \deg b_i + 1$, $\delta c_i = c'_i$ and $\delta c'_i = 0$; then $(A, d) \otimes (E(B), \delta)$ is a relative Sullivan algebra. Moreover, an easy computation shows $H^*(E(B), \delta) = \mathbf{k}$, so that the relative Sullivan algebra inclusion

$$(A, d) \to (A, d) \otimes (E(B), \delta)$$

is an acylic cofibration. Then define $g : (A, d) \otimes (E(B), \delta) \to (B, d)$ by $g = f$ on (A, d), $g(c_i) = b_i$ and $g(c'_i) = db_i$; then g is a well-defined surjective CDGA map, and we have a factorization

$$(A, d) \overset{\sim}{\hookrightarrow} (A, d) \otimes (E(B), \delta) \twoheadrightarrow (B, d)$$

of $f : (A, d) \to (B, d)$.

We omit the proof of M5(i), which is similar but slightly longer; see [BG76, pp. 21–22].

We now turn to M4(ii). Consider an acyclic cofibration $f : (A, d) \to (B, d)$; in the factorization we just obtained, the fibration $p : (A, d) \otimes (E(B), \delta) \twoheadrightarrow (B, d)$ is acyclic according to M2. Then, since f has a the left lifting property with respect to p, using the same technique as in proposition 3.1.5 on page 24 we can show that f is a retract of $(A, d) \overset{\sim}{\hookrightarrow} (A, d) \otimes (E(B), \delta)$. It is easily seen that this last map has the left lifting property with respect to all fibrations, and therefore so does f. $\square$

3.3 The homotopy relation for maps of CDGAs

Let $A \in \mathcal{G}$ be *cofibrant* (for instance a Sullivan algebra) and let $X \in \mathcal{G}$ be any CDGA (always fibrant). Then left and right homotopy both define an equivalence relation on $\mathcal{G}(A, X)$, and these relations agree (proposition 3.1.13 on page 27); we shall simply say that $f, g \in \mathcal{G}(A, X)$ are *homotopic* (in symbols $f \sim g$) if they are right or left homotopic, and denote by $[A, X]$ the set of homotopy classes of maps from A to X.

The following "lifting lemma" is just a specialization of 3.1.14 on page 27; we state it explicitly for later reference.

Proposition 3.3.1. *Let (A, d) be cofibrant and let $p : (Y, d) \xrightarrow{\sim} (X, d)$ be a weak equivalence (not necessarily a fibration). Then composition with p induces a bijection $[(A, d), (Y, d)] \to [(A, d), (X, d)]$.*

$$\begin{tikzcd} & (Y, d) \arrow[d, "p", "\wr"'] \\ (A, d) \arrow[r] \arrow[ur, dashed] & (X, d) \end{tikzcd}$$

Next, define $\Lambda(t,dt)$ to be the free CDGA on t and dt, with $\deg t = 0$, $\deg dt = 1$ and the obvious differential (which we omit from the notation); and define two CDGA maps $\varepsilon_0, \varepsilon_1 : \Lambda(t,dt) \to \mathbf{k}$ by

$$\varepsilon_0(t) = 0, \quad \varepsilon_1(t) = 1, \quad \varepsilon_0(dt) = \varepsilon_1(dt) = 0.$$

Then for any CDGA (X,d) the factorization

$$(X,d) \xrightarrow{\sim} (X,d) \otimes \Lambda(t,dt) \xrightarrow{(\mathrm{Id}_X \otimes \varepsilon_0, \mathrm{Id}_X \otimes \varepsilon_1)} (X,d) \times (X,d)$$

of the diagonal map exhibits $(X,d) \otimes \Lambda(t,dt)$ as a good path object for (X,d). Then, as a further immediate consequence of proposition 3.1.13 on page 27, we get:

Proposition 3.3.2. *Let (A,d) and (X,d) be CDGAs, with (A,d) cofibrant. Then two maps $f,g : (A,d) \to (X,d)$ are homotopic (in the model category sense) if and only if there is a map $H : (A,d) \to (X,d) \otimes \Lambda(t,dt)$ such that the composite*

$$(A,d) \xrightarrow{H} (X,d) \otimes \Lambda(t,dt) \xrightarrow{(Id_X \otimes \varepsilon_0, Id_X \otimes \varepsilon_1)} (X,d) \times (X,d)$$

is the map $(f,g) : (A,d) \to (X,d) \times (X,d)$.

Remark 3.3.3. With a suitable definition of the notion of rational differential form on a topological space, the algebra $\Lambda(t,dt)$ can be seen as the algebra of rational differential forms on the unit interval I. Our explicit homotopy relation then becomes the algebraic analogue of the usual homotopy relation for maps of topological spaces, and with this motivation in mind can be taken as definition, thus avoiding any reference to the model category structure. This is Sullivan's original approach, as exposed in [GM81] and [DGMS75].

This characterization allows one to prove the following through direct computation; see [FHT01, p. 152].

Proposition 3.3.4. *Let $f \sim g : (A,d) \to (B,d)$ be homotopic CGDA maps, with (A,d) cofibrant. Then $H(f) = H(g)$.*

3.4 Minimal models

We finally introduce the notion of a minimal model for a CDGA, which is in a sense the algebraic analogue of the Postnikov tower, and which constitutes the algebraic basis for the duality exposed in section 4.

We define the notion of minimal CDGA through the following equivalent properties. In general, we say that an element a of a CDGA (A,d) is *decomposable* if it is in $A^+ \cdot A^+$.

Proposition 3.4.1. *Let $(\Lambda V, d)$ be a free CDGA, with $V^0 = V^1 = 0$. Then the following properties are equivalent.*

(i) The differential d is decomposable, *i.e. $Im\, d \subseteq (\Lambda V)^+ \cdot (\Lambda V)^+$.*

(ii) The differential satisfies $d : V^k \to \Lambda(V^{\leq k-1})$, $k \geq 2$.

We say that a CDGA is minimal *if it is of the form* $(\Lambda V, d)$ *with* $V^0 = V^1 = 0$ *and if these properties hold.*

Note that property (ii) exhibits $(\Lambda V, d)$ as a Sullivan algebra with filtration $V(k) = V^{\leq k}$ (in particular, any minimal CDGA is fibrant-cofibrant). In other words, a minimal algebra is constructed by progressively adding "free generators" *in order of their degree* with the condition that the differential of a new generator has to be in the previously constructed part of the algebra.

Proof. (i) $\implies$ (ii). Consider $v \in V^k$; then $d(v) \in (\Lambda V)^{k+1}$. Since $V^1 = 0$, for degree reasons $d(v) \in (\Lambda V)^{\leq k-1} \cdot (\Lambda V)^{\leq k-1}$. Hence $d(v) \in \Lambda(V^{\leq k-1})$.

(ii) $\implies$ (i). Since $\Lambda(V^{\leq k-1})$ has no generators in degree $k+1$, d is decomposable on V^k. Therefore d is decomposable on all generators of $(\Lambda V, d)$ and the conclusion follows. □

We now define the notion of a *minimal model* for a CDGA.

Definition 3.4.2. Let (A, d) be a CDGA. A *minimal model* for (A, d) is a quasi-isomorphism

$$(\Lambda V, d) \xrightarrow{\sim} (A, d)$$

where $(\Lambda V, d)$ is minimal.

Before proceeding to the proof of the existence and uniqueness of such models in the simply-connected case, we need to discuss some properties of maps between minimal algebras.

First, let $(\Lambda V, d)$ be a free CDGA with $V^0 = 0$. The differential carries $(\Lambda V)^+ \cdot (\Lambda V)^+$ to $(\Lambda V)^+ \cdot (\Lambda V)^+$ (note that we regard this subspace as a subalgebra: it contains the unit); therefore we can form the *cochain complex of indecomposables* (V, d), that is, the quotient graded vector space

$$V = \Lambda V \,/\, ((\Lambda V)^+ \cdot (\Lambda V)^+)$$

with the induced differential. Moreover, a CDGA map $f : (\Lambda V, d) \to (\Lambda V', d')$ induces a map $Qf : (V, d) \to (V', d')$ between the associated cochain complexes of indecomposables; we shall call Qf the *linear part* of f.

If $(\Lambda V, d)$ is minimal, then the cochain complex of indecomposables is just $(V, 0)$; thus the linear part of a map $f : (\Lambda V, d) \to (\Lambda V', d')$ between minimal models is just a morphism of $\mathbf{k}$-vector spaces $Qf : V \to V'$. We now show that this linear map is a homotopy invariant.

Lemma 3.4.3. *Let* $f \sim g : (\Lambda V, d) \to (\Lambda V', d')$ *be homotopic maps between minimal CDGAs. Then* $Qf = Qg : V \to V'$.

Proof. Consider a homotopy $H : (\Lambda V, d) \to (\Lambda V', d') \otimes \Lambda(t, dt)$: we have $f = (\mathrm{Id} \otimes \varepsilon_0)H$, $g = (\mathrm{Id} \otimes \varepsilon_1)H$ with $\varepsilon_0(t) = 0$, $\varepsilon_1(t) = 1$, $\varepsilon_0(dt) = \varepsilon_1(dt) = 0$. Since $V^1 = 0$, all the elements of $(\Lambda V)^+$ are sent to $(\Lambda V')^+ \otimes \Lambda(t, dt)$, and so the decomposables $(\Lambda V)^+ \cdot (\Lambda V)^+$ are carried to $((\Lambda V')^+ \cdot (\Lambda V')^+) \otimes \Lambda(t, dt)$. As above, this allows us to define a map of cochain complexes

$$\tilde{H} : (V, 0) \to (V', 0) \otimes \Lambda(t, dt)$$

which is such that $Qf = (\mathrm{Id} \otimes \varepsilon_0)\tilde{H}$, $Qg = (\mathrm{Id} \otimes \varepsilon_1)\tilde{H}$. Now $\mathrm{Im}\,\tilde{H}$ is contained in the space of cocycles of the cochain complex $(V', 0) \otimes \Lambda(t, dt)$, that is, in the subspace

$$V' \otimes ((\mathbf{k} \cdot 1) \oplus (\Lambda(t)dt)).$$

But on this subspace $\varepsilon_0 = \varepsilon_1$; hence $Qf = Qg$. □

This allows us to prove the following.

Proposition 3.4.4. *Let $\phi : (\Lambda V, d) \xrightarrow{\sim} (\Lambda V', d')$ be a quasi-isomorphism between two minimal CDGAs. Then ϕ is an isomorphism.*

Proof. Since minimal algebras are fibrant-cofibrant, ϕ admits a homotopy inverse (proposition 3.1.16 on page 28); applying the previous lemma then shows that $Q\phi$ is a bijection, and an easy induction on V'^k allows us to deduce from this that ϕ itself is surjective (so a fibration). Thus axiom M4(i) yields a right inverse $\psi : (V', d') \to (V, d)$ for ϕ, which has to be injective. Now ϕ and ψ are (two-sided) homotopy inverses, so as above ψ is also surjective and hence an isomorphism; therefore so is ϕ. □

We finally prove the existence and uniqueness of minimal models in the simply-connected case, as promised.

Proposition 3.4.5. *Let (A, d) be a simply-connected CDGA. Then (A, d) admits a minimal model*

$$m : (\Lambda V, d) \xrightarrow{\sim} (A, d).$$

Moreover, $(\Lambda V, d)$ is unique up to isomorphism.

Proof. Existence. We inductively construct V^k, $d_{|V^k}$ and $m_k : (\Lambda(V^{\leq k}), d) \to (A, d)$ such that $H^i(m_k)$ is an isomorphism for $i \leq k$ and an injection for $i = k + 1$. The resulting $m : (\Lambda V, d) \xrightarrow{\sim} (A, d)$ will be a minimal model for (A, d).

First, for each generator of $H^2(A)$, pick some $a_i \in A^2$ representing it, add a generator v_i to V^2 with $d(v_i) = 0$ and set $m_2(v_i) = a_i$. Then $H^1(m_2)$ is an isomorphism because (A, d) and $(\Lambda V^2, d)$ are simply-connected, $H^2(m_2)$ is an isomorphism by construction and $H^3(m_2)$ is an injection because ΛV^2 has no elements of degree 3 (it is a polynomial algebra on generators of degree 2).

Now suppose $(\Lambda(V^{\leq k}), d)$ and m_k have been constructed. Since $H^{k+1}(m_k)$ is injective, we can make it an isomorphism as above: for each generator of $H^{k+1}(A)$ not already in the image of $H^{k+1}(m_k)$, we pick $a_i \in A^{k+1}$ representing it, add a generator v_i to V^{k+1} with $d(v_i) = 0$ and set $m_{k+1}(v_i) = a_i$. We then make $H^{k+2}(m_k)$ injective by adding more generators in V^{k+1} so as to kill its kernel. For each generator of $\mathrm{Ker}(H^{k+2}(m_k))$, pick $z_i \in (\Lambda V^{\leq k})^{k+2}$ representing it, and choose $b_i \in A^{k+1}$ such that $db_i = m_k(z_i)$; then add a generator w_i to V^{k+1} and set

$$dw_i = z_i, \quad m_{k+1}(w_i) = b_i.$$

(Notice that this definition of d satisfies the minimality condition.) This makes $H^{k+2}(m_{k+1})$ an injection and does not modify $H^{k+1}(m_{k+1})$ since the w_i are not cocyles. Therefore m_{k+1} has the desired properties.

Uniqueness. Suppose we are given two minimal models $m : (\Lambda V, d) \xrightarrow{\sim} (A, d)$ and $m' : (\Lambda V', d') \xrightarrow{\sim} (A, d)$ for (A, d).

$$\begin{tikzcd}
(\Lambda V, d) \arrow[dr, "\sim", "m"'] \arrow[dd, dashed, "\phi"'] & \\
& (A, d) \\
(\Lambda V', d') \arrow[ur, "\sim", "m'"'] &
\end{tikzcd}$$

Proposition 3.3.1 on page 30 gives us a map $\phi : (\Lambda V, d) \to (\Lambda V', d')$ such that $m'\phi \sim m$. Since ϕ is a quasi-isomorphism (proposition 3.3.4 on page 31), it is an isomorphism (proposition 3.4.4 on the previous page). $\square$

Let $\mathcal{G}_1$ be the full subcategory of $\mathcal{G}$ whose objects are the simply-connected CDGAs. The homotopy category of $\mathcal{G}_1$ — its localization with respect to the quasi-isomorphisms — is equivalent to the category $\pi\mathcal{G}_{1cf}$ with objects the fibrant-cofibrant simply-connected CDGAs and with arrows the homotopy classes of maps (proposition 3.1.17 on page 28); this category in turn is equivalent to its full subcategory on the minimal CDGAs. In other words, the homotopy category of simply-connected CDGAs is equivalent to the category of minimal CDGAs with arrows homotopy classes of maps.

Section 4

Differential forms and rational homotopy type

We first give a very sketchy account of Sullivan's construction for a topological space of a natural CDGA over $\mathbf{Q}$, the algebra of "PL forms" on a topological space. We then discuss the transgression map of a fibration to prepare the ground for our discussion of the correspondence between minimal models and rational Postnikov towers. After exposing this correspondence we briefly relate the smooth forms on a manifold and the PL forms, and conclude with a short discussion of real homotopy type.

4.1 Rational differential forms

The basic building block of Sullivan's construction is the algebra of rational differential forms on the standard n-simplex. This is the CDGA over $\mathbf{Q}$ given by:

$$\Lambda(t_0, \ldots, t_n, dt_0, \ldots, dt_n)/(\sum t_i = 1, \sum dt_i = 0)$$

where $\deg t_i = 0$ and $\deg dt_i = 1$, with the obvious differential. Note that this is just the algebra of $\mathbf{Q}$-polynomial differential forms on $\mathbf{R}^{n+1}$ with the additional relation, true on $\Delta^n \subseteq \mathbf{R}^{n+1}$, that the sum of the coordinates be 1 (and the derived relation).

From there, one can proceed in two different ways to define a rational form on a space X. The first one, which is as far as we know Sullivan's original construction, is to start with a triangularizable topological space X (for instance a smooth manifold), to choose a triangulation, which is a *simplicial complex*, and then to associate to each simplex a polynomial form as described in a way compatible with the restriction to faces; hence the name, often found in the literature, of "PL forms" (for Piecewise Linear). This process is very geometric, but has the drawback that one constantly has to worry about triangulations, subdivisions, simplicial approximations, and so on – this accounts for many technical complications in the Friedlander-Griffiths-Morgan lecture notes [GM81].

The other approach is to consider, for any topological space, the *simplicial set* $S_*(X)$ of singular simplices on X, and again to associate to every simplex a polynomial form in a way compatible with the face maps. This is our preferred method; we refer the reader to [BG76] or [FHT01, chapter 10]. We shall not delve any deeper into the construction; we limit ourselves to stating the following result. Here $C^*(\,\cdot\,;\mathbf{Q})$ denotes the singular cochain complex functor with coefficients in $\mathbf{Q}$.

Theorem 4.1.1. *There exists a contravariant functor A^* from the category of topological spaces to the category of CDGAs over $\mathbf{Q}$ such that for any topological space X, there is a natural map of cochain complexes*

$$A^*(X) \to C^*(X;\mathbf{Q})$$

inducing an isomorphism on the cohomology algebras.

Let X be a simply-connected topological space. We extend the terminology of 3.4 as follows.

Definition 4.1.2. A *minimal model* for X is a quasi-isomorphism

$$m : (\Lambda V, d) \xrightarrow{\sim} A^*(X)$$

where $(\Lambda V, d)$ is minimal.

We shall need one more fact about the functor A^*, namely, that it "preserves homotopy". Let X and Y be two simply-connected topological spaces, and let $m : (\Lambda V, d) \to A^*(X)$ and $m' : (\Lambda V', d') \to A^*(Y)$ be minimal models for X and Y. Then to a map $f : X \to Y$ we can associate $A^*(f) : A^*(Y) \to A^*(X)$ and from there a map $(\Lambda V', d') \to (\Lambda V, d)$ (use the lifting lemma 3.3.1 on page 30). We have the following (for a proof see [FHT01, p. 149]).

Proposition 4.1.3. *Let X and Y be simply-connected topological spaces, and let $m : (\Lambda V, d) \to A^*(X)$ and $m' : (\Lambda V', d') \to A^*(Y)$ be minimal models for X and Y respectively. Then the natural map*

$$\mathcal{T}_1(X,Y) \to \mathcal{G}_1((\Lambda V', d'), (\Lambda V, d))$$

induces a map

$$[X,Y] \to [(\Lambda V', d'), (\Lambda V, d)].$$

4.2 The transgression map

4.2.1 The definition through differential forms

Let $F \overset{i}{\hookrightarrow} E \overset{p}{\to} B$ be a fibration with B and F path-connected. Let A^* be the functor defined above, sending a space to its CDGA of rational differential forms. (Note however that the discussion below would apply just as well to the singular cochain functor C^*: commutativity is not required). We have the following diagram.

$$\begin{array}{ccc} A^*(F) & \xleftarrow{A^*(i)} & A^*(E) \\ & & \big\uparrow{\scriptstyle A^*(p)} \\ & & A^*(B) \end{array}$$

In this setting, a cohomology class in $H^{n-1}(F;\mathbf{Q})$ is said to be *transgressive* if it is possible to extend some form $\omega \in A^{n-1}(F)$ representing it to a form $\tau \in$

$A^{n-1}(E)$ such that $d\tau = p^*(\beta)$ for some closed form $\beta \in A^n(B)$ (by *closed* we mean that $d\beta = 0$, as is customary in the context of smooth differential forms). We shall denote the subgroup of transgressive elements by $T^{n-1} \subseteq H^{n-1}(F; \mathbf{Q})$.

The cohomology class $[\beta] \in H^n(B; \mathbf{Q})$ depends on several choices (the choice of a cochain in $C^{n-1}(F)$ representing our cohomology class, that of an extension in $C^{n-1}(E)$, that of an antecedent by p^*). The process just described therefore defines a map from T^{n-1} to some quotient B^n of $H^n(B; \mathbf{Q})$; this $t^n : T^{n-1} \to B^n$ is called the n-th *transgression map* of the fibration. We could generalize it to any coefficient group G by replacing $A^*(\,\cdot\,)$ with, say, $C^*(\,\cdot\,; G)$.

This definition seems strange at first sight, but is exactly the notion we shall need to relate principal fibrations and elementary extensions of algebras; its geometric content will become clear in 4.3. However, before we can proceed to this, we need to establish some properties of t^n and, in the case of a principal fibration, to relate it to the k-invariant.

4.2.2 The definition through relative cohomology

First, notice that we can avoid any reference to forms (or to cochains) in the definition of the transgression and phrase it in terms of cohomology classes only, in the following way. Let G be an abelian group (setting $G = \mathbf{Q}$ gives us the transgression defined above). Consider the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H^{n-1}(*; G) \arrow[r, "\delta"] \arrow[d] & H^n(B, *; G) \arrow[r, "j"] \arrow[d, "p_0^*"] & H^n(B; G) \arrow[r] \arrow[d, "p^*"] & \cdots \\
\cdots \arrow[r] & H^{n-1}(F; G) \arrow[r, "\delta"'] & H^n(E, F; G) \arrow[r, "j"'] & H^n(E; G) \arrow[r] & \cdots
\end{tikzcd}$$

where the lines are fragments of the cohomology long exact sequences for the pairs $(B, *)$ and (E, F), and p_0 is the obvious map of pairs associated to p. Then define:

- $T^{n-1} = \delta^{-1}(\operatorname{Im} p_0^*)$;
- $B^n = H^n(B; G)/(j(\ker p_0^*))$;
- $t^n(z) = j(r) + j(\ker p_0^*)$ where $z \in T^{n-1}$ and $p_0^*(r + \ker p_0^*) = \delta z$.

This definition has the advantage that it can be formulated over any coefficient ring. It also allows us to identify the transgression map t^n with a differential in the cohomology Leray-Serre spectral sequence associated to the fibration, as follows. We omit the proof; see [McC01, pp. 185–189].

Theorem 4.2.1. *Let $F \overset{i}{\hookrightarrow} E \overset{p}{\to} B$ be a fibration with B simply-connected and F path-connected, and let $\{E_r^{*,*},\ d_r\}$ be the associated cohomology Leray-Serre spectral sequence with coefficients in G. Then:*

(1) $E_n^{0,n-1} \cong T^{n-1} \subseteq H^{n-1}(F; G)$;

(2) $E_n^{n,0} \cong B^n$;

(3) These isomorphisms identify $d_n : E_n^{0,n-1} \to E_{n-1}^{n,0}$ with the n-th transgression map $t^n : T^{n-1} \to B^n$.

From this we immediately deduce that for a fibration with fibre a $K(\pi, n)$, the $(n+1)$-th transgression is a map

$$t^{n+1} : H^n(F; G) \to H^{n+1}(B; G);$$

indeed, $E_2^{0,q} = 0$ for $1 \leq q \leq n-1$.

4.2.3 The homology transgression

There is also a notion of transgression for homology, with a definition similar to our second definition of the cohomology transgression above. The n-th homology transgression is a map from a subgroup of $H_n(B; G)$ to a quotient of $H_{n-1}(F; G)$ and can be identified with the differential

$$d^n : E^n_{n,0} \to E^n_{0,n-1}$$

in the homology Leray-Serre spectral sequence; see [McC01, pp. 185 sqq.] for details.

4.2.4 The link with the k-invariant

Since principal (π, n)-fibrations are pullbacks of the "universal" (π, n)-fibration $K(\pi, n) \hookrightarrow PK(\pi, n+1) \to K(\pi, n+1)$, and since the transgression is natural with respect to (fibre-preserving) maps of fibrations (this is easily seen from the first definition we gave), we first turn to the study of the fibration $K(\pi, n) \hookrightarrow PK(\pi, n+1) \to K(\pi, n+1)$.

Consider the homotopy long exact sequence for this fibration. The only non-trivial map is the isomorphism

$$\partial : \pi_{n+1}(K(\pi, n+1), *) \to \pi_n(K(\pi, n), *)$$

which provides an explicit identification of the abstractly isomorphic homotopy groups of the fibre and of the base. (This map also is occasionally referred to as the *transgression* of the fibration.)

In our case the $(n+1)$-th (integral) homology transgression is an isomorphism

$$t_{n+1} : H_{n+1}(K(\pi, n+1); \mathbf{Z}) \to H_n(K(\pi, n); \mathbf{Z})$$

which may be seen, from its definition (see McCleary's big diagram [McC01, p. 185]), to make the following square commute

$$\begin{tikzcd}
\pi_{n+1}(K(\pi, n+1), *) \arrow[r, "h"] \arrow[d, "\partial"] & H_{n+1}(K(\pi, n+1); \mathbf{Z}) \arrow[d, "t_{n+1}"] \\
\pi_n(K(\pi, n), *) \arrow[r, "h"] & H_n(K(\pi, n); \mathbf{Z})
\end{tikzcd}$$

where the horizontal maps are the Hurewicz isomorphisms.

Since all lower homology groups are trivial, we may identify the cohomology transgression with coefficients in G

$$t^{n+1} : H^n(K(\pi, n); G) \to H^{n+1}(K(\pi, n+1); G)$$

with

$$t^{n+1} : \mathrm{Hom}(H_n(K(\pi,n);\mathbf{Z}),G) \to \mathrm{Hom}(H_{n+1}(K(\pi,n+1);\mathbf{Z}),G)$$

and then with the dual of the homology transgression t_{n+1}; in other words, t^{n+1} is composition by t_{n+1} in the diagram below.

$$\begin{tikzcd}
H_{n+1}(K(\pi,n+1);\mathbf{Z}) \arrow[r] \arrow[d, "t_{n+1}"] & G \arrow[d, equal] \\
H_n(K(\pi,n);\mathbf{Z}) \arrow[r] & G
\end{tikzcd}$$

Now consider cohomology with π coefficients, and recall that we defined (remark 1.2.4 on page 8) the fundamental class of $K(G,m)$ to be the class in $H^m(K(G,m);G)$ corresponding to the map $\kappa : H_m(K(G,m);\mathbf{Z}) \to G$ such that the composite

$$G \xrightarrow{\chi_{K(G,m)}} \pi_m(K(G,m),*) \xrightarrow{h} H_m(K(G,m);\mathbf{Z}) \xrightarrow{\kappa} G$$

is the identity. In our case we see that the diagram

$$\begin{tikzcd}
\pi \arrow[r, "\chi_{K(\pi,n+1)}"] \arrow[d, equal] & \pi_{n+1}(K(\pi,n+1),*) \arrow[r, "h"] \arrow[d, "\partial"] & H_{n+1}(K(\pi,n+1);\mathbf{Z}) \arrow[r, "\kappa"] \arrow[d, "t_{n+1}"] & \pi \arrow[d, equal] \\
\pi \arrow[r, "\chi_{K(\pi,n)}"'] & \pi_n(K(\pi,n),*) \arrow[r, "h"'] & H_n(K(\pi,n);\mathbf{Z}) \arrow[r, "\kappa"'] & \pi
\end{tikzcd}$$

commutes when the rightmost maps correspond to the appropriate fundamental classes: in other words, the $(n+1)$-th cohomology transgression for the universal (π,n)-fibration sends the fundamental class of the fibre to the fundamental class of the base.

Next, consider any principal (π,n)-fibration $p : E \to B$.

$$\begin{tikzcd}
K(\pi,n) \arrow[r] & E \arrow[r] \arrow[d, "p"'] & PK(\pi,n+1) \arrow[d] \\
& B \arrow[r, "k"'] & K(\pi,n+1)
\end{tikzcd}$$

The naturality of the transgression means that the square

$$\begin{tikzcd}
H^n(K(\pi,n);\pi) \arrow[r, equal] \arrow[d, "t'^{n+1}"'] & H^n(K(\pi,n);\pi) \arrow[d, "t^{n+1}"] \\
H^{n+1}(B;\pi) & H^{n+1}(K(\pi,n+1);\pi) \arrow[l, "H^{n+1}(k)"]
\end{tikzcd}$$

commutes (where t'^{n+1} is the $(n+1)$-th cohomology transgression for p). Since in effect we defined the k-invariant of p to be the pullback by k of the fundamental class of $K(\pi,n+1)$ (see remark 1.2.7 on page 9), we have just established that for a general principal (π,n)-fibration, the $(n+1)$-th cohomology transgression map *sends the fundamental class of the fibre to the k-invariant of the fibration.*

4.2.5 The rational case

Now consider a principal (V, n)-fibration $K(V, n) \hookrightarrow E \xrightarrow{p} B$, where V is a finite-dimensional rational vector space. The $(n+1)$-th transgression with rational coefficients for this fibration is a map

$$t^{n+1} : H^n(K(V, n); \mathbf{Q}) \to H^{n+1}(B; \mathbf{Q}).$$

Using the isomorphisms

$$V \xrightarrow[\chi_{K(V,n)}]{\sim} \pi_n(K(V, n), *) \xrightarrow[h]{\sim} H_n(K(V, n); \mathbf{Z}) \cong H_n(K(V, n); \mathbf{Q})$$

one sees that this t^{n+1} determines a map

$$V^* = \mathrm{Hom}(V, \mathbf{Q}) \to H^{n+1}(B; \mathbf{Q})$$

which is dual (over $\mathbf{Q}$) to a map

$$H_{n+1}(B) \to V$$

which corresponds to a cohomology class in $H^{n+1}(B; V)$. We claim that this class is the k-invariant of the fibration. This is proved using the previous result for the transgression with coefficients in V and noting that in this rational setting, the cohomology transgressions with coefficient in $\mathbf{Q}$ and V are dual (over $\mathbf{Q}$ and V respectively) to the same homology transgression. We leave it to the reader to draw the relevant commutative diagram and do not spell out the matter in more details.

4.3 Rational Postnikov tower and minimal model

Let V be a finite-dimensional $\mathbf{Q}$-vector space, and let $K(V, n) \overset{i}{\hookrightarrow} E \xrightarrow{p} B$ be a principal (V, n)-fibration. We construct a CDGA $(H^*(K(V, n); \mathbf{Q}) \otimes A^*(B), d)$ and a map

$$\rho : (H^*(K(V, n); \mathbf{Q}) \otimes A^*(B), d) \to A^*(E)$$

as follows. Recall our first definition of the transgression: for every class $[\omega] \in H^n(K(V, n); \mathbf{Q})$ we pick a form $\tau \in A^n(E)$ and a closed form $\beta \in A^{n+1}(B)$ such that $[A^n(i)(\tau)] = [\omega]$ and $d\tau = A^{n+1}(p)(\beta)$. Then the transgression is given by $t^{n+1}([\omega]) = [\beta]$. We define d so as to extend the differential of $A^*(B)$, and we set $d([\omega]) = \beta$. Then we define:

$$\rho([\omega] \otimes \gamma) = \tau \cdot A^k(p)(\gamma)$$

for $\gamma \in A^k(B)$. The transgression is precisely defined so that $d\rho = \rho d$: therefore ρ is a well-defined CDGA map. In this construction, $A^*(B)$ may be replaced by a quasi-isomorphic CDGA (C, d) in a straightforward way.

The important fact about ρ is that it induces an isomorphism on cohomology. With the ingredients we already have, [DGMS75] and [Sul77] seem to imply that the proof of this fact can be completed using a simple geometric argument; we have not been able to do so. Instead, we refer the reader to the more algebraic argument of [GM81, appendix A1], which is unfortunately much complicated by the fact that they work with simplicial complexes. Alternatively, our proposition easily follows from [FHT01, proposition 15.6 p. 199].

Proposition 4.3.1 (Chevalley-Hirsch-Koszul formula). *Let $K(V,n) \overset{i}{\hookrightarrow} E \overset{p}{\to} B$ be a principal (V,n)-fibration, with V a finite-dimensional $\mathbf{Q}$-vector space. Let $(B,d) \to A^*(B)$ be a quasi-isomorphism of CDGAs. Then the map*

$$\rho : (H^*(K(V,n);\mathbf{Q}) \otimes B, d) \to A^*(E)$$

constructed above is a quasi-isomorphism.

Now let X be a topological space of finite type; we can apply this proposition inductively to its rational Postnikov tower. This yields an inductively constructed graded vector space V^* and differential d such that the following diagram commutes.

$$\begin{tikzcd}
 & \vdots & \vdots \\
 & A^*(X_{4,\mathbf{Q}}) \arrow[dl, "A^*(j_4)"'] & (\Lambda(V^{\leq 4}),d) \arrow[l, "\sim"] \\
 & A^*(X_{3,\mathbf{Q}}) \arrow[u, "A^*(q_3)"'] \arrow[dl, "A^*(j_3)"] & (\Lambda(V^{\leq 3}),d) \arrow[l, "\sim"] \arrow[u] \\
A^*(X) & A^*(X_{2,\mathbf{Q}}) \arrow[l, "A^*(j_2)"] \arrow[u, "A^*(q_2)"'] & (\Lambda(V^{\leq 2}),d) \arrow[l, "\sim"] \arrow[u]
\end{tikzcd}$$

We have the following properties:

- The maps $A^*(j_k)$ are isomorphisms on cohomology in degrees $\leq k$;
- $V^k \cong \mathrm{Hom}(\pi_k(X) \otimes \mathbf{Q}, \mathbf{Q})$;
- the differential $d : V^k \to \Lambda V^{\leq k-1}$ is dual (modulo the appropriate identifications) to the k-invariant of q_k.

From the first of these and the uniqueness of the minimal model, it follows that the direct limit $(\Lambda V, d)$ is the minimal model of X. The two others then show how the minimal model of X encodes the rational homotopy type of X. In particular, the dimension of $\pi_n(X) \otimes \mathbf{Q}$ can be read off from the number of generators of degree n; but it is also possible, for example, to extract information about the Whitehead products (which are products $\pi_n(X) \times \pi_m(X) \to \pi_{n+m-1}(X)$, so a higher-order structure on the homotopy groups) from the differential — see [FHT01, p. 175 sqq.].

We now turn to the converse: given a minimal algebra $(\Lambda V, d)$ such that all V^k are finite-dimensional, we construct a rational Postnikov tower corresponding to it, and therefore (taking the direct limit and a CW model, as in corollary 2.4.4 on page 21) a rational space X and a quasi-isomorphism $A^*(X) \xrightarrow{\sim} (\Lambda V, d)$. This is immediate from the following observation.

Let B be a topological space, let $(C,d) \xrightarrow{\sim} A^*(B)$ be a quasi-isomorphism and let $(C,d) \otimes (\Lambda V, d)$ be a relative Sullivan algebra, where V is a finite-dimensional graded $\mathbf{Q}$-vector space concentrated in degree $n \geq 2$. Then d determines a map $V \to H^{n+1}(B;\mathbf{Q})$, which dualizes to a map $H_{n+1}(B) \to V^*$ defining a cohomology class $H^{n+1}(B;V^*)$. Choosing a $K(V^*, n+1)$, we then get a map $k : B \to K(V^*, n+1)$. Pulling back the universal $(V^*, n+1)$-fibration, we get a principal $(V^*, n+1)$-fibration with total space E whose transgression map may be identified with d. Then from the considerations above there is a quasi-isomorphism $(C, d) \otimes (\Lambda V, d) \to A^*(E)$ (remember that there is a canonical isomorphism $V^{**} \cong V$).

There is a similar result for maps. Recall that if X and Y are simply-connected topological spaces, and if $m : (\Lambda V, d) \to A^*(X)$ and $m' : (\Lambda V', d') \to A^*(Y)$ are minimal models for X and Y, then there is a map:

$$[X, Y] \to [(\Lambda V', d'), (\Lambda V, d)].$$

We have the following.

Proposition 4.3.2. *If A, X are two simply-connected rational CW complexes, and if $m : (\Lambda V, d) \to A^*(X)$ and $m' : (\Lambda V', d') \to A^*(Y)$ are minimal models for X and Y respectively, then the map*

$$[X, Y] \to [(\Lambda V', d'), (\Lambda V, d)].$$

is a bijection.

This is proved using the same general methods: from a map between the minimal algebras, we construct inductively a map between the Postnikov towers; but it is rather technical and requires a fair amount of obstruction theory. We refer the reader to [GM81, chapter XIV].

Now consider the functor from the homotopy category of simply-connected rational CW complexes of finite type to the homotopy category of minimal CDGAs over **Q** of finite type defined in the following way: a space X is sent to some minimal model of $A^*(X)$, and a homotopy class of maps $X \to Y$ is sent to the corresponding homotopy class of maps between the chosen minimal models. Our results mean that this functor is full, faithful and surjective on isomorphism classes of objects: it is an equivalence of categories.

Using the equivalences of categories we established in sections 2 and 3, this may be diversely rephrased, for example as: the rational homotopy category of simply-connected topological spaces of finite type is equivalent to the homotopy category of simply-connected CDGAs over **Q** of finite type.

4.4 Smooth differential forms and real homotopy type

Similarly to what we did in section 4.1, we can define an algebra of smooth differential forms on the standard n-simplex. Then, for any smooth manifold M, we can consider the simplicial set $S_*^\infty(M)$ of smooth singular simplices on M, and associate to each simplex a smooth form in a way compatible with the face maps. This defines a notion of piecewise smooth form on M and allows us to build $A^*_{\mathcal{C}^\infty}(M)$, a CDGA over **R** which can be used to relate the de Rham complex $\Omega^*(M)$ and $A^*(M) \otimes \mathbf{R}$. We refer the reader to [FHT01, chapter 11] for details, and for the proof of the following theorem.

Theorem 4.4.1. *There exists a contravariant functor $A^*_{\mathcal{C}^\infty}$ from the category of smooth manifolds to the category of CDGAs over $\boldsymbol{R}$ such that for any smooth manifold M, there are natural maps of CDGAs*

$$A^*(M) \otimes \boldsymbol{R} \to A^*_{\mathcal{C}^\infty}(M) \leftarrow \Omega^*(M)$$

inducing isomorphisms on cohomology.

In particular, the minimal models of $A^*(M) \otimes \mathbf{R}$ and $\Omega^*(M)$ are isomorphic. In fact, we see that if $m : (\Lambda V, d) \to A^*(M)$ is the minimal model for $A^*(M)$, then the minimal model for $\Omega^*(M)$ is $(\Lambda V, d) \otimes \mathbf{R}$. This is what justifies calling the minimal model of $\Omega^*(M)$ the ***real homotopy type*** of M: it is the real form of the algebra containing all the rational homotopy-theoretic information about M. We do not know of any suitable "geometric" notion of real homotopy type; as far as we know, there is no "real Postnikov tower". See [DGMS75] for a discussion.

It is then possible to compute the rank of $\pi_n(M) \otimes \mathbf{Q}$ from $\Omega^*(M)$ (recall that every compact smooth manifold has finite-dimensional cohomology): it is just the number of generators of degree n in the minimal model of $\Omega^*(M)$.

We very much regret that lack of time prevented us from adding computations and examples. One of the great advantages of rational homotopy theory is its computational power, and we did not do justice to it in this work. We refer the reader to [GM81, chapter XIII] and to [FHT01] for a broader overview of the possible applications.

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