Are Euclid's Diagrams 'Representations'? On an argument by Ken Manders

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Summary. In his well-known paper on Euclid's geometry, Ken Manders (2008) sketches an argument against conceiving the diagrams of the *Elements* in 'semantic' terms, that is, against treating them as representations—resting his case on Euclid's striking use of 'impossible' diagrams in some proofs by contradiction. This paper spells out, clarifies and assesses Manders's argument, showing that it only succeeds against a particular semantic view of diagrams and can be evaded by adopting others, but arguing that Manders nevertheless makes a compelling case that semantic analyses ought to be relegated to a secondary role for the study of mathematical practices.

Introduction

Should the diagrams of Euclid's geometry be conceived of in *semantic* terms, that is, as *representations* of some kind? On the basis of Euclid's pervasive use of seemingly 'impossible' diagrams in proofs by contradiction, Ken Manders (2008) argues for a negative answer: a semantic approach to mathematical diagrams is not fruitful, he claims, neither for Euclid nor in general. This paper aims at clarifying and evaluating Manders's argument, a task which requires distinguishing different senses in which Euclid's diagrams may be called 'representations'.

I first show that, despite seemingly being about semantic accounts of diagrams *in general*, Manders's argument actually targets a *very specific* thesis about the relation between the text of Euclid's propositions and proofs on the one hand, and the corresponding diagram on the other (Section 1): roughly, the thesis that the diagram is (or depicts) what the text *is about*—that the text *talks* about lines and circles while the diagram directly (though perhaps approximately) *displays* them. Against this thesis, which I dub the 'classical view' of diagrams, Manders's observation that reductio proofs often rely on apparently incorrect diagrams indeed seems cogent: for instance, in some reductio proofs, the text talks about properties of circles but the diagram shows (indeed, *has to* show) something else, namely ellipses or other irregular closed curves, because circles satisfying the conditions imposed by the text cannot exist. Section 2 then explores an objection (hinted at but not really discussed by Manders) that a fairly straightforward refinement of the classical view, according to which the diagram only depicts *some* of the properties ascribed by the text to the objects it discusses, may allow evading the argument. I show that the case of reductio proofs, while no longer sufficient to provide a

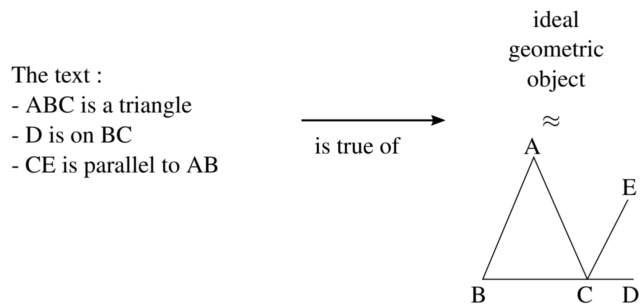

Fig. 1. The 'classical view' of diagrams

downright refutation of such a refinement, nevertheless diminishes the classical view's appeal and plausibility.

I then introduce another approach to Euclid's diagrams—inspired by Barwise and Etchemendy's work on diagrammatic reasoning—which certainly deserves to be called 'representational' or 'semantic', but in a different sense than the classical view (Section 3): in it, diagrams and text are placed on an equal footing, and both are given a semantics in a common domain. I argue that this approach easily accommodates reductio proofs, so that Manders's argument, while successful against a limited target, cannot warrant sweeping conclusions about semantic accounts of diagrams in general. Finally, Section 4 returns to the bigger picture: I conclude that, although semantic analyses of Euclid's diagrams remain possible despite reductio proofs, they might not bring much to the study of a mathematical practice such as Euclid's. In the end, Manders's broader point—that semantic analyses, even if possible, should be relegated to a secondary role—remains compelling.

1 Reductio proofs and the 'classical view' of diagrams

While Manders starts his discussion of 'semantic' approaches to Euclid's diagrams in very general terms,[1] the target of his argument is actually a quite specific thesis. The goal of this first section is to spell out this target, which I shall dub the 'classical view' of diagrams, and to explain why Manders's discussion of reductio proofs is a cogent objection to it.

Here is how Manders introduces the thesis he sets out to refute:

> Long-standing philosophical difficulties, on the nature of geometric objects and our knowledge of them, arise from *the assumption that the geometrical text is in an ordinary sense true of the diagram or a 'perfect counterpart'*.[2]

Roughly, this 'assumption' may be pictured as on Fig. 1; I shall refer to it as the 'classical view' of diagrams, although it is probably best thought of not as a single well-defined view, but as a family of related ones—depending on whether concrete diagrams or ideal objects take

[1] 'Artifacts in a practice that gives us a grip on life are sometimes thought of in semantic terms—say, as representing something in life. There is, of course, an age-old debate on how geometrical diagrams are to be treated in this regard.' (Manders 2008: 84).

[2] Manders (2008: 84); my emphasis.

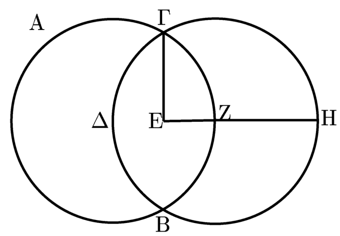

Fig. 2. Diagram for Proposition III.5 of Euclid's *Elements* found in Codex B, as reproduced by Saito (2011: 54)

center stage, and on how the relation of approximation that may obtain between them is understood. The word 'representation' is notoriously slippery and there is little to be gained, for our purposes, from a terminological discussion about whether—on such an account—diagrams should properly be called representations, and if so representations of what; but notice that the crucial semantic relation here holds between the text and the diagram, and in fact goes *from* the text *to* the diagram. The diagram itself may secondarily be taken to 'represent' the ideal version of itself that it approximates (assuming one decides to defend such a version of the classical view), but this second relation is of no import for Manders's argument.

On the face of it, something like the classical view has considerable plausibility. Elementary geometry would be about 'perfect' or 'ideal' geometrical configurations, to which the diagrams we actually draw give us some kind of approximate access; the fact that in practice, our lines have thickness, tend not to be perfectly straight, and so on, does not for the most part threaten the reliability of our inferences.

Against this classical view, Manders brings up the puzzling case of proofs by contradiction:

> [A] genuinely semantic relationship between the geometrical diagram and text is incompatible with the successful use of diagrams in proof by contradiction: reductio contexts serve precisely to assemble a body of assertions which patently could not together be true; hence no genuine geometrical situation could in a serious sense be pictured in which they were. This simple-minded objection has nothing to do specifically with geometry: proofs by contradiction never admit of semantics in which each entry in the proof sequence is true (in any sense which entails their joint compatibility).[3]

To clarify what is at stake here, let us look at an example, namely Proposition III.5 (i.e., Proposition 5 of Book III of Euclid's *Elements*), which asserts that 'if two circles cut one another, they will not have the same centre.'[4] In his proof, Euclid introduces two circles ABΓ and ΓΔH that cut one another in two points B and Γ, assumes that they have the same center E, and goes on to show that these hypotheses are contradictory. On the face of it, the conclusion means that it is impossible to come up with a correct diagram of two concentric circles that cut one another in two points. Nevertheless, as is always the case in Euclid, the *Elements* do provide a diagram—but this diagram cannot and does not conform to the hypotheses: while it does show two circles cutting one another and also displays a point E, this point does not look remotely like the center of either circle. (Admittedly, the vagaries of textual transmission make

[3] Manders (2008: 84).

[4] See Heath (1908: II: 12) = Vitrac (1990: I: 399–400) = Heiberg (1883: I: 176–177).

it impossible to know for sure what kind of diagrams Euclid himself may have used; taking advantage of recent historical scholarship, I have chosen to reproduce sketches of the diagrams of 'Codex B', one of the oldest extant manuscripts of the *Elements*.[5]) Thus it is clear—here as in every *reductio* proofs of the *Elements*—that the assumptions in the text cannot all be true of the diagram.

The reason this is, according to Manders, a serious problem is that it undermines the classical view's main virtue, to wit, the fact that it nicely explains why Euclid-style geometry is able to unproblematically rely on diagrams in the course of its proofs:

> If diagram imperfections only were in play, one might well hold that the function of diagrams could fruitfully be approached by first elaborating a notion of perfect geometricals of which the text is literally true, then treating diagrams actually drawn in geometrical demonstrations as approximations to perfect ones; finally deriving from all this an understanding of the bearing of the imperfect diagram on inferences in the text. But no detour through ontology and semantics which treats of truth in a diagram in a sense which entails joint compatibility of all claims in force in the reductio context can speak to the difficulty with the role of diagrams in reductio arguments, which are pervasive in Euclid.[6]

Let us spell out the argument here. All the geometrical diagrams we draw are (at least a little) off. Accordingly, it would be hopeless, say, to try and determine whether the angles of a triangle are exactly equal by way of measurements performed on some concrete drawing of it. On the other hand, it is plausible to think that *some* properties of our diagrams are not impacted by such imperfections, and that they adequately reflect the properties of the ideal geometrical objects we are aiming at. (Such, for instance, is the position defended by Panza (2012).) In fact, it is well-known that Euclid's proofs often rely on their diagrams;[7] the most famous example of this is Proposition I.1 (Fig. 3), which shows how to construct an equilateral triangle on a given line.[8] Modern criticisms notwithstanding, authors like Panza and Manders argue that—in the framework of Euclid's plane geometry if not by later standards—it is safe and legitimate to conclude on the basis of the diagram that the two circles intersect, and in

[5] A few elements of context about this choice are in order. As recently shown by Ken Saito, Byzantine manuscripts of Euclid—which, save for isolated fragments, are the oldest we have—typically display diagrams that differ significantly from those of modern editions, including critical ones. (For an introduction, see Saito and Sidoli 2012, Saito 2009: 817–825 or Saito 2012—the latter discussing the very Proposition III.5 taken as an example here. For a fuller overview of diagrams in manuscript sources of the first books of Euclid's *Elements*, see Saito 2006; 2011.) The difference is of particular interest in the case of *reductio* proofs, as the manuscript diagrams are often much more blatantly 'wrong' or 'impossible' than those of modern editions. This is why I have chosen, in this paper, to reproduce the diagrams from 'Codex B', a 888 C.E. manuscript which, though removed from Euclid himself by almost 1200 years, is one of the oldest still extant. (The letters standardly used to refer to manuscripts of the *Elements* go back to Heiberg's authoritative 19th-century critical edition of the Greek text; for a list, see Heiberg 1883: I: V–X or Saito 2006: 95–96.)

[6] Manders (2008: 85–86).

[7] For a survey of the role of Euclid's diagrams in his proofs, see for instance Netz (1999: 175–182).

[8] Heath (1908: I: 241–242) = Vitrac (1990: I: 194–195) = Heiberg (1883: I: 10–13).

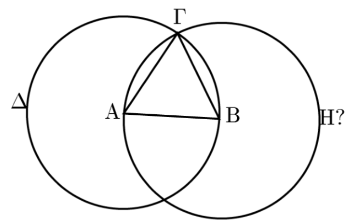

Fig. 3. Diagram for Proposition I.1 of Euclid's *Elements* in Codex B (Saito 2006: 97)

general that Euclid's geometry licenses reading off from diagrams *some* properties that are unaffected by standard drawing imperfections.[9]

Once the inferential reliance of Euclid's geometry on its diagrams is recognized, Manders's challenge to what I called the 'classical view' arises from the observation that *reductio proofs too* rely on diagrams—despite their diagrams being blatantly wrong. Again, Proposition III.5 can help us pinpoint the issue. After introducing the circles ABΓ and ΓΔH, their intersections B and Γ, and their common center E (see Fig. 2 above), Euclid reasons as follows:

> [L]et EΓ be joined, and let EZH be drawn through at random. Then, since the point E is the centre of the circle ABΓ, EΓ is equal to EZ. Again, since the point E is the centre of the circle ΓΔH, EΓ is equal to EH. But EΓ was proved equal to EZ also; therefore EZ is also equal to EH, the less to the greater: which is impossible.[10]

This proof relies on the line EZH, which is 'drawn through at random' and is taken to meet ABΓ in Z and ΓΔH in H (in this order). Importantly, in Euclid's geometry *nothing but the diagram* can provide us with the existence of a line satisfying these conditions—a modern diagram-free version of this proof would require an additional axiom, just as in the case of Proposition I.1.

Manders's argument should by now be clear. Reductio proofs in the *Elements*—just like direct proofs—essentially rely on their diagrams. But whereas, in the case of direct proofs, this reliance can be justified by taking the diagram to adequately reflect some of the properties of a corresponding perfect geometrical object (to wit, those properties that are not impacted by the coarseness of our drawing), this answer appears unavailable for reductio proofs, since what such proofs show is precisely that geometrical objects obeying their hypotheses cannot exist. Hence, unless one is prepared to claim that diagrams play a fundamentally different role in direct proofs and in reductio proofs—and that the classical view is adequate to the former but not to the latter—the classical view seems refuted.

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[9] In general, Manders calls 'co-exact' those properties that can be read off from diagrams in Euclid; as for Panza, the fact that diagrams of Euclidean geometry allow attributing some of their properties to the corresponding geometrical objects is what he calls their 'local role', and those properties that geometrical objects are taken to inherit from their diagrammatic representations are what he calls 'diagrammatic attributes' (see Panza 2012: in part. 72–82).

[10] Trans. from Heath (1908: II: 12), where (for consistency with the Codex B diagram) I have replaced Heath's Roman letters with Heiberg's Greek letters. See also Vitrac (1990: I: 400) = Heiberg (1883: I: 177).

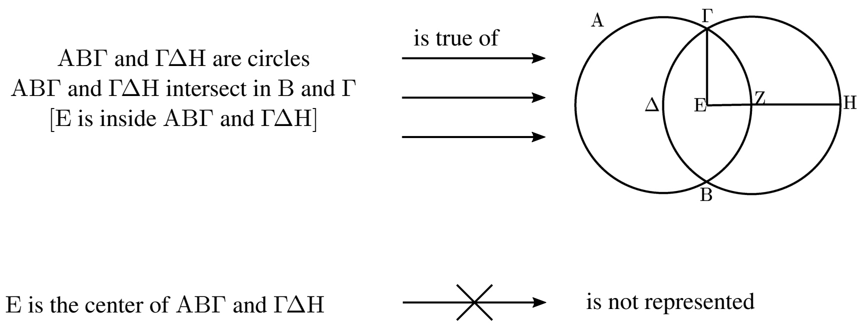

Fig. 4. A partial version of the classical view of diagrams, on the example of Proposition III.5

2 A partial version of the classical view

The argument just presented against the classical view gives rise to a simple objection. It is clear that in propositions proved by contradiction, like III.5, the hypotheses of the reductio cannot *all* be true of the diagram. But *some* of them will be, and that may be enough to account for whatever inferences are drawn from the diagram—in much the same fashion as for usual proofs. Manders concedes as much:

> It does not follow that there could not be a picturing-like relationship between the diagram and *some* claims in force within a reductio context.[11]

This section discusses the prospects of such a 'partial' classical view and puts it to the test of Proposition III.5 as well as of thornier cases, Propositions III.2 and III.10. We shall see that while, strictly speaking, the argument of the previous section is not enough to knock down the partial classical view, it still puts significant pressure on it.

Let us first return to Proposition III.5. To rescue the classical view of diagrams from the reductio objection, the idea is to turn it into a 'partial' variant—pictured on Fig. 4—according to which only *some* of the claims in the text would be true of the diagram. In this case, while the diagram does display two circles cutting one another in two points—this part of the hypotheses is accurately shown—point E is not their center—this hypothesis is simply not reflected by the diagram. This makes the inferential reliance on the diagram unproblematic. Indeed, the diagram adequately represents two circles as well as a point *that is inside both of them*; on this unassailable basis, one can construct Z and H. One can then derive a contradiction by reference to the *additional hypothesis* (not reflected by the diagram) that E is the center of both circles—but crucially, no further reference to the diagram is required for this.

However, the reductio problem raised by Manders retains some force, for two different reasons. First, there are reductio proofs that are harder to analyze than III.5. Consider for instance III.2—'if on the circumference of a circle two points be taken at random, the straight line joining the points will fall within the circle'[12] (Fig. 5)—or III.10—'A circle does not cut

[11] Manders (2008: 85); KM's emphasis. The claims 'in force' within a reductio context refer to the hypotheses under which one arrives at a contradiction; the terminology here comes from an analogy with natural deduction, in which inferences are relative to a context defined by the undischarged assumptions under which it is made.

[12] Heath (1908: II: 8–9) = Vitrac (1990: I: 394–395) = Heiberg (1883: I: 168–171).

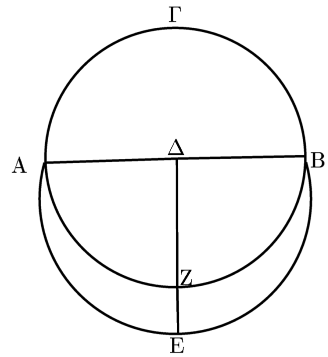

Fig. 5. Diagram of Proposition III.2 in Codex B (Saito 2011: 52)

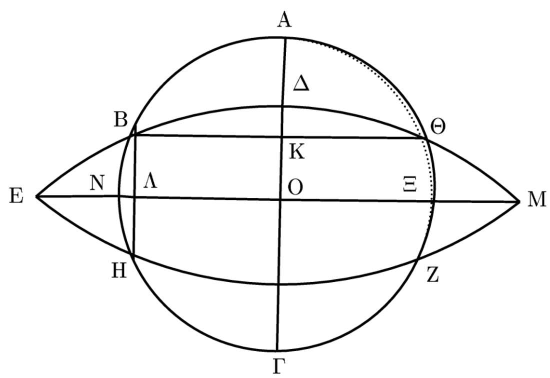

Fig. 6. Diagram of Proposition III.10 in Codex B (Saito 2011: 59)

a circle at more points than two'[13] (Fig. 6). In the first case, the straight line AB cannot be diagrammed as straight; in the second, at least one of the circles cannot be a circle. Thus, in order for the same strategy as above to go through, the textual claims that one would have to assume are not represented concern the very identity of some of the objects in the diagram (e.g., that AB is a straight line, or that ABΓ and ΔEZ are circles). The problem is that, by contrast with III.5, what remains when such fundamental claims are removed is not a run-of-the-mill Euclidean diagram made up of straight lines and circles, but involves objects that are not normally discussed by Euclid, namely arbitrary (non straight) lines and arbitrary (non circular) closed curves. Admittedly, this is not a knock-down argument against the (partialized) classical view: if the point of the classical view, and of its partial variant, was to defend the reliance of diagrams in proofs—by arguing that diagrams adequately reflect some properties of corresponding ideal geometrical objects—then examples such as Propositions III.2 and III.10 do not make this strategy impossible. However, they do make it more costly, because it forces one to admit that, in the course of a proof, diagrams can allow discerning properties of geometrical objects—like arbitrary curved lines—that go beyond the usual circles and straight lines that Euclid's geometry is ostensibly about.

[13] Heath (1908: II: 23–24) = Vitrac (1990: I: 412–413) = Heiberg (1883: I: 192–195).

A second difficulty for the partial classical view concerns the norms governing partial representation. It is well and good to suggest, as done above, that the diagram of III.5 does not depict E as the center of both circles but merely as a point inside them, or that the diagram of III.2 does not depict the line AB as straight, but merely as a line. Euclid's text, however, does not say anything about what the diagram should or should not depict. Tellingly, in order to make the 'partial' interpretation of III.5 fully explicit on Fig. 4, I have been led to introduce a phantom claim—that E is inside ΑΒΓ and ΓΔΗ—which is stated nowhere in the text. In other words, while the classical view of diagrams introduced in the previous section was simple and clear, its partial variant seems to obey subtle and implicit norms: what claims is one allowed not to represent on the diagram? What kind of distortions on circles and lines are allowable?[14]

This second difficulty is presumably why, in Manders's view, conceding that the diagram is not a straightforward (though perhaps approximate) model of the text is the crucial step. Once this is granted, one is forced to accept that the way diagrams operate in reductio proofs is not self-evident and requires a deeper analysis:

> Thus one is forced back to a direct attack on the way diagrams are used in reductio argument; the problem of the relationship between diagram and geometric inference here turns out to be one of standards of inference not reducible in a straightforward way to an interplay of ontology, truth, and approximate representation. But once this is admitted, there seems to be no reason why direct inferential analysis of diagram-based geometrical reasoning should not be the approach of choice to characterizing geometrical reasoning overall, with or without reductio.[15]

The 'direct inferential analysis' Manders advocates focuses on the way diagrams are used rather than on what they might be taken to represent. Explaining in detail how he does this is beyond the scope of this paper, but roughly, his main strategy is to explain the norms of the practice in terms of the requirement—seemingly crucial to Euclid's kind of mathematical practice—that geometers should be able to reach agreement on controversial cases and adjudicate disputes conclusively. In a nutshell, he argues that the reason the norms of Euclidean practice do not license observing, say, equality of lengths on a diagram is that this cannot be done reliably: different geometers, constructing two diagrams according to the same rules, might reach different conclusions. On the other hand, properties that can be reliably reproduced by different practitioners—for instance that two circles constructed as in Proposition I.1 intersect—can unproblematically be read off from diagrams.

3 A semantics for diagrams

As discussed in the previous sections, Manders's argument targets a very specific account of the semantic role of Euclid's diagrams, to wit, the 'classical view' according to which the diagrams are what the text is about (or perhaps approximations of the objects that the text is about). But—regardless of whether, as discussed in the previous section, his refutation of the classical view can ultimately be evaded—other 'semantic' accounts of diagrams are possible, which his discussion does not address. This section explores one such account, in which the

[14] As Rabouin (2015: 115–118, 126–131) shows, the kinds of distortions that reductio proofs require easily produce incorrect results in other situations: some form of *selective* control over diagram distortions is clearly going on; see Manders (2008: 109–118) for further discussion.

[15] Manders (2008: 86).

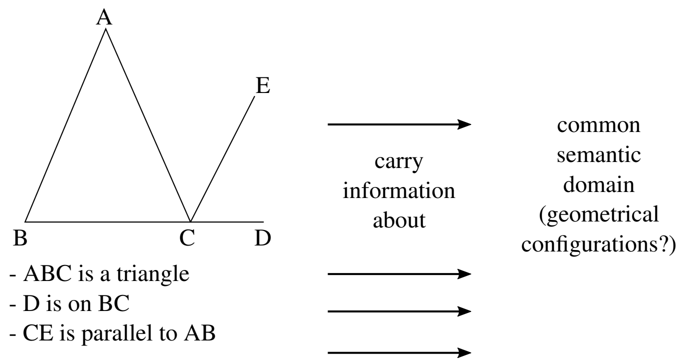

Fig. 7. A different semantic view of geometrical diagrams

diagrams, rather than being—so to speak—the semantics of the text (as was the case in the above), are given a semantics themselves. I then argue that the case of reductio proofs does not raise particular problems for such an account.

As a first observation, note that the straightforward conclusion that follows from Manders discussion of *reductio* proofs is, in his own words, that diagrams should be treated as '*textual* components of a traditional geometrical text or argument, rather than semantic counterparts'[16]—but that textual components of a proof *are* usually given a semantics. Indeed, Manders's conclusion is perfectly compatible with a view according to which both text *and* diagrams are treated as ways to represent a common subject-matter, that is, as admitting a semantics in a common domain. Roughly, the resulting view could be pictured as on Fig. 7. Notice that here, the relevant semantic relation does not relate the text with the diagram, as in the 'classical view' discussed earlier, but relates both text and diagram with a common subject-matter. Such a picture of diagrams as information-bearers underlies much of the diagrammatic reasoning program initiated by Barwise and Etchemendy;[17] this is, for instance, how Shin (1994) approaches Venn diagrams. In a similar spirit, two formal diagrammatic systems for Euclid's geometry have been produced in recent decades that approach diagrams as text-like proof symbols and define for them a formal semantics of the kind meant here: the systems of Miller (2007) and Mumma (2006; 2019).[18]

To get a better grasp on what such a semantics of diagrams would look like—and importantly, on how it could help make sense of *reductio* proofs—let us make a detour through a reductio proof in which no diagram is required. Consider Proposition VII.22, from Euclid's books on number theory:[19] 'The least numbers of those which have the same ratio as them are

[16]Manders (1996: 391). (This quotes comes, not from his most famous 2008 paper, but from a previous publication on the topic; his view on this did not change, however.)

[17]See, in particular, the collective volume Allwein and Barwise (1996).

[18]The original version of Mumma's system did not define a formal semantics for its diagrams, but this is possible and is done in Mumma (2019).

[19]As a matter of fact, the propositions from Euclid's arithmetical books also contain diagrams of sorts, which represent by way of lines the numbers discussed in the text; but, in contrast to the geometrical case, these diagrams do not play much of a role in proofs. See, e.g., Mueller (1981: 67).

prime to one another.'[20] Euclid proves this by *reductio*. He starts from two numbers A and B and makes two distinct hypotheses about them, from which he then derives a contradiction: (1) that they are the smallest among those that have the same ratio as them (call this hypothesis 'p'); (2) that they are not prime to one another (call this 'q'). It follows from q that A and B have a common factor, hence that there are smaller numbers C and D that have the same ratio, which contradicts p.

Now, both p and q taken on their own can unproblematically be given a semantics, and so can their conjunction $p \wedge q$. For instance, in a suitable model-theoretic framework, such a semantics might roughly be determined by: (1) for p, the set of pairs of numbers (A, B) such that they are the smallest among those having the same ratio; (2) for q, the set of pairs of numbers (A, B) that are not relatively prime; (3) for $p \wedge q$, the empty set—precisely because the two hypotheses are, in fact, incompatible. Of course, the fact that the semantics of $p \wedge q$ reflects the incompatibility of p and q does not in and of itself preclude using p, q or $p \wedge q$ in the course of a (reductio) proof, since there are no syntactic principles that would allow deducing this incompatibility immediately.

With this example in mind, let us go back to a reductio proof involving one of the 'impossible' diagrams discussed earlier—say, Proposition III.10. My claim is that it can be treated in a similar fashion. In very broad terms, and without entering into a precise discussion of how Euclidean diagrams may be formalized, there would be at least two different strategies to give a semantics to the diagram of III.10 (Fig. 6). First, in the spirit of our discussion in Section 2, one could take this diagram to represent two closed curves $\mathrm{AB\Gamma}$ and $\Delta\mathrm{EZ}$ intersecting in four points, and nothing more. The diagram would then play the role of, say, hypothesis p from the preceding example, while q would correspond to the further (and incompatible) claim that $\mathrm{AB\Gamma}$ and $\Delta\mathrm{EZ}$ are circles. A second, perhaps less convoluted reading would take the diagram at face value and accept that it does represent two *circles* intersecting in four points; on this reading, it would be comparable to the conjunction $p \wedge q$ rather than to p (or q)—i.e., it would have an empty set of models as semantics, but would not self-destruct, because nothing in the practice of Euclidean geometry licenses immediately reading off the contradictoriness of such a diagram from its appearance (indeed, one could hardly practice Euclidean geometry at all if one were allowed to reject any diagram as contradictory if some circle *appeared* imperfect). The 'semantic' contradictoriness of such a diagram—that is, the fact that it admits no models because it involves incompatible assumptions—would no more preclude its use in proofs than the incompatibility of p and q precludes assuming $p \wedge q$ and reasoning from there. This may still strike us as strange, but, as the comparison with our number-theoretic example shows, the strangeness in question is about the semantic analysis of reductio proofs *in general* and has nothing specific to do with Euclid's geometry: reductios involving seemingly impossible diagrams then become no more (but also no less) semantically problematic than any other reductio proofs. They certainly license no conclusions unique to the semantics of Euclid's diagrams.

Thus, approaching diagrams in 'semantic terms' is possible, and in fact fully compatible with Manders's diagrams-as-text account of Euclidean practice; but, instead of establishing a semantic relation between text and diagram, this involves endowing with a semantics both diagram and text, now placed on an equal footing.

[20] Heath (1908: II: 323–324) = Vitrac (1990: II: 328) = Heiberg (1883: II: 234–237). In modern terms, if two integers A and B are such that there are no *smaller* integers C and D such that $\frac{C}{D} = \frac{A}{B}$, then A and B are relatively prime.

4 By way of conclusion: what is a semantics good for?

In sum, Manders's reductio argument, though framed as an attack against any kind of 'semantic' approach to Euclid's diagrams, is in fact only cogent against a particular account, the one I called the classical view (in passing, we have seen that even this view may be defended, more or less convincingly, with appropriate contortions). The idea of treating diagrams—just like the sentences of Euclid's text—as representations that admit of a semantics is unaffected, in principle, by the case of reductio proofs.

There is a sense, however, in which this discussion misses the deeper lesson of Manders's study. It may well be that giving a semantics to Euclidean diagrams is possible; his analysis, however, suggests that it would be rather futile. As he puts it:

> If this order of analysis [a direct inferential analysis of diagram-based geometrical reasoning] proves fruitful, ontological and semantic considerations will seem decidedly less central to the philosophical project of appreciating geometry as a means of understanding. For in their then remaining role of making the standards of geometrical reasoning seem appropriate, ontological-and-semantic pictures will have to compete with other types of considerations which we will find have potential to shape a reasoning practice.[21]

According to Manders, what accounts for the way diagrams are used in Euclid is not at bottom what they *represent*, but rather pressures and constraints of a different kind—centrally, the requirement that practitioners be able to straightforwardly reach agreement. This, of course, does not preclude coming up with some kind of semantics (perhaps along the lines suggested in Section 3) that would account for the norms of the practice as one finds them—but it would serve no purpose save '[making] the standards of geometrical reasoning seem appropriate' *after the fact*, that is, legitimizing norms whose real source is elsewhere.

Manders's conclusion, then, is that the historian of a mathematical practice should focus on its inferential norms; one may add a semantic analysis later on, but trying to start from it is liable to lead astray rather than bring any benefit. What matters is not so much to argue for the impossibility of a 'representational' approach to mathematical diagrams as to *marginalize* such an approach in favor of an inferential analysis.

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