Mémoire de M2: Schramm Loewner Evolutions and Conformal Loop Ensembles

Yijun WAN

Contents

1 Schramm Loewner evolution **2**
1.1 Definitions . . . 2
1.1.1 Conformal map and half-plane capacity . . . 2
1.1.2 Continuity and local growth property . . . 3
1.1.3 Chordal Loewner Chain . . . 3
1.1.4 Schramm-Loewner Evolution . . . 4
1.2 Advanced properties of SLE . . . 5
1.2.1 SLE as a complex Bessel process . . . 5
1.2.2 Transience . . . 8
1.2.3 Phases of SLE . . . 10
1.2.4 Upper bounds of dimensions . . . 13
1.2.5 Phase transition of $\tau(z)$ at $\kappa = 4$ . . . 15

2 Continuity properties of Loewner chains **16**
2.1 Continuity properties of $W \mapsto f$ and $W \mapsto g$ . . . 17
2.2 Continuity of $\gamma \mapsto g$ . . . 18
2.3 Continuity of $\gamma \mapsto W$ . . . 19

3 Convergence of interfaces in site percolation model **19**
3.1 Definitions . . . 19
3.2 Cardy's formula and conformal invariance . . . 20
3.3 Characterization of percolation interface scaling limit . . . 23

4 CLE percolations **24**
4.1 Construction . . . 25
4.2 Markovian construction . . . 27
4.2.1 Explorations . . . 29
4.3 Loop-soup construction . . . 29
4.3.1 Notations . . . 29
4.3.2 Preliminaries on excursion measures . . . 30
4.4 Bubble soup and loop soup . . . 33
4.4.1 Brownian bubble measure and bubble soup . . . 33
4.4.2 Brownian loop measure and loop soup . . . 33
4.5 Decompositions of bubble measure and loop measure . . . 35
4.6 Relations between bubbles and loops . . . 37
4.6.1 Some estimates on Poisson kernels . . . 37
4.7 Parametrization . . . 40

5 CLE: construction via loop-soups **40**
5.1 Markovian characterization of CLE 40
5.2 Loop-soup percolation 41
5.3 Relation between c and κ 43
5.4 Identifying the critical intensity c_0 44
5.5 An intermediate way to measure the size of sets 44

1 Schramm Loewner evolution

1.1 Definitions

In this section, we will review some basic facts concerning Loewner's equation and Schramm-Loewner evolution.

1.1.1 Conformal map and half-plane capacity

Let $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}z > 0\}$ to be the upper half-plane. A bounded subset $A \subset \mathbb{H}$ is called a *compact* $\mathbb{H}-$*hull* if $A = \overline{A} \cap \mathbb{H}$ and $\mathbb{H} \setminus A$ is a simply connected domain. For any compact $\mathbb{H}-$hull A, there exists a conformal map $g_A : \mathbb{H} \setminus A \to \mathbb{H}$. Note that the conformal isomorphism is not unique. For all isomorphisms σ and all $\lambda \in (0, \infty)$, the map $z \mapsto \lambda\sigma(z)$ is also an isomorphism. We specify a particular one such that $|g_A(z) - z| \to 0$ as $z \to \infty$.Moreover, for any compact $\mathbb{H}-$hull K, $r \in (0, \infty)$ and $x \in \mathbb{R}$, set $rK = \{rz : z \in K\}$ and $K + x = \{z + x : z \in K\}$. Then rK and $K + x$ are compact $\mathbb{H}-$hulls and

$$g_{rK}(z) = rg_K(z/r), \quad g_{K+x}(z) = g_K(z - x) + x.$$

The half-plane capacity hcap$A$ is defined by

$$\mathrm{hcap}(A) := \lim_{z\to\infty} z(g_A(z) - z).$$

More precisely speaking, there is a unique conformal isomorphism $g_A : \mathbb{H} \setminus A \to \mathbb{H}$ such that there exists $a_{\gamma'} \in \mathbb{R}_+$, $g_{\gamma'}(z) = z + \frac{a_{\gamma'}}{z} + O(|z|^{-2})$ as $|z| \to \infty$.

The half-plane capacity has some nice properties:

- If $r > 0, x \in \mathbb{R}M$, then

$$\mathrm{hcap}(rA) = r^2\mathrm{hcap}A, \qquad \mathrm{hcap}(A + x) = \mathrm{hcap}(A).$$

- If B is also a compact $\mathbb{H}-$hull and $A \subset B$, then

$$\mathrm{hcap}(B) = \mathrm{hcap}(A) + \mathrm{hcap}(g_A(B \setminus A)).$$

  Especially, the half-plane capacity increases as the set grows.

In the following, we will consider only compact $\mathbb{H}-$hulls given by simple curves γ.

1.1.2 Continuity and local growth property

Define $\mathrm{rad}(K) = \inf\{r \geq 0 : K \subset r\overline{\mathbb{D}} + x \text{ for some } x \in \mathbb{R}\}$.

Proposition 1.1. *Let K be a compact $\mathbb{H}-$hull, then*

$$|g_K(z) - z| \leq 3rad(K), z \in H = \mathbb{H} \setminus K.$$

Proof. By the scaling and translation argument, it suffices to consider the case where $K \subset \overline{\mathbb{D}}$ and $\mathrm{rad}(K) = 1$. Recall that $g_K(z) - z$ is a bounded holomorphic function on H. Fix $z \in H$ and consider a complex Brownian motion $(B_t)_{t\geq 0}$ starting from z. For $t < T = T(H)$ (the exit time of $(B_t)_{t\geq 0}$ through H), $M_t = g_K(B_t) - B_t$ is a continuous and bounded local martingale, and $M_t \to g_K(B_T) - B_T$ as $t \uparrow T$. Hence by optimal stopping,

$$g_K(z) - z = \mathbb{E}_z[g_K(B_T) - B_T].$$

We admit here that $|g_K(x) - x| \leq 3$ for all $x \in \mathbb{R}$ if $K \subset \overline{\mathbb{D}}$ and $\mathrm{rad}(K) = 1$. □

Let $(K_t)_{t\geq 0}$ be an increasing family of compact $\mathbb{H}_h ulls$. ($(K_t)_{t\geq 0}$ is said to be *increasing* if K_s is strictly contained in K_t whenever $s < t$.) Set $K_{t+} = \cap_{s>t} K_s$ and, for $s < t$, set $K_{s,t} = g_{K_s}(K_t \setminus K_s)$. $(K_t)_{t\geq 0}$ is said to have the *local growth property* if $\mathrm{rad}(K_{t,t+h}) = \inf\{r \geq 0 : K_{t,t+h} \subset r\overline{\mathbb{D}} + x \text{ for some } x \in \mathbb{R}\} \to 0$ as $h \downarrow 0$ uniformly on compacts in t. This is a type of continuity condition for the growth of $(K_t)_{t\geq 0}$ but note that $K_t \setminus K_s$ can be large even when $K_{s,t}$ is small.

Proposition 1.2. *Let $(K_t)_{t\geq 0}$ be an increasing family of compact $\mathbb{H}-$hulls having the local growth property. Then for all t, there is a unique $\xi_t \in \mathbb{R}$ such that $\xi_t \in \overline{K_{t,t+h}}$ for all $h > 0$, and the process $(\xi_t)_{t\geq 0}$ is continuous.*

Proof. For fixed $t \geq 0$, the sets $\overline{K_{t,t+h}}$ are compact and decreasing when $h \downarrow 0$. By the local growth property, they have a unique common element $\xi_t \in \mathbb{R}$. For $t \geq 0$ and $h > 0$, choose $z \in K_{t+2h} \setminus K_{t+h}$ and set $w = g_{K_t}(z)$ and $w' = g_{K_t}(z)$. Then $w \in K_{t,t+2h}$ and $w' \in K_{t+h,t+2h}$ with $w' = g_{K_{t,t+h}}(w)$. Hence

$$|\xi_t - w| \leq 2\mathrm{rad}(K_{t,t+2h}),\ |\xi_{t+h} - w'| \leq 2\mathrm{rad}(K_{t+h,t+2h}),\ |w - w'| \leq 3\mathrm{rad}(K_{t,t+h}).$$

Therefore

$$|\xi_{t+h} - \xi_t| \leq 2\mathrm{rad}(K_{t+h,t+2h}) + 3\mathrm{rad}(K_{t,t+h}) + 2\mathrm{rad}(K_{t,t+h}) \to 0$$

as $h \downarrow 0$, uniformly on compacts in t. □

1.1.3 Chordal Loewner Chain

Chordal Loewner chain is the solution of a type of Loewner equation that describes the evolution of a curve growing from the boundary to the boundary in a domain of $\mathbb{C}$. By conformal mapping, we consider the special case where the domain is the upper half-plane $\mathbb{H}$. Loewner's equation (also called Loewner evolution) is a differential equation for a family of Riemann mappings from upper half-plane $\mathbb{H}$ with a growing slit γ (from 0 to ∞) to $\mathbb{H}$.

If γ is parametrized by half-plane capacity such that

$$\lim_{z\to\infty} z(g_t(z) - z) = 2t,$$

we denote by *Loewner chain* g_t the unique conformal isomorphism from the unbounded component of $\mathbb{H}\setminus\gamma(0,t]$ satisfying $|g_t(z)-z|\to 0$ as $z\to\infty$. Denote by *driving term* $\xi_t\in\mathbb{R}$ the image of the tip $\gamma(t)$ (since the tip can be identified with a prime end), then g_t satisfies the following differential equation, which we call the chordal Loewner equation:

$$\frac{d}{dt}g_t(z)=\frac{2}{g_t(z)-\xi_t},\qquad g_0(z)=z. \tag{1.1}$$

Conversely, suppose that a continuous function $\xi(\cdot):[0,\infty)\to\mathbb{R}$ is given in advance, for $z\in\mathbb{H}$, one can also solve the differential equation (1.1) to obtain the solution $g_t(z)$ till the moment $T_z:=\sup\{t>0:|g_t(z)-\xi_t|>0\}$ and set $K_t:=\{z\in\mathbb{H}:T_z\le t\}$. Then for $t>0$, $g_t(z)$ is a conformal map from $\mathbb{H}\setminus K_t$ to $\mathbb{H}$. The family $(g_t)_{t\ge0}$ describes the evolution of hulls $(K_t)_{t\ge0}$ corresponding to $\xi_\cdot$ and growing from the boundary to ∞. Therefore, we have a one-to-one correspondence between $\xi_\cdot$ and $(K_t)_{t\ge0}$. A growing family of hulls $(K_t)_{t\ge0}$ is *generated by a curve* γ if $\mathbb{H}\setminus K_t$ is the unbounded component of $\mathbb{H}\setminus\gamma[0,t]$ for all $t\ge0$. If $\xi_\cdot$ is sufficiently nice, then $(K_t)_{t\ge0}$ is generated by a curve γ with $\gamma(0)\in\mathbb{R}$, $\lim_{t\to\infty}\gamma(t)=\infty$.

For any Loewner chain $g_t, t\ge0$, we try to define the generating curve γ as $\gamma(t)=\lim_{\epsilon\downarrow0}f_t(W(t)+i\epsilon)$, where $f_t=g_t^{-1}$ defined on $\mathbb{H}\setminus\gamma[0,t]$. The function γ, if it exists, is called the *trace* of the Loewner chain.

1.1.4 Schramm-Loewner Evolution

In the half-plane, a deterministic curve corresponds to a deterministic driving term ξ_t, and random γ should correspond to random ξ_t. For the sake of conformal invariance, the most feasible candidate is the Brownian motion.

Definition 1.3. Chordal Schramm-Loewner evolution with parameter $\kappa>0$ (abbreviated as chordal SLE(κ) in half-plane) is a curve generated by driving term $\sqrt{\kappa}B_t$ in Loewner equation, where $(B_t)_{t\ge0}$ is the standard Brownian motion. Namely, the Loewner chain corresponding to SLE(κ) is

$$\frac{\partial}{\partial t}g_t(z)=\frac{2}{g_t(z)-\sqrt{\kappa}B_t},\qquad g_0(z)=z,\quad z\in\mathbb{H},.$$

The existence and uniqueness in law of SLE(κ) is obtained from Loewner's theorem. More precisely, the case ξ_t is a Brownian motion of any diffusivity κ except $\kappa=8$ was proved by Rohde and Schramm. The case $\kappa=8$ was then added by Lawler, Schramm and Werner when working on LERW and UST[LSW01].

It is not hard to check that SLE is scale invariant and has some kind of Markov property.

Theorem 1.4. *Let $(K_t)_{t\ge0}$ be a random increasing family of compact $\mathbb{H}-$hulls with local growth property and parametrized by half-plane capacity. Then $(K_t)_{t\ge0}$ is an SLE if and only if it satisfies:*

- *scaling invariance: for all $\lambda>0$, $(\lambda K_{\lambda^{-2}t})_{t\ge0}$ has the same distribution as $(K_t)_{t\ge0}$*

- *domain Markov property: set* $K_t^{(s)} = g_{K_s}(K_{s+t} \setminus K_s) - \xi_s$. *Then* $(K_t^{(s)})_{t\geq 0}$ *has the same distribution as* $(K_t)_{t\geq 0}$ *and is independent of* $\mathcal{F}_s = \sigma(\xi_r; r \leq s)$

Proof. Write $(\xi_t)_{t\geq 0}$ for the driving term of the Loewner chain corresponding to $(K_t)_{t\geq 0}$ and note that $(\xi_t)_{t\geq 0}$ is continuous. For $\lambda \in (0,\infty)$ and $s \in [0,\infty)$, define $\xi_t^\lambda = \lambda \xi_{\lambda^{-2}t}$ and $\xi_t^{(s)} = \xi_{s+t} - \xi_s$. Then the Loewner chain corresponding to $(K_t^\lambda)_{t\geq 0}$ has driving term $\xi_t^\lambda = \lambda \xi_{\lambda^{-2}t}$ and the Loewner chain corresponding to $(K_t^\lambda)_{t\geq 0}$ has driving term $\xi_t^{(s)} = \xi_{s+t} - \xi_s$. Hence $(K_t)_{t\geq 0}$ has the domain Markov property if and only if $(\xi_t)_{t\geq 0}$ has stationary independent increments. Also $(K_t)_{t\geq 0}$ is scale invariant if and only if the law of $(\xi_t)_{t\geq 0}$ is invariant under Brownian scaling. By Levy-Khinchin theorem, $(\xi_t)_{t\geq 0}$ has both these properties if and only if it is a Brownian motion of some diffusivity $\kappa \in [0,\infty)$, that is to say, if and only if $(K_t)_{t\geq 0}$ is an SLE. □

1.2 Advanced properties of SLE

For each κ, the trace γ exists and is a random curve such that the hulls $(K_t)_{t\geq 0}$ of SLE(κ) are generated by γ almost surely. We will admit this and define SLE(κ) to be its trace without mentioning it.

1.2.1 SLE as a complex Bessel process

The Bessel process of dimension n started from $x > 0$ is the real-valued process X given by

$$X_t = ||B_t||,$$

where $||\Delta||$ denotes the Euclidean norm in $\mathbb{R}^n$ and B is an n-dimensional Wiener process (Brownian motion) with $||B_0|| = x$. The n-dimensional Bessel process is the solution to the stochastic differential equation

$$dX_t = dB_t + \frac{n-1}{2}\frac{dt}{X_t} \tag{1.2}$$

where Z is a 1-dimensional Brownian motion. Note that this SDE makes sense for any real parameter n. Therefore we can define Bessel processes in any dimension to be the solution to (1.2).

Consider $z \in \overline{\mathbb{H}}$ with $z \neq 0$. Let g_t be a chordal SLE(κ) with driving process $W_t = -\sqrt{\kappa}B_t$, where the minus sign is for convenience. Define the precesses

$$\hat{Z}_t = g_t(z) - W_t, \qquad Z_t = \hat{Z}_t/\sqrt{\kappa}.$$

By Loewner equation, these processes have Ito differentials

$$d\hat{Z}_t = \frac{2}{\hat{Z}_t}dt + \sqrt{\kappa}dB_t, \quad dZ_t = \frac{2/\kappa}{Z_t}dt + dB_t$$

respectively. Therefore $(Z_t)_{t\in[0,T)}$, where T is the maximal time such that the solution exists and is positive for any $t \in [0,T)$, could be called as a $\delta(\kappa)-$dimensional complex Bessel process started from $\frac{z}{\sqrt{\kappa}}$, where $\delta(\kappa) = 1+\frac{4}{\kappa} \in (1,\infty)$.

In the next proposition, we list some properties of (real-valued) Bessel processes.

Proposition 1.5. *Let $\delta \in \mathbb{R}$ and let $(X_t)_{t\in[0,T)}$ be a $\delta-$dimensional Bessel process started from $x > 0$, that is, $(X_t)_{t\in[0,T)}$ is the unique solution of*

$$dX_t = \frac{\delta - 1}{2X_t}dt + dB_t, \quad X_0 = x.$$

Then for $x, y \in (0,\infty), x < y$,

1. $\mathbb{P}[T < \infty] = 1$ *if and only if* $\delta < 2$,
2. $\mathbb{P}[T = \infty] = 1$ *if and only if* $\delta \geq 2$,
3. $\mathbb{P}[\inf_{0\leq t<T} X_t > 0] = 1$ *if and only if* $\delta \geq 2$,
4. $\mathbb{P}[\lim_{t\to T} X_t = 0] = 1$ *when* $\delta < 2$.

Remark 1.6. As we saw earlier, the Euclidean norm of a $d-$dimensional Brownian motion is a $d-$dimensional Bessel process. By this proposition, when $d \geq 2$, the Brownian motion of dimension d won't hit the origin. In the case $d = 2$, the Brownian motion can get arbitrarily close to zero, but doesn't hit it in finite time.

Proof. These claims can be proven using the fact that for $\delta-$dimensional Bessel process $(X_t)_{t\in[0,T)}$, the process $M_t = X_t^{2-\delta}$ when $\delta \neq 2$ or $M_t = \log X_t$ when $\delta = 2$ is a local martingale for $t < T$. Since, by Ito's formula, when $\delta = 2$,

$$\begin{aligned} dM_t = d\log X_t &= \frac{dX_t}{X_t} - \frac{1}{2X_t^2}d\langle X\rangle_t \\ &= \frac{dt/2X_t + dB_t}{X_t} - \frac{1}{2X_t^2}dt \\ &= \frac{dB_t}{X_t}, \end{aligned}$$

and when $\delta \neq 2$

$$\begin{aligned} dM_t = dX_t^{2-\delta} &= (2-\delta)(X_t)^{1-\delta}dX_t + \frac{(2-\delta)(1-\delta)}{2X_t^{\delta}}d\langle X\rangle_t \\ &= \frac{(2-\delta)(\delta-1)}{2X_t^{\delta}}dt + (2-\delta)(X_t)^{1-\delta}dB_t + \frac{(2-\delta)(1-\delta)}{2X_t^{\delta}}dt \\ &= (2-\delta)(X_t)^{1-\delta}dB_t. \end{aligned}$$

Notice that Bessel processes are scale invariant so that they satisfy Brownian scaling $\lambda X_{t/\lambda^2} \overset{d}{=} X_t$ for all $\lambda > 0$.

For $t < T$, notice that $X_t - B_t = x + \frac{\delta-1}{2}\int_0^t X_s^{-1}ds$. Since $X_s > 0$ for all $s \in [0,T)$, it follows that

$$\begin{cases} X_t \geq B_t \text{ when } \delta \geq 1 \\ X_t \leq B_t \text{ when } \delta \leq 1 \end{cases} \tag{1.3}$$

For $a \in (0,x), b \in (x,\infty), c \in (0,x)\cup(x,\infty)$, let $\tau_c = \inf\{[0,T) : X_t = c\}$ and $\tau_{a,b} = \tau_a \wedge \tau_b$. Define also $\tau_0 = \lim_{a\to 0}\tau_a$ and $\tau_{0,b} = \lim_{a\to 0}\tau_{a,b}$ which exist since

they are decreasing functions of a. By the comparaison inequalities (1.3), we can show that for any $a \in [0, x)$ and $b \in (x, \infty)$, the stopping time $\tau_{a,b}$ is finite almost surely. For exemple, when $\delta \geq 1$, $\tau_{a,b}$ is bounded from above by the almost surely finite time that the Brownian motion hits b. The same argument works for $\delta < 1$.

Since $(M_{t\wedge\tau_{a,b}})_{t\geq 0}$ is a bounded martingale and $\tau_{a,b}$ is an almost surely finite stopping time, by the optional stopping theorem,

$$f(x) = f(a)\mathbb{P}[X_{\tau_{a,b}} = a] + f(b)\mathbb{P}[X_{\tau_{a,b}} = b],$$

where $f(x) = x^{2-\delta}$ when $\delta \neq 2$ and $f(x) = -\log(x)$ when $\delta = 2$. Then the expression for the exiting time

$$\mathbb{P}[\tau_a = \tau_{a,b}] = \frac{f(x) - f(b)}{f(a) - f(b)}$$

leads to the following conclusions:

- When $\delta < 2$ ($\kappa > 4$), $\lim_{a\to 0} f(a) = 0$. Thus

  $$\mathbb{P}[\tau_0 = \tau_{0,b}] = 1 - \frac{f(x)}{f(b)} > 0.$$

  Notice that $\tau_0 = T$. Thus

  $$\mathbb{P}[T < \infty] = \lim_{b\to\infty} \mathbb{P}[\tau_0 = \tau_{0,b}] = 1.$$

- When $\delta < 2$ ($\kappa < 4$), then choose $r_n > 1$ such that $\mathbb{P}[\tau_{r_n x} = \tau_{0,r_n x}] = 2^{-n}$. Then by scale invariance and the Markov property of the Bessel process,

  $$\mathbb{P}[X_t \geq n^{-1} \text{ for some } t > \tau_{n^{-1}r_n^{-1}}] = \mathbb{P}[\tau_{r_n x} = \tau_{0,r_n x}, x = n^{-1}r_n^{-1}] = 2^{-n}$$

  for all $n \in \mathbb{Z}_+$ such that $n^{-1}r_n^{-1} < x$. Consequently $\limsup_{t\to T} X_t = 0$.

- When $\delta \geq 2 (\kappa \leq 4)$, $\lim_{a\to 0} f(a) = \infty$, thus

  $$\mathbb{P}[\tau_0 = \tau_{0,b}] = 0.$$

  It is then possible to argue that almost surely, $\tau_{0,b} = \tau_b$ tends to ∞ as $b \to \infty$, since there is no blow-up of the solution in finite time by the uniqueness and existence theorem of SDEs. This shows that $T = \tau_0 = \infty$ almost surely.

- When $\delta > 2$, since almost surely

  $$\lim_{b\to\infty} \tau_b = \infty \text{ and } \lim_{b\to\infty} f(b) = 0,$$

  it holds that

  $$\mathbb{P}[\tau_{2^{-n}} < \infty] = f(x)/f(2^{-n}) = x^{2-\delta}2^{-(\delta-2)n}$$

  for any $n \in \mathbb{Z}_+$ such that $2^{-n} < x$, which is summable over n. Thus by Borel-Cantelli lemma,

  $$\inf_{t\geq 0} X_t > 0 \text{ almost surely.}$$

All the claims follows evidently. □

1.2.2 Transience

We will show in this subsection that a chordal SLE(κ) curve γ is transient, in the sense that $|\gamma(t)| \to \infty$ as $t \to \infty$. In particular, $\overline{\gamma[0,\infty)} = \gamma[0,\infty)$ when γ is transient.

Lemma 1.7. *When* $0 < \kappa \leq 4$, $\mathbb{P}[dist(\overline{\gamma[0,\infty)}, [x,x']) > 0] = 1$ *for any* $0 < x < x'$ *or* $x < x' < 0$.

Proof. Suppose first that $\kappa \in (0,4)$. By symmetry and scale invariance of SLE(κ), it is enough to show that $\mathbb{P}(\mathrm{dist}(\overline{\gamma[0,\infty)}, [1,x]) > 0) = 1$ for all $x > 1$. Let $0 < \delta < 1/4$ and define $\sigma_\delta = \inf\{t \geq 0 : \mathrm{dist}(\gamma(t), [1,x]) \leq \delta\}$. Let us consider the event $\sigma_\delta < \infty$. Let R_{σ_δ} be the union of the right-hand side of $\gamma[0,\sigma_\delta]$ and $[0,1/2]$ and $h(z)$ be the bounded harmonic function on $H_{\sigma_\delta} = \mathbb{H} \setminus \gamma[0,\sigma_\delta]$ that has boundary value 1 on R_{σ_δ} and 0 elsewhere. By conformal invariance of harmonic functions, after applying the conformal map $\varphi \circ g_{\sigma_\delta}$ where $\varphi : z \mapsto \frac{z - g_{\sigma_\delta}(1/2)}{z + g_{\sigma_\delta}(1/2)}$ maps $\mathbb{H}$ conformally onto the unit disc which sends $g_{\sigma_\delta}(iy)$ to 0, we see that for $y > 0$, $h(iy)$ has the expression

$$\begin{aligned} h(iy) &= \frac{1}{2\pi}|\arg\frac{g_{\sigma_\delta}(1/2) - g_{\sigma_\delta}(iy)}{g_{\sigma_\delta}(1/2) + g_{\sigma_\delta}(iy)} - \arg\frac{W_{\sigma_\delta} - g_{\sigma_\delta}(iy)}{W_{\sigma_\delta} + g_{\sigma_\delta}(iy)}| \\ &= \frac{1}{2\pi}|\arg\frac{g_{\sigma_\delta}(1/2)/y - i + O(1/y^2)}{g_{\sigma_\delta}(1/2)/y + i + O(1/y^2)} - \arg\frac{W_{\sigma_\delta}/y - i + O(1/y^2)}{W_{\sigma_\delta}/y + i + O(1/y^2)}| \\ &= \frac{1}{\pi y}(g_{\sigma_\delta}(1/2) - W_{\sigma_\delta}) + O(\frac{1}{y^2}) \end{aligned}$$

as $y \to \infty$ since $g(iy) = iy + O(1/y)$.

On the other hand, we can write the harmonic measure $h(iy)$ as the probability that a Brownian motion started from iy exits H_{σ_δ} through R_{σ_δ}. On this event the Brownian motion has to intersect the vertical line connecting $[1,x]$ to $\gamma(\sigma_\delta)$. Let $x_0 = \mathrm{Re}\gamma(\sigma_\delta)$. Then the probability that a complex Brownian motion started from iy will hit the right-hand side of the segment $[x_0, x_0 + i\delta]$ before hitting the left-hand side of $\gamma[0,\sigma_\delta]$ or real axis, which is less than the probability that a complex Brownian motion started from iy hit the segment $[x_0, x_0 + i\delta]$ before hitting real axis is $O(\frac{\delta}{y})$ as $y \to \infty$. This can be derived similarly as above by mapping to the unit disc.

Using the latter harmonic measure as an upper bound for the former one, we conclude that $g_{\sigma_\delta} - W_{\sigma_\delta} = O(\delta)$ uniformly almost surely. Because the infimum of Bessel process is positive a.s., there exists a positive random variable δ_0 such that $\sigma_\delta = \infty$ for all $\delta \in (0, \delta_0)$. The claim follows for $\kappa \in (0,4)$.

For $\kappa = 4$, the above proof does not apply since the infimum of Bessel process with corresponding dimension 2 is zero. Let us handle this differently. Let $0 < y < x < \infty$ and let $\overline{\gamma}(t)$ denote the complex conjugate of $\gamma(t)$, and set $D_t := \mathbb{C} \setminus (\gamma[0,t] \cup \overline{\gamma}[0,t] \cup (-\infty, y])$. Using Schwarz reflection, g_t can be conformally extended to D_t. Note that $g_t(D_t) = \mathbb{C} \setminus (-\infty, g_t(y)]$. By the Koebe quarter theorem applied to g_t^{-1}, the distance from x to ∂D_t is at least $\frac{1}{4}(g_t(x) - g_t(y))/g_t'(x)$. Consequently, it suffices to show that a.s.

$$\sup_{t \geq 0} g_t'(x)/(g_t(x) - g_t(y)) < \infty.$$

Set $Q(t) = Q_{x,y}(t) := \log g_t'(x) - \log(g_t(x) - g_t(y))$. Then it reduces to verifying that $\sup_{t\geq 0} Q(t) < \infty$. The time derivative of Q i.s

$$\partial_t Q(t) = -2Y_x(t)^{-2} + 2Y_x(t)^{-1}Y_y(t)^{-1}.$$

Since $Y_x(t) \geq Y_y(t)$ for all $t \geq 0$, $Q(t)$ is increasing in t. Define

$$G(s) := \log s \log(1+s) - \frac{1}{2}\log^2(1+s) + \mathrm{Li}_2(-s),$$

where $\mathrm{Li}_2(z) = \int_z^0 s^{-1}\log(1-s)ds$. G satisfies that

$$G'(s) = \frac{\log s}{1+s} - \frac{\log(1+s)}{1+s},$$

$$G''(s) = 1/s(1+s)^2 - \frac{\log s}{(1+s)^2} + \frac{\log(1+s)}{(1+s)^2}.$$

Also G satisfies

$$s(1+s)^2 G''(s) + s(1+s)G'(s) = 1.$$

exo: $Q(t) - G_t$ is a local martingale, where

$$G_t := G(\frac{g_t(x) - g_t(y)}{g_t(y) - \xi(t)})$$

Consequently, there is an increasing sequence of stopping times t_n tending to ∞ a.s. such that $\mathbb{E}[Q(t_n)] = \mathbb{E}[G_{t_n}] + Q(0) - G_0$. Since $\sup\{G(x) : x > 0\} < \infty$,

$$\limsup_{n\to\infty} \mathbb{E}[Q(t_n)] < \infty.$$

Because Q is increasing in t, it follows that $\mathbb{E}[\sup_t Q(t)] < \infty$ and hence $\sup_t Q(t) < \infty$ a.s., which completes the proof. □

Lemma 1.8. *Suppose that $\kappa > 4, \kappa \neq 8$, and let $t > 0$. Then a.s. there is $\epsilon > 0$ such that $\{z \in \mathbb{H} : |z| < \epsilon\} \subset K_t$.*

Proof. It is sufficient to show that there exists $t > 0$ such that

$$\delta := \mathbb{P}[0 \notin \overline{H_t}] > 0.$$

Since by conformal invariance, δ does not depend on t, and these events $\{0 \notin \overline{H_t}\}$ are monotone in t, it follows that

$$\mathbb{P}[\forall t > 0, 0 \notin \overline{H_t}] = \delta.$$

Then Blumenthal's $0-1$ law gives that $\delta = 1$. The detailed proof is omitted here, see [RS03] □

Theorem 1.9 (Transience)**.** *For all $\kappa > 0$, the SLE(κ) trace $\gamma(t)$ is transient a.s. That is,*

$$\mathbb{P}[\liminf_{t\to\infty} |\gamma(t)| = \infty] = 1.$$

Proof. Let $T_1 = \inf\{t \geq 0 : |\gamma(t)| = 1\}$. First consider the case $\kappa \in [0, 4]$. In this range, γ is a simple path. Then a.s. there are two limit points x_-, x_+ for $g_{T_1}(z)$ as $z \to 0$ in H_1. Note that $g_{T_1}(\gamma[T_1, \infty))$ has the same distribution as $\gamma[0, \infty)$ translated by W_{T_1}. Almost surely, the distance from the SLE(κ) curve $g_{T_1}(\gamma[T_1, \infty)) - W_{T_1}$ to $[-n, -\frac{1}{n}] \cup [\frac{1}{n}, n]$ is positive for all $n \in \mathbb{N}$. Hence it will stay as a positive distance from x_- and x_+, which means that $0 \notin \overline{\gamma[T_1, \infty)}$ a.s. Consequently, there exists a random variable $r > 0$ such that $|\gamma(t)| \geq r$ for all $t \geq T_1$. Combined with scaling invariance and Borel-Cantelli lemma, we can construct a sequence of random variables $0 < R_1 < R_2 < ...$ such that $R_k \to \infty$ almost surely and γ doesn't enter $B(0, R_{k-1})$ after hitting $\partial B(0, R_k)$.

Now suppose that $\kappa > 4$. There is a random variable $r > 0$ such that $\{z \in \mathbb{H} : |z| < r\} \subset K_1$. The scaling property of SLE shows that

$$\mathbb{P}[\{z \in \mathbb{H} : |z| < r\sqrt{T_1}\} \subset K_{T_1}] = \mathbb{P}[\{z \in \mathbb{H} : |z| < r\} \subset K_1] = 1.$$

Now let the random radius be $r\sqrt{T_1}$ and the same argument as in the case $\kappa \in [0, 4]$ gives the transience. □

1.2.3 Phases of SLE

The next theorem summarizes the important facts on random curve γ.

Theorem 1.10. *Let the random curve $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ be SLE(κ). Then*

- *For all $0 < \kappa \leq 4$, γ is simple and $\gamma(0, \infty) \cap \mathbb{R} = \emptyset$.*
- *For all $4 < \kappa < 8$, γ is not simple. In fact, it is not simple on any interval: for any $0 \leq t_1 < t_2$, there exists $t_1 < s_1 < s_2 < t_2$ such that $\gamma(s_1) = \gamma(s_2)$. However, γ is not space-filling: for any $z \in \mathbb{H}$, dist$(z, \gamma[0, \infty)) > 0$ or equivalently $z \notin \gamma[0, \infty)$ almost surely.*
- *For all $\kappa \geq 8, \gamma$ is not simple on any interval. Moreover, γ is space-filling: $z \in \gamma[0, \infty)$ almost surely.*

We will first show that γ is simple for $\kappa \in (0, 4]$ and non-simple curve for $\kappa \in (4, \infty)$. Remember that $z \in K_t$ if and only if $\tau(z) \leq t$, where $\tau(z)$ is the maximal time such that the solution of Loewner equation with initial value z exists. Notice also that for $0 < x_1 < x_2$ or for $x_2 < x_1 < 0$, it holds that $W_t < g_t(x_1) < g_t(x_2)$ or $g_t(x_2) < g_t(x_1) < W_t$ for all $0 < t < \tau(x_1) \wedge \tau(x_2)$ and consequently, $\tau(x_1) < \tau(x_2)$.

Now we use the fact that for all $x \in \mathbb{R}\backslash\{0\}$, the process $X_t = (g_t(x) - W_t)/\sqrt{\kappa}$ is a Bessel process of dimension $\delta = 1 + \frac{4}{\kappa}$. Proposition 1.5 applied to X_t shows that $\mathbb{P}[\tau(x) < \infty] = 1$ when $\kappa > 4$ and 0 when $\kappa \in (0, 4)$.

proof: phase transition from simple to non-simple at $\kappa = 4$. Let $s > 0$ and let x_- and x_+ be the two images of 0 under the map $g_s - W_s$. By Proposition 1.5 and conformal Markov property, $\hat{\gamma}(t) = g_s(\gamma(s + t)) - W_s, t \geq 0$, intersect the real axis only at 0. In particular it doesn't intersect $[x_-, 0) \cup (0, x_+]$. Since $f_s = g_s^{-1}$ is continuous to the boundary by Caratheodory principle (since the tip of γ can be identified with a prime end of the unbounded component), this implies that

$$\gamma[0, s] \cap \gamma[s, \infty) = \{\gamma(s)\}$$

almost surely. In fact, we can show that this holds almost surely for all s by taking the intersection on all rationals and then by continuity to all s. Thus γ is simple.

Let's then show that $\mathrm{SLE}(\kappa), \kappa > 4$, is not simple on any interval. Suppose on the contrary tha $\mathrm{SLE}(\kappa), \kappa > 4$, is simple. Since $\tau(1)$ is finite almost surely, for fixed $t > 0$, by scaling $\mathbb{P}[\tau(x) \leq t] \to 1$ as $x > 0$ tends to 0. Therefore for all $t > 0$ and $x > 0$

$$\mathbb{P}[\gamma[0,t] \cap (0,x] \neq \emptyset] = 1.$$

For any $0 \leq t_1 < u < t_2$. Let $\hat{x}_- \leq 0 \leq \hat{x}_+$ be such that the image of $\gamma[t_1, u]$ under $g_u - W_u$ is $[\hat{x}_-, \hat{x}_+]$. Then by domain Markov property we can find $u < s_2 < t_2$ such that $g_u(\gamma(t_2)) - W_u \in [\hat{x}_-, 0) \cup (0, \hat{x}_+]$. Therefore there exists $t_1 \leq s_1 < u$ such that $\gamma(s_1) = \gamma(s_2)$, which leads to a contradiction. □

Then we study the phase transition at $\kappa = 8$ by analyzing the distance from z to $\overline{\gamma[0, \tau(z))}$ or $\gamma[0, \infty)$. To do so, we first need a conformally covariant [1] version of the distance to the boundary. For any simply connected domain $U \subset \mathbb{C}$, $U \neq \mathbb{C}$, and for any point $z_0 \in U$, let ψ be the unique conformal map from U onto $\mathbb{D}$ such that $\psi(z_0) = 0$ and $\psi'(z_0) > 0$. Then the *conformal radius of U from z_0* is defined as

$$\rho(z_0, U) = \psi'(z_0)^{-1}.$$

The conformal radius is proportional to the radius by Koebe quarter Theorem,

$$\frac{1}{4}\rho(z_0, U) \leq \mathrm{dist}(z_0, \partial U) \leq \rho(z_0, U).$$

For any fixed $z_0 \in \overline{\mathbb{H}}$, let $Z_t = g_t(z_0) - W_t, t \in [0, \tau(z_0))$ and let X_t and Y_t be the real and imaginary parts of Z_t respectively, and let $W_t = -B_t$. Since the conformal map from $H_t = \mathbb{H} \setminus K_t$ normalized at z_0 is $\psi(z) = e^{i\theta} \frac{g_t(z) - g_t(z_0)}{g_t(z) - \overline{g_t(z_0)}}$ such that $\psi'(z_0) > 0$, the conformal radius of H_t from z_0 is equal to

$$\rho(z_0, H_t) = |\psi'(z_0)|^{-1} = \frac{|g_t(z_0) - \overline{g_t(z_0)}|}{|g_t'(z_0)|} = \frac{2Y_t}{g_t'(z_0)}$$

when $t < \tau(z_0)$. The proportionality to the inradius implies that

$$\frac{1}{2}\mathrm{dist}(z_0, \mathbb{R} \cup \overline{\gamma[0,\infty)}) \leq \lim_{t \uparrow \tau(z_0)} \frac{Y_t}{g_t'(z_0)} \leq 2\mathrm{dist}(z_0, \mathbb{R} \cup \overline{\gamma[0,\infty)}).$$

By taking the real and imaginary parts on both sides of Loewner equation, it follows that

$$dX_t = \frac{2X_t}{X_t^2 + Y_t^2} dt + \sqrt{\kappa} dB_t \tag{1.4}$$

$$dY_t = -\frac{2Y_t}{X_t^2 + Y_t^2} dt \tag{1.5}$$

$$d \log Z_t = (2 - \kappa/2) \frac{dt}{Z_t^2} + \sqrt{\kappa} \frac{dB_t}{Z_t} \tag{1.6}$$

$$\partial_t \log |g_t'(z)| = \frac{1}{2} \partial_t \log g_t'(z) \cdot \overline{g_t'(z)} = -2 \frac{X_t^2 - Y_t^2}{(X_t^2 + Y_t^2)^2} \tag{1.7}$$

[1] Conformally covariant here means that the transformation rule under conformal maps is simple

Using (1.5) and (1.7) write that

$$\partial_t \log \rho(z_0, H_t) = \partial_t \log \frac{2Y_t}{|g_t'(z)|} = -4\frac{Y_t^2}{(X_t^2+Y_t^2)^{-2}}.$$

Define

$$S(t) := \log \rho(z_0, \mathbb{H}) - \log \rho(z_0, \mathbb{H}_t) = 4\int_0^t \frac{Y_s^2}{(X_s^2+Y_s^2)^2} ds,$$

then it follows that

$$\rho(z_0, H_t) = 2y_0 \exp(-S(t)),$$

where $y_0 = \mathrm{Im} z_0$.

phase transition from non-space-filling to space-filling at $\kappa = 8$. Define a time change using $S(t)$ and set $\sigma(s) = S^{-1}(s)$. Write $a_t = \frac{2Y_t}{X_t^2+Y_t^2}$ and define $\sigma(s) = \inf\{t \geq 0 : S(t) \geq s\}$. (1.6) gives that

$$d\theta_t = (\kappa-4)X_tY_t\frac{dt}{(X_t^2+Y_t^2)^2} - \sqrt{\kappa}\frac{Y_t}{X_t^2+Y_t^2}dB_t.$$

Define the Brownian motion after time change $\tilde{B}_s = -\int_0^{\sigma(s)} a_t dB_t$. We use the definition $\hat{\theta}_s = 2\theta_{\sigma(s)}$, then

$$\begin{aligned}
d\hat{\theta}_s &= 2d\int_0^{\sigma(s)} d\theta_t \\
&= d(\int_0^{\sigma(s)} (\kappa-4)\frac{X_tY_tdt}{(X_t^2+Y_t^2)^2} - \sqrt{\kappa}\int_0^{\sigma(s)} a_t dB_t) \\
&= d(\int_0^s (\kappa-4)\frac{X_{\sigma(r)}dr}{2Y_{\sigma(r)}} + \sqrt{\kappa}\int_0^s d\tilde{B}_r) \\
&= \frac{\kappa-4}{2}\cot(\tilde{\theta_s}/2)ds + \sqrt{\kappa}d\hat{B}_s
\end{aligned}$$

where $\theta_t := \arg Z_t$, and

$$\rho(z_0, H_{\sigma(s)}) = 2y_0 y^{-s}$$

The solution to this stochastic equation exists, independent of the radius of the initial point and is unique until the exit time from interval $(0, 2\pi)$

$$\hat{\tau} = \sup\{s \geq 0 : \hat{\theta}_u \in (0, 2\pi) \text{ for all } u \in [0, s]\}.$$

If $\hat{\tau}$ is finite, then $\lim_{t\to\hat{\tau}} \hat{\theta}_t$ exists and belongs to $\{0, 2\pi\}$. Since $\theta_t \in (0, \pi)$ for all $t < \tau(z_0)$ (the flow is invertible, so $\mathbb{R}$ has preimage in $K_t \cup \mathbb{R}$), we have that $S(t) < \hat{\tau}$ for all $t < \tau(z_0)$ and

$$S(\tau(z_0)) := \lim_{t\to\tau(z_0)} S(t) \leq \hat{\tau}.$$

We can compare $\hat{\theta}_s$ to Bessel processes asymptotically when $\hat{\theta}_s \approx 0$ or $\hat{\theta}_s \approx 2\pi$. Namely, $\cot(\hat{\theta}_s/2)$ is close to $\hat{\theta}_s/2$ or $2/(2\pi - \hat{\theta}_s)$ respectively. Then the corresponding Bessel process has dimension $\delta = (3\kappa - 8)/\kappa$. This implies that $\hat{\tau}$ is almost surely finite for $\kappa < 8$ and almost surely infinite for $\kappa \geq 8$.

When $\kappa < 8$, for any $z \in \mathbb{H}$, almost surely

$$0 < \mathrm{dist}(z, \overline{K^-_{\tau(z)}}) \leq \mathrm{dist}(z, \overline{\gamma[0, \tau(z))})$$

, where $K^-_{\tau(z)} := \cup_{\epsilon>0} K_{\tau(z)-\epsilon}$. By Markov property applied to stopping time $\tau(z)$, $\gamma[\tau(z), \infty)$ is SLE(κ) in domain $\mathbb{H} \setminus K_{\tau(z)}$, independent of $\gamma[0, \tau(z))$, $\mathrm{dist}(z, \gamma) = \mathrm{dist}(z, \overline{\gamma[0, \tau(z))}) > 0$, which means γ is not space-filling.

When $\kappa \geq 8$, since almost surely γ is transient and we will show later that $S(\tau(z_0)) = \hat{\tau} = \infty$ for SLE(κ), γ is space-filling a.s.. □

Remark 1.11. The proof above implies that when $\kappa < 8$, for any $z \in \mathbb{H}$ almost surely $\mathrm{dist}(z, \gamma[0, \tau(z))) > 0$, since $\lim_{t \to \tau(z_0)} \rho(z_0, H_t) = \lim_{t \to \tau(z_0)} 2y_0 \exp(-S(t)) = 2y_0 \exp(-\hat{\tau})$ and $\hat{\tau} < \infty$ for $\kappa < 8$.

Lemma 1.12. *Let $z \in \mathbb{H}$ and γ be a SLE(κ) curve. Let $(g_t)_{t \geq 0}$ be the Loewner chain corresponding to γ with driving term $(W_t)_{t \geq 0}$. Then*

$$\lim_{t \to \tau(z)} \theta_t \in \{0, \pi\} \text{ and } S_{\tau(z)} = \hat{\tau}.$$

Proof. First, we claim that if $\lim_{t \to \tau(z)} \arg(g_t(z) - W_t) \in \{0, \pi\}$, then $\hat{\tau} \leq S(\tau(z))$, which implies that $S_{\tau(z)} = \hat{\tau}$. If $\kappa \in (0, 4]$, $\tau(z) = \inf \emptyset$. Let $r = |z|$ and $R > r$. By the transience, there exists $s > 0$ such that $|\gamma(t)| \geq R$ for all $t \geq s$. By symmetry, we can suppose that z is to the right of $\gamma[0, s]$ in the sense that z can be connected by a path in $(\mathbb{H} \setminus \gamma[0, s]) \cap B(0, r)$ to the "right side" of $\gamma[0, s]$. Notice that then z is to the right of $\gamma[0, t]$ in the same sense for all $t \geq s$.

Denote by L_t the union of $\mathbb{R}_{<0}$ and the "left side" of $\gamma[0, t]$. We can write using the harmonic measure $\arg(g_t(z) - W_t) = \pi \mathrm{HM}(z, L_t, \mathbb{H} \setminus \gamma(0, t])$. By the weak-Beurling estimate of harmonic measure

$$0 \leq \arg(g_t(z) - W_t) \leq C(\frac{r}{R})^\alpha$$

with some universal constants $C > 0$ and $\alpha > 0$. The claim follows by taking R to ∞. When z is to the left of $\gamma[0, s]$, the same proof gives that $\lim_{t \to \infty} \arg(g_t(z) - W_t) = \pi$.

When $\kappa \in (4, 8)$, $\tau(z) < \infty$ and $\mathrm{dist}(z, \gamma[0, \tau(z)]) > 0$. When z is to the right of $\gamma[0, \tau(z)]$, then the harmonic measure of the union of $\mathbb{R}_{<0}$ and the left-hand side of $\gamma[0, t]$ tends to zero as $t \to \tau(z)$. Similarly when z is to the left of $\gamma[0, \tau(z)]$.

When $\kappa \in (8, \infty)$, if it would happen that $S(\tau(z)) < \infty$, then by the above argument $\lim_{t \to \tau(z)} \theta_t \in \{0, \pi\}$ and therefore $\hat{\tau} \leq S(\tau(z)) < \infty$, which would lead to a contradiction. □

1.2.4 Upper bounds of dimensions

Recall the time change in the previous section. The notion $(U)_{z_0}$ is used for the connected component of z_0 in U.

Proposition 1.13. *For all κ, $\rho(z_0, (\mathbb{H} \setminus \gamma(0, \infty))_{z_0}) = 2y_0 e^{-\hat{\tau}}$.*

In this part, we will write $z_0 = r\exp(i\hat{\theta}_0/2)$, $r > 0$, $\hat{\theta}_0 \in [0, 2\pi]$. Define

$$F(\hat{\theta}_0, u) := \mathbb{P}[\rho(z_0, \mathbb{H} \setminus \gamma(0,\infty)) \le 2y_0 e^{-u}] = \mathbb{P}[\hat{\tau} \ge u].$$

Its conditional version given $\hat{\mathcal{F}}_s = \mathcal{F}_{\sigma(s)}$, can be written in the form

$$\mathbb{P}[\hat{\tau} \ge u | \hat{\mathcal{F}}_s] = F(\hat{\theta}_s, u - s) \tag{1.8}$$

by the conformal Markov property of SLE(κ). The left-hand side of (3.4) is a martingale with respect to s. Therefore F satisfies, provided F is smooth,

$$\frac{\partial F}{\partial t} = LF$$

$$\mathbb{P}[\overline{\gamma[0,\infty)} \cap \overline{B(z_0, r)} \neq \emptyset] \asymp (\frac{r}{\mathrm{Im} z_0})^{\lambda} \sin(\arg z_0)^{\beta} \tag{1.9}$$

where $A \asymp B$ means that there exists positive constants c_1 and c_2 such that $c_1 B \le A \le c_2 B$.

For a non-empty bounded Borel set $K \subset \mathbb{C}$, let N_ϵ be the number of sets of form $[(j-1)\epsilon, j\epsilon] \times [(k-1)\epsilon, k\epsilon]$ that intersect K, where $(j,k) \in \mathbb{Z}^2$. The upper and lower box-counting dimensions (or Minkowski) dimensions are defined respectively as

$$\dim_{\overline{M}}(K) = \limsup_{\epsilon \downarrow 0} \frac{\log N_\epsilon}{\log \frac{1}{\epsilon}}, \dim_{\overline{M}}(K) = \liminf_{\epsilon \downarrow 0} \frac{\log N_\epsilon}{\log \frac{1}{\epsilon}}.$$

If the upper and lower limits are equal, then $\lim_{\epsilon \downarrow 0} (\log N_\epsilon)/(\log \frac{1}{\epsilon})$ exists and is called the *box-counting dimension*. It always holds that the (upper and lower) box-counting dimension is not less than the Hausdorff dimension of K. Hence any upper bound for the box-counting dimension is an upper bound for the Hausdorff dimension.

Return to our case SLE(κ) curve, $0 < \kappa < 8$. By (1.9), there exists some $C > 0$ and $\lambda > 0$,

$$\mathbb{P}[\overline{\gamma[0,\infty)} \cap \overline{B(z_0, r)} \neq \emptyset] \le C(\frac{r}{\mathrm{Im} z_0})^{\lambda}$$

for any $z_0 \in \mathbb{H}$ and $r > 0$. If $\gamma[0,\infty)$ intersects the box $R_{j,k} := [(j-1)2^{-n}, j2^{-n}] \times [(k-1)2^{-n}, k2^{-n}]$, then $\mathrm{dist}((j-\frac{1}{2})2^{-n} + i(k-\frac{1}{2})2^{-n}, \gamma[0,\infty)) < 2^{-n-\frac{1}{2}}$. Hence

$$\begin{aligned}
\mathbb{E}N_{2^{-n}} &= \sum_{-2^n < j \le 2^n, 0 < k \le 2^n} \mathbb{P}[\gamma[0,\infty) \cap R_{j,k} \neq \emptyset] \\
&\le \mathrm{const} 2^{-n\lambda} \sum_{-2^n < j \le 2^n, 0 < k \le 2^n} (k - \frac{1}{2})^{-\lambda} \\
&\le \mathrm{const} 2^{(2-\lambda)n}
\end{aligned}$$

where the constant depends only on C and λ.

By Chebyshev inequality, for each $\delta > 0$,

$$\mathbb{P}[N_{2^{-n}} \ge 2^{(2-\lambda+\delta)n}] \le \mathrm{const} 2^{-\delta n}.$$

Since these probabilities are summable over n, by Borel-Cantelli lemma, there exists a random integer $n_0(\delta)$ such that $N_{2^{-n}} < 2^{(2-\lambda+\delta)n}$ for $n > n_0(\delta)$. Thus

$$\limsup_{n\to\infty}(\log N_{2^{-n}})/(n\log 2) \le 2-\lambda+\delta.$$

Since δ is arbitrary and $\epsilon \mapsto N_\epsilon$ is non-decreasing,

$$\limsup_{\epsilon\downarrow 0}(\log N_\epsilon)/(\log\frac{1}{\epsilon}) \le 2-\lambda.$$

Therefore, the upper box-counting dimension is at most $2-\lambda = 1+\frac{\kappa}{8}$. Thus we have shown

$$\dim_{\overline{M}}(([-1,1]\times[0,1])\cap\gamma[0,\infty)) \le 1+\frac{\kappa}{8},$$

which implies that $\dim_{\mathcal{H}}(\gamma[0,\infty)) \le 1+\frac{\kappa}{8}$.

1.2.5 Phase transition of $\tau(z)$ at $\kappa = 4$

We will investigate in this subsection whether $\tau(z)$ is finite or infinite, that is, whether or not z belongs to $\cup_{t\ge 0}K_t$.

Proposition 1.14. *Let $Z_t = g_t(z) - W_t$ When $\kappa > 4$, for any $z \in \mathbb{H}$, $\tau(z) < \infty$ almost surely. And when $\kappa \in (0,4]$, for any $z \in \mathbb{H}$, $\tau(z) = \infty$ almost surely.*

Remark 1.15. In fact, in range $\kappa > 4$, combined with the transience, a stronger statement that a.s. for all $z \in \overline{H}$, $\tau(z) < \infty$ holds as well.

Proof. Let $Z_t = g_t(z) - W_t, z \in \mathbb{H}$ and $B_R = \{z \in \mathbb{C} : |z| < R\}$. Denote by σ_R the exit time of $(Z_t)_{t\ge 0}$ from B_R, $\sigma_R \le \tau(z)$. First, we will show that $\sigma_R < \infty$ almost surely for all κ.

To see this, let $X_t = \mathrm{Re}Z_t$ and write

$$dX_t = \sqrt{\kappa}B_t + \frac{2X_t}{|Z_t|^2}.$$

For each $k \in \mathbb{Z}_{\ge 0}$ and $R > 0$, let $E_{k,R}$ be the event that

$$\min_{t\in[k,k+1]} W_t \le W_{k+1} - R \text{ or } \max_{t\in[k,k+1]} W_t \ge W_{k+1} + R.$$

On the event $E_{k,R}$, denote by $\eta_{k,R}$ the maximal s such that $|W_s - W_{k+1}| = R$. By reflection symmetry, the distribution of $(B_t : t > \eta_{k,R})$ is the same as that of $(2B_{\eta_{k,R}} - B_t : t > \eta_{k,R})$. It is not hard to see that

$$\mathbb{P}[W_{\eta_{k,R}} - W_{k+1} = m_1 R, m_2 X_{\eta_{k,R}} \ge 0 | E_{k,R}] \tag{1.10}$$
$$= \mathbb{P}[W_{\eta_{k,R}} - W_{k+1} = m_1 R | E_{k,R}]\mathbb{P}[m_2 X_{\eta_{k,R}} \ge 0 | E_{k,R}] \tag{1.11}$$
$$= \frac{1}{2}\mathbb{P}[m_2 X_{\eta_{k,R}} \ge 0 | E_{k,R}] \tag{1.12}$$

for all $m_1, m_2 = \pm 1$. Also by Markov property $E_{k,R}$ is independent of $\sigma(B_t : t \le k)$,

$$\mathbb{P}[E_{k,R}] = \mathbb{P}[E_{1,R}] > 0$$

for all k and R. Notice that all events $E_{k,R}, k \in \mathbb{Z}_{\geq 0}$ are independent and $\sum_{k\in\mathbb{Z}_{\geq 0}} \mathbb{P}[E_{k,R}] = \infty$, thus by the Borel-Cantelli lemma,

$$\mathbb{P}[\cap_{n=0}^{\infty} \cup_{k=n}^{\infty} E_{k,R}] = 1.$$

We claim that the conditional probability of $\sigma_R \leq k+1$ given $E_{k,R}$ is at least $\frac{1}{2}$. Again by Borel-Cantelli, $\sigma_R < \infty, a.s.$ If $\tau(z) \leq k+1$, then $\sigma_R \leq k+1$. If $\tau(z) > k+1$ and either one of the events $E_+ = W_{k+1} - W_{\eta_{k,R}} = R, X_{\eta_{k,R}} \leq 0$ or $E_- = W_{k+1} - W_{\eta_{k,R}} = -R, X_{\eta_{k,R}} \geq 0$ occurs, then

$$|X_{k+1}| = |X_{\eta_{k,R}} + 2\int_{\eta_{k,R}}^{k+1} X_t |Z_t|^{-2} dt - (W_{k+1} - W_{\eta_{k,R}})| \geq R$$

since on E_+ we have $W_t - W_{\eta_{k,R}} > 0$ and on E_- we have $W_t - W_{\eta_{k,R}} < 0$ for all $t \in [\eta_{k,R}, k+1]$. It follows that the probability of $\sigma_R \leq k+1$ given $E_{k,R}$ is at least $\frac{1}{2}$ by (1.10).

Next notice that when $\kappa > 4$, $Z_t^{1-4/\kappa}$ is a local martingale, in the sense that its real and imaginary parts are all local martingales. Let $\alpha = 1 - \frac{4}{\kappa}$ and $\mu = e^{i\pi(1-\alpha)/2}$, then $h(z) = \mathrm{Im}(\mu z^\alpha)$ is a positive function on $\overline{\mathbb{H}} \setminus \{0\}$ and there exists a constant $c \in (0,1)$ such that $c|z|^\alpha \leq h(z) \leq |z|^\alpha$ for all $z \in \overline{\mathbb{H}}$. Moreover $h(Z_t)$ is a real-valued local martingale.

If we apply the optional stopping theorem to bounded martingale $h(Z_{t\wedge\sigma_R})$ and the stopping time σ_R, which is almost surely finite, then

$$h(z) = \mathbb{E}[h(Z_\sigma)|\sigma_R < \tau(z)]\mathbb{P}[\sigma_R < \tau(z)].$$

Therefore $\mathbb{P}[\sigma_R < \tau(z)] \leq c^{-1}(\frac{|z|}{R})^\alpha$. Thus

$$\mathbb{P}[\tau(z) < \infty] \geq 1 - \mathbb{P}[\sigma_R < \tau(z)] \geq 1 - c^{-1}(\frac{|z|}{R})^\alpha.$$

Since R is chosen arbitrarily, it follows that $\mathbb{P}[\tau(z) < \infty] = 1$.

When $\kappa \in (0,4]$, for all $z \in \mathbb{H}$, almost surely, $K_t = \gamma[0,t]$ and $\mathrm{dist}(z, \overline{\gamma[0,\tau(z))}) > 0$, therefore $\tau(z) = \infty$.

□

2 Continuity properties of Loewner chains

In this part, we study regularity properties of curves in the half-plane capacity parametrization and their convergence with respect to the uniform norm on compact time intervals. When we consider random curves, we are interested, in probability theoretic sense, in so called *weak convergence of probability measures* or equivalently *convergence of random curves in distribution* with respect to the above topology.

First we state the geometric description of Caratheodory convergence of a sequence of compact sets.

Theorem 2.1 (Carathéodory kernel theorem)**.** *For a given point z_0 and a sequence of simply connected domains $(\Omega_n)_{n\in\mathbb{Z}_{>0}}$ such that $z_0 \in \Omega_n \neq \mathbb{C}$, define conformal and onto maps $\phi_n : \mathbb{D} \to \Omega_n$ such that $\phi_n(0) = z_0$ and $\phi_n'(0) > 0$. Then the uniform convergence of $(\phi_n)_{n\in\mathbb{Z}_{>0}}$ is equivalent to the kernel convergence of $\Omega_n \to \Omega$ as $n \to \infty$ with respect to z_0 in the sense that*

- *either $\Omega = \{z_0\}$ or Ω is a domain $\neq \mathbb{C}$ with $z_0 \in \Omega$ such that every $z \in \Omega$ has a neighborhood that lies in Ω_n for large enough n.*
- *for each $z \in \partial U$ there is a sequence z_n in $\partial\Omega_n$ such that $z_n \to z$ as $n \to \infty$.*

2.1 Continuity properties of $W \mapsto f$ and $W \mapsto g$

Lemma 2.2. *For each $\delta > 0$ and $T > 0$, there exists a constant $C(T, \delta)$ such that the following holds. Let $g_k(t, z), k = 1, 2$ be the solutions of Loewner equation $\frac{\partial}{\partial t} g(t, z) = \frac{2}{g(t,z) - W(t)}$ with continuous driving terms $(W_k(t))_{t \in [0,T]}, k = 1, 2$ respectively. Then*

$$g_1(T, z_1) - g_2(T, z_2) \leq C(T, \delta)(||W_1 - W_2||\infty, [0, T] + |z_1 - z_2|)$$

for any z_1, z_2 such that $Img_k(T, z_k) > \delta > 0, k = 1, 2$.

Proof. Fix $\delta > 0, T > 0$ and $z_k \in \mathbb{H}, k = 1, 2$, such that $\mathrm{Im} z_k > \delta, k = 1, 2$. Let $g_k(t, z), k = 1, 2$ be the solutions of reverse Loewner equation with continuous driving terms $(W_k(t))_{t \in [0,T]}$, which we also consider to be fixed. Write $\varphi(t) = g_1(t, z_1) - g_2(t, z_2)$. Then

$$\partial\varphi(t) = \zeta(t)(\varphi(t) - D(t)),$$

where $\zeta(t) = -\frac{2}{(g_1(t,z_1) - W_1(t))(g_2(t,z_2) - W_2(t))}$ and $D(t) = W_1(t) - W_2(t)$. We can also write $\partial_t(e^{-\int_0^t \zeta(s)ds}\Psi(t)) = -\zeta(t)e^{-\int_0^t \zeta(s)ds}D(t)$. Hence

$$\varphi(t) = e^{\int_0^t \zeta(s)ds}\varphi(0) - \int_0^t \zeta(u)e^{\int_u^t \zeta(s)ds}D(u)du.$$

We find using $|e^{\int_0^t \zeta(s)ds}| \leq e^{\int_0^t |\zeta(s)|ds}$ that

$$\begin{aligned} |\int_0^t \zeta(u)e^{\int_u^t \zeta(s)ds}D(u)du| &\leq \|D\|_{\infty,[0,T]} \int_0^t |\zeta(u)|e^{\int_u^t |\zeta(s)|ds}du \\ &= \|D\|_{\infty,[0,T]}(e^{\int_0^t |\zeta(s)|ds} - 1). \end{aligned}$$

By Cauchy-Schwarz inequality,

$$\int_0^t |\zeta(s)|ds \leq \sqrt{I_1 I_2},$$

where $I_k = \int_0^t \frac{2ds}{|g_k(s,z_k) - W_k(s)|^2}, k = 1, 2$. On the other hand,

$$\partial_t \log \mathrm{Im} g_k(t, z_k) = \frac{2dt}{|g_k(t, z_k) - W_k(t)|^2}$$

and therefore $I_k = \log \frac{\mathrm{Im} z_k}{\mathrm{Im} g_k(t,z_k)} \leq \log \frac{\mathrm{Im} z_k}{\max\{\delta, \sqrt{((\mathrm{Im} z_k)^2 - 4t)^+}\}}$. Here we used the upper bound $\partial_t \mathrm{Im} g_k(t, z_k) \geq -2(\mathrm{Im} g_k(t, z_k))^{-1}$ to derive an lower bound for $\mathrm{Im} g_k(t, z_k)$. Thus from the bounds above, we establish the bound

$$\begin{aligned} |\Psi(t)| &\leq \log \frac{\mathrm{Im} z_k}{\max\{\delta, \sqrt{((\mathrm{Im} z_k)^2 - 4t)^+}\}} |\Psi(0)| + (\log \frac{\mathrm{Im} z_k}{\max\{\delta, \sqrt{((\mathrm{Im} z_k)^2 - 4t)^+}\}} - 1)\|D\|_{\infty,[0,T]} \\ &\leq C(T, \delta)(|z_1 - z_2| + \|W_1 - W_2\|_{\infty,[0,T]}) \end{aligned}$$

□

Let f to be the inverse of g defined on $[0,T]\times\mathbb{H}$ for any $T>0$. The above lemma gives the continuous dependency of solutions of Loewner equations on the driving term (i.e. continuity of the mappings $W\mapsto f$ and $W\mapsto g$).

Proposition 2.3. *The mappings $W\mapsto f$ is continuous with respect to the convergence in the Caratheodory sense. More specifically, for any compact $G\subset\mathbb{H}$, there exists a constant $C>0$ such that if $f_k, k=1,2$ are two (inverse) Loewner chains, then*

$$\|f_1-f_2\|_{\infty,[0,T]\times G}\leq C\|W_1-W_2\|_{\infty,[0,T]}$$

Proposition 2.4. *Let K_0 be a hull and $G\subset\mathbb{H}\setminus K_0$ be a compact set. Then there exists a constant $C>0$ such that if $g_k, k=1,2$ are two Loewner chains such that $K_k(T)\subset K_0$ for $k=1,2$, then*

$$\|g_1-g_2\|_{\infty,[0,T]\times G}\leq C\|W_1-W_2\|_{\infty,[0,T]}.$$

2.2 Continuity of $\gamma\mapsto g$

The mapping from $(\gamma(t))_{t\in[0,T']}$ to $g(T',\cdot)$ for any fixed T' is continuous by Theorem 2.1 with respect to the convergence in the Caratheodory sense. Let us extend this to continuity of the mapping $(\gamma(t))_{t\in[0,T']}$ to $(g(t,\cdot))_{t\in[0,T']}$ by an argument using compactness.

Proposition 2.5. *The mapping $\gamma\mapsto g$ is continuous with respect to the convergence in Caratheodory sense.*

Proof. Assume that there are $\epsilon>0,\gamma,\gamma_n,t_n\in[0,T']$ and $z_n\in G$, where G is any compact set in $\mathbb{H}$ such that γ_n converges to γ uniformly on $[0,T']$ but

$$\liminf_n |g_n(t_n,z_n)-g(t_n,z_n)|>\epsilon.$$

By compactness we can suppose that t_n,z_n converge to $t\in[0,T]'$ and z respectively as $n\to\infty$. By local uniform convergence of curves and absolute continuity of γ in a compact interval, $(\gamma_n(t))_{t\in[0,t_n]}$ converges to $(\gamma(t))_{t\in[0,t]}$ with respect to Hausdorff topology. Specially, $g_n(t_n,\cdot)$ converges to $g(t,\cdot)$ and $g(t_n,\cdot)$ converges to $g(t,\cdot)$ as $n\to\infty$. Therefore $\lim_{n\to\infty}|g_n(t_n,z_n)-g(t_n,z_n)|=0$ (along a subsequence), which leads to a contradiction. □

Combined with the continuous dependency of the solution of Loewner equation , we have

Corollary 2.6. *Suppose that γ_n is a sequence of curves and W_n are their corresponding driving terms. If γ_n tends to γ and W_n tends to W uniformly on $[0,T]$ as $n\to\infty$, then γ is driven by W, that is, the Loewner chain g of γ satisfies the Loewner equation with driving term W.*

Proof. Let g_n be the Loewner chain corresponding to γ_n and W_n, both the convergence of γ_n and W_n implies the convergence of g_n in the Caratheodory sense. Moreover, the limit is the Loewner chain corresponding to both γ and W. □

2.3 Continuity of $\gamma \mapsto W$

Theorem 2.7. *Let $X \subset C([0,T),\mathbb{C})$ be the set of curves in the upper half-plane that generate a Loewner chain. The mapping $(\gamma(t))_{t\in[0,T)} \mapsto (W_t)_{t\in[0,T)}$ from X to $C([0,T),\mathbb{R})$ is continuous. However, it is not uniformly continuous.*

Proof. It is sufficient to verify that for any $T' \in (0,T)$ and a sequence γ_n that converges to γ in the uniform norm on $[0,T']$, the corresponding driving terms W_n of γ_n converges to a limit W in the uniform norm on $[0,T']$ and γ is driven by W. Since γ_n tends to γ, the family γ_n is equicontinuous on $[0,T']$, therefore the family W_n is equicontinuous on $[0,T']$. It is also uniformly bounded and we can apply Arzela-Ascoli theorem to find a converging subsequence, whose limit is denoted by W. By Corollary 2.6, W is the driving term of γ on $[0,T']$. By taking a sequence T'_n that increases to T, we can suppose that convergence of W_n. To prove the second claim, let γ_n be the broken line $0, i, i+\frac{1}{n}, \frac{1}{2}i+\frac{1}{n}$ parametrized with the half-plane capacity and let $\tilde{\gamma}_n(t) = -\overline{\gamma}_n(t)$, i.e. the reflection with respect to the $y-$axis. Let the corresponding driving terms be W_n and $\tilde{W}_n$.

There exists $c > 0$ and T' such that the capacity of γ_n and $\tilde{\gamma}_n$ is greater than T' and $W_n(T') > c$ and $\tilde{W}_n(T') < -c$, otherwise γ_n would be sufficiently close to the $y-$axis (under half-plane capacity parametrization). Hence $\|\gamma_n - \tilde{\gamma}_n\|_{\infty,[0,T']} = \frac{2}{n}$ and $\|W_n - \tilde{W}_n\|_{\infty,[0,T']} > 2c$. This shows that the map $(\gamma(t))_{t\in[0,T)} \mapsto (W_t)_{t\in[0,T)}$ is not uniformly continuous. □

3 Convergence of interfaces in site percolation model

In this section, we establish the convergence of random curves to SLEs in a specific example case of site percolation model on triangular lattice. Most of the proofs are based on estimates for this discrete model.

3.1 Definitions

Let $\mathbb{L}_{\text{tri}}$ be the triangular lattice, which is an infinite graph such that its faces tile the plane and the tiling if formed from translates of triangles. Let $G = (V,E)$ be a subgraph of $\mathbb{L}_{\text{tri}}$. Let p be a parameter and consider a family of random variables, one for each site of G, taking values in the set {open,closed}. We say that a site $v \in V$ is open or closed depending on the value that random variables takes. To define the law precisely,

$$\mathbb{P}[v \text{ is open}] = p, \qquad \mathbb{P}[v \text{ is closed}] = 1-p$$

for each $v \in V$ and that these random variables are independent. A probability space with percolation model can be constructed as a product space $\{\text{open,closed}\}^V$ with a product measure of laws of Bernoulli random variables. Denote by $\mathbb{P}_p$ this probability measure to stress the dependency on the parameter.

The model is called the *site percolation* on G. The closed sites model a random media and the open sites form cavities through which a fluid can flow.

Fundamental questions in percolation model are therefore connectivity properties of the subgraph of open sites. For small values of p, we expect that it is rare to see large clusters of open sites and that for large p, almost surely there exists an open infinite cluster in an infinite graph.

We wish to define an *exploration process* that explores a part of a percolation configuration. This process will be a simple path on the dual graph of $\mathbb{L}_{\text{tri}}$, which is a hexagonal lattice, that keeps open sites of $\mathbb{L}_{\text{tri}}$ on its left and closed sites on its right. The type of domain on which we will define the exploration is a simply-connected domain such that $\partial\Omega^\delta \neq \emptyset$ is a path on $\mathbb{L}_{\text{hex}}$ for each lattice mesh $\delta > 0$. Let $\tilde{V}$ be the set of sites on $\delta\mathbb{L}_{\text{tri}}$ that lie inside Ω^δ and let V_1 be the set of those $v \in \tilde{V}$ such that the hexagon corresponding to v has at least on common edge with $\partial\Omega^\delta$. The set V_1 is connected and we can interpret it as a unique non-self-crossing closed path π (with counterclockwise orieatation). Let $V = \tilde{V} \setminus V_1$. The set V is now the one where we are going to put the percolation configuration. Let a and b be two distinct vertices on $\partial\Omega_\delta$.

Now if we have defined a percolation configuration on V. To define the exploration process, pick for both a and b one of the neighboring edges that cross π and denote the two halves of π as (ab) and (ba). If the boundary condition is imposed in the way that (ab) is closed and (ba) is open, then the *exploration process* is defined to be the unique simple path from a to b on the hexagonal lattice such that all the hexagons on its right are closed and all on its left are open.

3.2 Cardy's formula and conformal invariance

Consider a Jordan domain Ω with four points a, b, c and d on the boundary. The $5-$tuple (Ω, a, b, c, d) with four points a, b, c, d on $\partial\Omega$ is called a *topological rectangle*. The arc (ab) denotes the counterclockwise arc of $\partial\Omega$ from a to b. We extend this definition to subdomains of the hexagonal lattice $\delta\mathbb{T}$ of mesh size $\delta > 0$. In this case, we use $(\Omega^\delta, a^\delta, b^\delta, c^\delta, d^\delta)$ to denote the topological rectangle. In 1992, Langlands, Pouliot and Saint-Aubin observed numerically that the probability $\mathcal{C}^\delta(\Omega, a, b, c, d)$ of having a path of adjacent open sites between the boundary arcs (ab) and (cd) of a finite piece of a lattice of mesh size δ converges as $\delta \to 0$ towards a limit which is invariant under conformal map, that is, it is the same for (Ω, a, b, c, d) and $(\Omega', a', b', c', d')$ if they are images of each other by a conformal map. Notice that the existence of such a crossing property can be expressed in terms of properties of a well-chosen interface. Cardy then proposed an explicit formula for the limit in the same year. In 2001, Smirnov proved Cardy's formula rigorously for critical site percolation on triangular lattice, hence rigorously providing a concrete example of a conformally invariant property of the model. It implies the conformal invariance of crossing probabilities of the scaling limit in the percolation model.

Set $j = e^{2\pi i/3}$. Let $\mathbf{T}$ be the equilateral triangle with vertices $1, j$ and j^2.

Theorem 3.1. *(Smirnov) Consider a family of topological rectangles $(\Omega^\delta, a^\delta, b^\delta, c^\delta, d^\delta)_{\delta>0}$ converging to (Ω, a, b, c, d) in the Caratheodory sense. Then*

$$\lim_{\delta\to 0} \mathbb{P}_{\frac{1}{2}}[(a^\delta b^\delta) \overset{\Omega^\delta}{\longleftrightarrow} (c^\delta d^\delta)] = x.$$

where d is mapped to $x + j^2(1-x)$ by the conformal map from Ω to $\mathbf{T}$ mapping a to 1, b to j and c to j^2.

Before we dive into the proof of Cardy's formula, let us first highlight the connection between site percolation on triangular lattice and the so-called loop $O(1)$ model on hexagonal lattice. A *loop configuration* on a domain Ω is a collection of non intersecting loops. Equivalently, it is an even subgraph of Ω, i.e. a graph with even degree everywhere. We denote the set of loop configurations by $\mathcal{E}_\Omega^\emptyset$. Each percolation configuration can be seen as a loop configuration by considering all the edges separating two faces with different colors. We can therefore associate a percolation configuration in Ω with a loop configuration in $\mathcal{E}_\Omega^\emptyset$. We deduce that $|\mathcal{E}_\Omega^\emptyset| = 2^N$, where N is the number of faces in Ω.

We extend the definition of loop configurations to include paths from mid-edges to mid-edges. More precisely, let $\mathcal{E}_\Omega^{uv,rs}$ be the set of configurations with loops together with one self-avoiding path from u to v and one self-avoiding path from r to s avoiding the loops.

Fix z to be a mid-edge of Ω and three mid-edges a, b, c on $\partial\Omega$. Define $F_a(z) := 2^{-N}|\mathcal{E}_\Omega^{az,bc}|$ and similarly F_b and F_c. Define the *parafermionic observable with three marked points* in the domain Ω

$$F(z) = F_a(z) + jF_b(z) + j^2F_c(z).$$

Remark 3.2. We have that $F_a(z) + F_b(z) + F_c(z) = 1$ since

$$|\mathcal{E}_\Omega^{az,bc}| + |\mathcal{E}_\Omega^{bz,ca}| + |\mathcal{E}_\Omega^{cz,ab}| = |\mathcal{E}_\Omega^\emptyset| = 2^N.$$

The first identity is given by the fact that for any $\eta_0 \in \mathcal{E}_\Omega^{az,bc} \cup \mathcal{E}_\Omega^{bz,ac} \cup \mathcal{E}_\Omega^{cz,ab}$ to $\mathcal{E}_\Omega^\emptyset$.

Lemma 3.3. *For any vertex v in Ω, if p, q and r denote the three mid-edges around v, indexed in counterclockwise order,*

$$(p-v)F(p) + (q-v)F(q) + (r-v)F(r) = 0. \tag{3.1}$$

Proof. For any mid-edge z in Ω, fix a path γ from b to c and let $\mathcal{E}_\Omega^{az,bc}[\gamma]$ and $\mathcal{E}_\Omega^{bc}[\gamma]$ be the sets of loop configurations in $\mathcal{E}_\Omega^{az,bc}$ and $\mathcal{E}_\Omega^{bc}$ with path γ from b to c. We say that

$$|\mathbb{E}_\Omega^{az,bc}[\gamma]| = \begin{cases} 0 & \text{if } z \text{ and } a \text{ are separated by } \gamma \\ |\mathbb{E}_\Omega^{bc}[\gamma]| & \text{otherwise.} \end{cases}$$

In fact, the first line is trivial and the second simply follows from the fact that for any path φ from a to z in the complement of γ, the map $\eta \mapsto \eta\Delta\varphi$ is an involution from $\mathcal{E}_\Omega^{az,bc}[\gamma]$ to $\mathcal{E}_\Omega^{bc}[\gamma]$. Together with the observation that if γ does not go through v, we have a one-to-one correspondence between $F_a(p)$, $F_a(q)$ and $F_a(r)$ by adding or deleting mid-edges around v, we deduce that

$$(p-v)F_a(p)+(q-v)F_a(q)+(r-v)F_a(r) = (p-v)T_a(p)+(q-v)T_a(q)+(r-v)T_a(r),$$

where $T_a(p)$ is equal to 2^{-N} times the number of configurations in $\mathcal{E}^{ap,bc}$ for which the path from b to c visits v. Similarly,

$$\begin{aligned}
&(p-v)jF_b(p) + (q-v)jF_b(q) + (r-v)jF_b(r) \\
&= (p-v)jT_b(p) + (q-v)jT_b(q) + (r-v)jT_b(r) \\
&(p-v)j^2F_c(p) + (q-v)j^2F_c(q) + (r-v)j^2F_c(r) \\
&= (p-v)j^2T_c(p) + (q-v)j^2T_c(q) + (r-v)j^2T_c(r)
\end{aligned}$$

Any configuration in $\mathcal{E}^{ap,bc}$ is naturally in bijection with configurations in $\mathcal{E}^{bq,ca}$ and $\mathcal{E}^{cr,ab}$ with path visiting v by switching the two mid-edges incident to v. From that, we deduce $(p-v)T_a(p)+(q-v)jT_b(q)+(r-v)j^2T_c(r)=0$. Then the result follows by summing the equations above. □

Remark 3.4. The result of the above lemma can be understood in the following way. Coefficients in (3.1) are three cubic roots as unity multiplied by $p-v$, so the left-hand side can be seen as a discrete integral along an elementary contour on the dual lattice in the following sense. For a closed path $\mathbf{z}=(z_i)_{i\leq n}$ of vertices in the triangular lattice $\mathbb{T}$, define the discrete integral of a function F on mid-edges by

$$\sum_{i=0}^{n-1} F(\frac{z_i+z_{i+1}}{2})(z_{i+1}-z_i)$$

The discrete integral along any closed path vanishes by decomposing a closed contour into a sum of elementary triangles.

Remark 3.5. $F_a(z)$ is equal to 2^{-N} times the number of configurations in $\mathcal{E}^{bc}_\Omega$ for which the path from b to c does not disconnect a from z, which is the probability of event γ does not disconnect z from a.

Proof of Cardy's formula. Consider the domain Ω^δ and three mid-edges $a^\delta, b^\delta, c^\delta$ on $\partial\Omega^\delta$. Define $F_a^\delta, F_b^\delta, F_c^\delta$ and F^δ as before. Extend F^δ to the interior of the polygon $\partial\Omega^\delta$ by convexly interpolating between mid-edges. F^δ maps Ω^δ to $\mathbb{T}$, where $\mathbf{T}$ is the equilateral triangle with vertices $1, j$ and j^2. Furthermore, $F_b^\delta(z)$ (resp. $F_c^\delta(z)$ and $F_a^\delta(z)$) is equal to 0 on $(c^\delta a^\delta)$ (resp. $(a^\delta b^\delta)$ and $(b^\delta c^\delta)$). Consequently $F^\delta(z)\in[j^2,1]$ (resp. $[1,j]$ and $[j,j^2]$) for $z\in(c^\delta a^\delta)$ (resp. $(a^\delta b^\delta)$ and $(b^\delta c^\delta)$). In other words, $\partial\Omega^\delta$ is mapped to $\partial\mathbf{T}$.

By Remark 3.5, for $z\in(c^\delta a^\delta)$, let d^δ be the nearest mid-edge on $\partial\Omega^\delta$, we have that

$$F_a^\delta(d^\delta)=\mathbb{P}[(a^\delta b^\delta)\overset{\Omega^\delta}{\longleftrightarrow}(c^\delta d^\delta)]=1-F_c^\delta(z)$$

In fact, the first identity comes from the fact that $\mathcal{E}^{ad,bc}_\Omega$ is in bijection with percolation configurations having a crossing from $(a^\delta b^\delta)$ to $(c^\delta d)$, and the second from Remark 3.2. Using Lemma 3.6 below, we may consider a subsequential limit F of $(F^\delta)_{\delta>0}$. By Lemma 3.1 and Morera's theorem, F is holomorphic. The previous observation gives that F is the holomorphic map from $(\mathbf{\Omega}, a, b, c)$ onto $(\mathbf{T}, 1, j, j^2)$ mapping $\partial\mathbf{\Omega}$ in an injective way. We deduce that F is the conformal map from $\mathbf{\Omega}$ to $\mathbf{T}$ sending a, b and c to $1, j$ and j^2 respectively, and

$$F(z)=x+j^2(1-x)=\lim_{\delta\to0}[F_a^\delta(d^\delta)+j^2(1-F_a^\delta(d^\delta))],$$

then the result follows readily by the conformal invariance of holomorphic function with certain Dirichlet boundary condition. □

Lemma 3.6. *The family $(F^\delta)_{\delta>0}$ is precompact for the uniform convergence on any compact subset of $\overline{\Omega}\setminus\{a,b,c\}$.*

Proof. By Russo-Seynour-Welsh theorem,

$$|F_a^\delta(z)-F_a^\delta(z')|\leq\mathbb{P}_{1/2}[\gamma \text{ disconnects exactly one of } z \text{ or } z' \text{ from } a]\leq C|z-z'|^\alpha.$$

This implies that F_a^δ is equicontinuous and the result follows by Arzela-Ascoli.. □

Corollary 3.7. *By the conformal invariance for the crossing probability of scaling limit of percolation configuration and the explicit form of Schwarz–Christoffel map from half-plane to triangle.*

3.3 Characterization of percolation interface scaling limit

We aim to define a curve γ in a bounded simply connected domain Ω from a boundary point a to another boundary point b. We need further a third boundary point c and a point z inside Ω to characterize the law of γ. We discretzie Ω by choosing a sequence $(\Omega_{\delta_n}, a_{\delta_n}, b_{\delta_n})$, where δ_n is the lattice mesh tending to 0 as n tends to ∞ and Ω_{δ_n} is a bounded, simply connected domain such that $\partial\Omega_{\delta_n}$ is a lattice path on $\delta_n\mathbb{L}_{\text{hex}}$ and $a_{\delta_n}, b_{\delta_n}$ are boundary points which are assumed to be sites on $\delta_n\mathbb{L}_{\text{hex}}$. We will drop n in the following if no confusion arises. Denote by $\Pi_\epsilon(\Omega, (a,b), (cd))$ the probability that there is an open cluster (of faces) joining the arc (ab) to the arc (cd) on the boundary. It is known that there is a limit $\Pi := \lim_{\delta\to 0}\Pi_\delta$, which is conformally invariant (depends only on the conformal modulus of the configuration Ω, a, b, c, d), and satisfies Cardy's formula in half-plane:

$$\Pi(\mathbb{C}_+, (1-u,1), (\infty, 0)) = \frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})\Gamma(\frac{4}{3})} u^{\frac{1}{3}} {}_2F_1(\frac{1}{3}, \frac{2}{3}; \frac{4}{3}; u) =: F(u),$$

where ${}_2F_1$ is the hypergeometric function and the explicit form is given by the Schwarz-Christoffel map.

Recall that we can draw an interface on the lattice given Dobrushin boundary condition. Parameterize the interface γ_δ by half-plane capacity, and add the part $\gamma_\delta[0,t]$ to the domain boundary. Note that it has open vertices on one side (arc $(\gamma_\delta a)$) and closed on another (arc $(a\gamma_\delta(t))$). Add two points x, y on the boundary, making Ω a topological rectangle such that we can consider the crossing probability $\Pi_\delta(\Omega, (a,x), (b,y))$. Then any open crossing from the arc (by) to the arc (ax) inside Ω is either disjoint from $\gamma_\delta[0,t]$, or hits its "open" arc $(\gamma_\delta(t)a)$. In either case, it produces an open crossing from the arc (by) to the arc $(\gamma_\delta(t)x)$ inside $\Omega \setminus \gamma_\delta[0,t]$, and the converse also holds. Therefore for every realization of $\gamma_\delta[0,t]$ the crossing probability conditioned on $\gamma_\delta[0,t]$ coincides with crossing probability in the slit domain $\Omega \setminus \gamma_\delta[0,t]$:

$$\Pi_\delta(\Omega, (a,x), (b,y) | \gamma_\delta[0,t]) = \Pi_\delta(\Omega \setminus \gamma_\delta[0,t], (\gamma_\delta(t), x), (b,y)),$$

- an analogue of the domain Markov property. Therefore, stopping the curve at times $t < s$ and we can write the total probability theorem for every realization of $\gamma_\delta[0,t]$

$$\Pi_\delta(\Omega \setminus \gamma_\delta[0,t], (\gamma_\delta(t), x), (b,y)) \tag{3.2}$$
$$= \mathbb{E}_{\gamma_\delta[t,s]}(\Pi_\delta(\Omega \setminus \gamma_\delta[0,s], (\gamma_\delta(s), x), (b,y)) | \gamma_\delta[0,t]). \tag{3.3}$$

Some a priori estimates show that the identity above (3.2) holds for the (subsequential) scaling limit μ (strictly speaking there is an error term in case the interface touches the arcs (ax) or (ya) before time s, but it decays very fast as we move x and y away from a). Therefore we can write (3.2) for the limit curve γ with Loewner parametrization as

$$\Pi(\Omega \setminus \gamma[0,t], (\gamma(t), x), (b,y))$$
$$= \mathbb{E}_{\gamma[t,s]}(\Pi(\Omega \setminus \gamma[0,s], (\gamma(s)x), (b,y)) | \gamma[0,t]).$$

for almost every realization of $\gamma[0,t]$. Plugging in exact values of conformal images, the formula above implies that

$$(\frac{g_t(x)-W_t}{g_t(x)-g_t(y)}) = \mathbb{E}_{\gamma[t,s]}(F(\frac{g_s(x)-W_s}{g_s(x)-g_s(y)})|\gamma[0,t]), \tag{3.4}$$

since

$$\begin{aligned}\Pi(\mathbb{H}\setminus\gamma[0,t],(\gamma(t),x),(\infty,y)) &= \Pi(\mathbb{H},(-\frac{g_t(y)-W_t}{g_t(x)-g_t(y)},1),(\infty,0))\\ &= F(\frac{g_t(x)-W_t}{g_t(x)-g_t(y)}).\end{aligned}$$

Therefore $(F(\frac{g_t(x)-W_t}{g_t(x)-g_t(y)}))_{t\geq 0}$ is a martingale with respect to the filtration generated by $\sigma(\gamma[0,t])$. Since we want to extract the information about W_t, we fix the ratio $\frac{x}{x-y} := \frac{1}{3}$ (y=-2x) and let x tend to infinity. Using the normalization $g_t(z) = z + \frac{2t}{z} + O(\frac{1}{z^2})$ at infinity, writing Taylor expansion for F, and plugging in values of derivatives of F at $\frac{1}{3}$, we obtain the following Ito expansion for the martingale:

$$\begin{aligned}F(\frac{g_t(x)-W_t}{g_t(x)-g_t(y)}) &= F(\frac{x-W_t+\frac{2t}{x}+O(\frac{1}{x^2})}{(x+\frac{2t}{x}+O(\frac{1}{x^2}))-(-2x+\frac{2t}{-2x})+O(\frac{1}{x^2})})\\ &= F(\frac{1}{3}-\frac{W_t}{3}\frac{1}{x}+\frac{3}{t}\frac{1}{x^2}+O(\frac{1}{x^3}))\\ &= F(\frac{1}{3})-\frac{W_t}{3}F'(\frac{1}{3})\frac{1}{x}+(\frac{t}{3}F'(\frac{1}{3})+\frac{W_t^2}{3^2\cdot 2}F''(\frac{1}{3}))\frac{1}{x^2}+O(\frac{1}{x^3})\\ &= A-\frac{B}{x}W_t-\frac{C}{x^2}(W_t^2-6t)+O(\frac{1}{x^3})\end{aligned}$$

Plugging the above expression in (3.4) and compare the coefficients in the series, we conclude that

$$\mathbb{E}[W_s|W[0,t]]=0, \quad \mathbb{E}[W_s^2-6s|W[0,t]]=W_t^2-6t.$$

Thus W_t is a continuous (by Loewner's theorem) process such that both

$$W_t \text{ and } W_t^2-6t$$

are martingales so by Levy's characterization of the Brownian motion $W_t = \sqrt{6}B_t$. Therefore SLE(6) is the scaling limit of the critical percolation interface.

4 CLE percolations

Conformal loop ensembles are random collections of loops in a simply connected domain, whose laws are characterized by a natural conformal invariance property. The set of points not surrounded by any loop is a canonical random connected fractal set - a random and conformally invariant analog of the Sierpinski carpet or triangle.

aim: a direct relationship between the conformal loop ensembles with simple loops (CLE_κ for κ in $(\frac{8}{3},4)$, whose loops are Schramm's SLE_κ-type curves) and

the corresponding conformal loop ensembles with non-simple loopf ($\mathrm{CLE}_{\kappa'}$ with $\kappa' := 16/\kappa \in (4,6)$, whose loops are $\mathrm{SLE}_{\kappa'}$-type curves.) This correspondence is the continuum analog of the Edwards-Sokal coupling between the $q-$state Potts model and the associated FK random cluster model, and its generalization to non-integer q.

4.1 Construction

A simple loop is a subset of $\mathbb{C}$ which is homeomorphic to the circle S^1. Equivalently, we identify a simple loop with a homeomorphism from S^1 to a subset of $\mathbb{C}$, modulo reparametrization. (In other words, we will identify two homeomorphisms if one of them is obtained by a bijective reparametrization of the other one; note that our loops are not oriented.)

Let $\mathcal{C}$ be the set of homeomorphisms from S^1 to subsets of $\mathbb{C}$ and let $\overline{\mathcal{C}}$ denote the closure of $\mathcal{C}$ with respect to the $?C^\infty$ metric. A **quasisimple loop** is an element of $\overline{\mathcal{C}}$, modulo monotone reparametrization. A quasisimple loop may intersect itself, but it cannot "cross" itself transversely. For simple loops, the winding number around a point $z \in \mathbb{C} \setminus \eta(S^1)$ is equal to 0 or ± 1, and it remains true for quasisimple loops since this property holds under uniform limits provided that the limit fails to intersect z.

We define the distance between quasisimple loops L_1 and L_2 by

$$d(L_1, L_2) = \inf ||\zeta_1 - \zeta_2||_\infty,$$

where the infimum is taken over pairs $\zeta_1 : S^1 \to \mathbb{R}^2$ and $\zeta_2 : S^1 \to \mathbb{R}^2$ of parametrizations of L_1 and L_2.

Let D be a bounded domain. Then the set Ω_D of all discrete subsets of the set of quasisimple loops in $\overline{D}$ is a metric space under the Hausdorff metric induced by the metric $d(\cdot,\cdot)$ described above. Denote $\mathcal{F}_D$ the Borel $\sigma-$algebra on Ω_D. Most natural functions of Ω_D can be shown to be $\mathcal{F}_D$ measurable, such as he number of loops completely surrounding a fixed disc, or the number of loops intersecting two disjoint open sets, or the event that the outermost loop surrounding a fixed point is a simple loop.

The most natural definition of a random loop ensemble is a random variable whose law is a probability measure on $(\Omega_D, \mathcal{F}_D)$. Unfortunately, when defining $\mathrm{CLE}(\kappa)$ in arbitrary domains, it may not be enough to consider quasisimple loops. For example, consider the case that D is a non-Jordan domain such as the square $(0,2)\times(0,2)$ minus the set $\{n^{-1} : n \in \mathbb{Z}_+\}\times(0,1)$, on which any loop in the closure of D which is space filling in D cannot be a continuous closed curve since its y coordinates must oscillate between 0 and 1 infinitely many times. Even when the original domain is a Jordan domain, one may worry a priori that as we construct loops through an exploration process we may create domains which are no longer Jordan domains. It will therefore be convenient to slightly expand our definition of loops.

A **pinned loop** in $\mathbb{H}$ is a quasisimple loop L in $\overline{\mathbb{H}}$ which intersects the origin at 0 and has the property that if $\eta : S^1 \to \overline{\mathbb{H}}$ is a parameterization of the loop then $\eta^{-1}\mathbb{H}$ is dense. Roughly speaking, we now wish to define a "conformal loop" in $\mathbb{C}$ to be the image of a pinned loop under a conformal map ϕ from $\mathbb{H}$ to another domain $D \subset \mathbb{C}$. Note that the parametrization $\phi \circ \eta$ of this image is well-defined on an open dense subset U_0 of S^1 but may not extend continuously

to all of S^1. If U is the set of all points s in S^1 for which $\phi \circ \eta$ can be extended to a continuous function in a neighborhood of s, then it is clear that U is the largest open set to which $\phi \circ \eta$ can be continuously extended. Formally, a **conformal loop** is a map ζ from a dense open subset U of S_1 into $\mathbb{C}$ modulo monotone reparametrization such that

1. There exists a conformal map ψ from $\mathbb{H}$ to a superset of $\zeta(U)$ with the property that $\phi^{-1} \circ \zeta$ is the restriction to U of a function $\eta \in \overline{\mathcal{C}}$ with $\eta^{-1}\mathbb{H} \subset U$

2. The set U is maximal, i.e. that there exists no open proper subset U' of U such that ζ can be extended to a continuous function of U'.

If $L = (\zeta, U)$ is a conformal loop, we will sometimes abuse notation slightly and use L to denote the corresponding set $\overline{\zeta(U)} \subset \mathbb{C}$ of points on the loop. A point z in $\mathbb{C} \setminus L$ is ...

initial approach to defining CLE(κ) will be to define a coupling of $\text{SLE}^{\mu}_{\beta}(\kappa; \kappa - 6)$ processes called a continuum exploration tree and to show that this process almost surely determines a countable collection of *conformal loops*. We will ... in the following paragraph.

Fix initial values $W_0 = O_0 = 1 \in \partial\mathbb{D}$. An $\text{SLE}^{\mu}_{\beta}(\kappa; \kappa - 6)$ targeted at $a_1 \in \mathbb{D}$ and an $\text{SLE}^{\mu}_{\beta}(\kappa; \kappa - 6)$ targeted at another point $a_2 \in \mathbb{D}$ can be coupled in such a way that the corresponding growth processes $K_t^{a_1}$ and $K_t^{a_2}$ agree (after a time change) up to the first time t at which a_1 and a_2 are separated, i.e. $a_1 \in K_t^{a_2}$ and $a_2 \in K_t^{a_1}$ and evolve independently of one another after that time. In other words, we may construct this coupling by first sampling an $\text{SLE}^{\mu}_{\beta}(\kappa; \kappa - 6)$ process $K_t^{a_1}$ targeted at a_1, and then sampling an $\text{SLE}^{\mu}_{\beta}(\kappa; \kappa - 6)$ process $K_t^{a_2}$ targeted at a_2 conditioned on it agreeing with the first sample (up to time change) until the first time that a_1 and a_2 are separated. Because we have not proved that radial $\text{SLE}^{\mu}_{\beta}(\kappa; \kappa - 6)$ is a continuous path, we say it is a growth process.

We can define a similar coupling inductively for sets $a_1, \cdots, a_k$ with $k > 2$ as follows. First we choose the growth processes $K_t^{a_j}$ for all $j < k$ and then conditioned on these processes, we choose $K_j^{a_k}$ conditioned on having it agree with each $K_t^{a_j}$ (after a time-change) up until the first time a_k is separated from a_j. We can interpret this as an $\text{SLE}^{\mu}_{\beta}(\kappa; \rho)$ process that "branches" at each of the finitely many times that a pair a_i and a_j becomes separated for the first time.

Correspondingly, we are able to repeat this procedure infinitely many times - for some countable dense sequence $a_1, a_2, a_3, \cdots$ of points in $\mathbb{D}$ - to obtain a coupling of $\text{SLE}^{\mu}_{\beta}(\kappa; \kappa - 6)$ growth process K^{a_i} from 1 to i. (Formally, if Ω^{a_i} is the space of continuous driving parameters W for growth processes targeted at a_i and $\mathcal{F}^{a_i}$ is the smallest $\sigma-$algebra which makes W_t measurable for each fixed t, then this coupling is a random variable in $\prod \Omega^{a_i}$ whose law is measurable with respect to the product $\sigma-$algebra generated by $\vee \mathcal{F}^{a_i}$.) Note that for every $i \neq j$, the $K_t^{a_i}$ and $K_t^{a_j}$ agree almost surely (after a time change) until the first time t that a_i and a_j are separated, after which they evolve independently. We may view this collection of $K_t^{a_j}$ as a single random growth process with branches at each of the countably many times that some a_i becomes separated from some a_j for the first time.

Given a family of growth process $\{K_t^{a_i}\}_{i\geq 1}$ chosen from such a coupling, we may almost surely uniquely (up to time change) define for each point $z \in \overline{\mathbb{D}}$, a growth process K_t^z such that for each a_i, the processes $K_t^{a_i}$ and K_t^z agree (after a time change) until the first time a_i and z are separated. ? The joint law of processes K_t^z (defined for all $z \in \overline{\mathbb{D}}$) is independent of our choice of countable dense set $\{a_i\}$.

The complete collection of processes K_t^z, $z \in \overline{\mathbb{D}}$, is called **branching** SLE($\kappa, \kappa-6$), or the **continuum exploration tree.** We will construct a family of conformal loops from a continuum exploration tree.

Fix $z \in \mathbb{D}$. Write $\theta_t = \text{arg}W_t - \text{arg}O_t$, where the branch of arg is chosen so that θ_t is continuous in t and $\theta_0 = 0$. Using the definition for SLE($\kappa, \kappa-6$), we see that the difference $\theta_t = \text{arg}W_t - \text{arg}O_t$ is a real-valued process that evolves according to the diffusion

$$d\theta_t = \frac{\kappa - 4}{2} \cot(\frac{\theta_t}{2})dt + \sqrt{\kappa}dB_t$$

is between those times t for which θ_t is an integer multiple of 2π.

We will now construct a sequence L_j^z of nested conformal loops surrounding z, such that each L_{j+1}^z is surrounded by L_j^z. First we call a time t a **loop closure time** if it is the first time that θ_t hits a particular integer multiple of 2π after the last previous time s that it hit a different multiple. ?2π or π Denote by t_j^z the jth such time and by s_j^z the corresponding value of s. Denote by A_j^z the component of $\mathbb{D} \setminus K_t$ that contains z.

The continuous path γ_j^z begins at 0 (at time s_j^z) and ends when it reaches the first point $\gamma_j^z(t_j^z)$, where t_j^z is the first time that $\phi(z)$ is separated from the origin. Conditioned on this γ_j^z, the law of the remainder of the loop $\phi(L_j^z)$ is that of an SLE(κ) in $\mathbb{H} \setminus \gamma_j^z([s_j^z, t_j^z])$ started at $\gamma_j^z(t_j^z)$ and ending at 0. Then extend $\gamma_j^{z'}$ in the same way for a different value z'. The union of all such extensions is the complete loop $\phi(L_j^z)$, a quasisimple closed loop in $\overline{\mathbb{H}}$ which begins and ends at 0.

We will see that the process K_t between times s_j^z and t_j^z traces part of a conformal loop. To begin to describe that loop, let $\phi : \mathbb{D} \setminus K_{s_j^z} \to \mathbb{H}$ be the composition of $g_{s_j^z}$ with a conformal map from $\mathbb{D}$ to $\mathbb{H}$ that sends $W_{s_j^z}$ to zero.

We now claim that for all $t \in [s_j^z, t_j^z]$, the set $\mathbb{H} \setminus \phi K_t$ can be written, almost surely, as the unbounded component of $\zeta([s_j^z, t])$ where $\zeta : [s_j^z, t_j^z] \to \overline{\mathbb{H}}$ is a continuous path which extends continuously to its endpoints. To see this, note that the corresponding SLE(κ) in a Jordan domain is almost surely continuous. For each fixed $s \in (s_j, t_j)$, the law of the evolution of K_t after time s is that of an SLE(κ). If h_s is a conformal map from $\mathbb{H} \setminus \phi(K_s)$ to $\mathbb{H}$ (with the hydrodynamic normalization at infinity), then this implies that the growth of $h_s(\phi(K_t))$ is indeed given by a continuous function ζ_s on $[s, t_j]$. We can then set $\zeta(t) = h_s^{-1}(\zeta_s(t))$

4.2 Markovian construction

A **simple loop** in the complex plane is the image of the unit circle in the plane under a continuous injective map, modulo reparametrization. Note that a simple loop γ separates the plane into two connected components that we call its interior int(γ). In this section, we will denote by $\mathcal{L}$ the space of simple loops

and use the $\sigma-$algebra Σ generated by all the events of the type $\{O \subset \text{int}(\gamma)\}$ when O takes over the set of open sets in the plane. Note that this $\sigma-$field is also generated by the events of the type $\{x \in \text{int}(\gamma)\}$ for all x in a countable dense subset Q of the plane. For a simple loop γ, $O \subset \text{int}(\gamma)$ if $O \cap Q \subset \text{int}(\gamma)$. We will consider collections $\Gamma = (\gamma_j)_{j \in J}$ (J is at most countable) of simple loops. We say that $(\gamma_j)_{j \in J}$ is a **simple loop configuration** in the bounded simply connected domain D if the following condition hold:

- For each $j \in J$, the loop γ_j is a simple loop in D.
- For each $j \neq j' \in J$, the loops γ_j and $\gamma_{j'}$ are disjoint.
- For each $j \neq j' \in J$, γ_j is not in the interior of $\gamma_{j'}$: the loops are not nested.
- For each $\epsilon > 0$, only finitely many loops γ_j have a diameter greater than ϵ. We call this the **local finiteness** condition.

All these conditions are measurable with respect to the $\sigma-$field discussed above.

We are going to discuss random simple loop configurations with some special properties. We will say that the random simple loop configuration $\Gamma = (\gamma_j)_{j \in J}$ in the unit disc $\mathbb{U}$ is a **Conformal Loop Ensemble** (CLE) if it satisfies the following properties:

- Non-triviality: the probability that $J \neq \emptyset$ is positive.
- Conformal invariance: the law of Γ is invariant under any conformal transformation from $\mathbb{U}$ onto itself.
- Restriction: suppose that U is a simply connected subset of the unit disc. Define
$$I = I(\Gamma, U) = \{j \in J : \gamma_j \nsubseteq U\}$$
and $J^* = J^*(\Gamma, U) = J \setminus I = \{j \in J : \gamma_j \subset U\}$. Define the (random) set
$$U^* = U^*(U, \Gamma) = U \setminus \cup_{j \in I} \overline{\text{int}(\gamma_j)}.$$
The restriction property is that (for all U), the conditional law of $(\gamma_j)_{j \in J^*}$ given U^* (or alternatively given the family $(\gamma_j, j \in I)$) is P_{U^*}.

By conformal invariance, it is feasible to define the law of the loop-ensemble Γ_D in any simply connected domain $D \neq \mathbb{C}$ as the law of the image of Γ under any given conformal map Φ from $\mathbb{U}$ onto D. We can also define the law of a loop-ensemble in any open domain $D \neq \mathbb{C}$ that is the union of disjoint open simply connected sets by taking independent loop-ensembles in each of the connected components of D, which we will denote by P_D. The set U^* constructed in the restriction property is an open subset of $\mathbb{U}$ (not necessarily simply connected) by the local finiteness condition. This set $U*$ is a open subset of $\mathbb{U}$ by the local finiteness condition.

We will list some simple consequences of the CLE definition.

Proposition 4.1. *Suppose that $\Gamma = (\gamma_j)_{j \in J}$ is a CLE in $\mathbb{U}$, then*

1. *For any given $z \in \mathbb{U}$, there almost surely exists a loop γ_j in Γ such that $z \in int(\gamma_j)$.*

2. *J is almost surely infinite.*

3. *Let $M(\theta)$ denote the set of configurations $\Gamma = (\gamma_j, j \in J)$ such that for all $j \in J$, the radius $[0, e^{i\theta}]$ is never locally "touched without crossing" by γ_j (in other words, θ is never a local extremum of the $arg(\gamma_j)'s$). Then for any θ, Γ is almost surely in $M(\theta)$. If we now define ...*

4. *For any $r < 1$, the probability that $r\mathbb{U}$ is entirely contained in the interior of one single loop is positive.*

5. *The restriction property continues to hold if we replace the simply connected domain $U \subset \mathbb{U}$ with the union U of countably many disjoint simply connected domains $U_i \subset \mathbb{U}$.*

Proof. to be completed □

4.2.1 Explorations

CLE is sometimes defined to include only the outermost loops, so that the collection of loops is non-nested (no loop is contained in another). Such a CLE is called a simple CLE to distinguish it from a full or nested CLE. The law of a full CLE can be recovered from the law of a simple CLE as follows. Sample a collection of simple CLE loops, and inside each loop sample another collection of simple CLE loops. Infinitely many iterations of this procedure gives a full CLE.

4.3 Loop-soup construction

First we define a natural conformally invariant measure on unrooted Brownian loops in the plane. We will start by some bridge-type Brownian measures on curves. These are measures on Brownian paths with prescribed starting point and prescribed terminal point. Since we are interested in conformally invariant properties, the standard bridges with prescribed time duration are not well-suited. In this section, we will deal with parametrized planar curves.

4.3.1 Notations

Let $\mathcal{K}$ be the set of all parametrized continuous planar curves γ defined on a time-interval $[0, t_\gamma]$. We consider $\mathcal{K}$ as a metric space with the metric

$$d_{\mathcal{K}}(\gamma, \gamma^1) = \inf_\theta [\sup_{0 \le s \le t_\gamma} |s - \theta(s)| + |\gamma(s) - \gamma^1(\theta(s))|],$$

where the infimum is over all increasing homeomorphisms $\theta : [0, t_\gamma] \to [0, t_{\gamma^1}]$. Note that $\mathcal{K}$ under this metric does not identify curves that are the same modulo time-parametrization.

If μ is any measure on $\mathcal{K}$, we let $|\mu| = \mu(K)$ denote the total mass. If $0 < |\mu| < \infty$, then we let $\mu^\# = \frac{\mu}{|\mu|}$ be μ normalized to be a probability measure.

If D is a domain, we say that γ is in D if $\gamma(0, t_\gamma) \subset D$; note that we do not require the endpoints of γ to be in D. Let $\mathcal{K}(D)$ be the set of $\gamma \in \mathcal{K}$ that are in D. In the following, $\mu_D(z, w)$ will be a measure on Brownian paths that remain in the domain D, starting at z and ending at w, but this notation will differ a bit depending on whether z, w are boundary or interior points of the domain D.

4.3.2 Preliminaries on excursion measures

Let $\mu(z,\cdot;t)$ denote the law of standard complex Brownian motion $(B_s, 0 \le s \le t)$ with $B_0 = z$. We can write

$$\mu(z,\cdot;t) = \int_{\mathbb{C}} \mu(z,w;t) dA(w),$$

where A denotes area and $\mu(z,w;t)$ is a measure supported on $\gamma \in \mathcal{K}_z^w$ with $t_\gamma = t$. In other words, $\mu(z,w;t)$ is $|\mu(z,w;t)|$ times the law $\mu^{\#}(z,w;t)$ of the Brownian bridge from z to w in time t, where

$$|\mu(z,w;t)| = (2\pi t)^{-1} \exp\{\frac{-|z-w|^2}{2t}\}.$$

The measure $\mu(z,w)$ is defined by

$$\mu(z,w) = \int_0^\infty \mu(z,w;t) dt.$$

This is a $\sigma-$finite measure (the integral explodes at infinity so that the total mass of large loops is infinite; when $z = w$, it also diverges at 0). The measure $\mu(z,z)$ is an infinite measure on Brownian loops that start and end at z, which can be also written as

$$\mu(z,z) = \int_0^\infty \frac{1}{2\pi t} \mu^{\#}(z,z;t) dt,$$

where $\mu^{\#}(z,z;t)$ is the usual probability measure of a Brownian bridge from z to z.

If D is a domain and $z, w \in D$, we define $\mu_D(z,w)$ to be $\mu(z,w)$ restricted to $\mathcal{K}(D)$. For fixed z, w, the family $\{\mu_D(z,w), \{z,w\} \subset D\}$ satisfies the restriction property. If $z \neq w$ and if the domain is such that a Brownian motion in D eventually exists D, then

$$|\mu_D(z,w)| = G_D(z,w) < \infty,$$

where G_D denotes the (positive) Green's function. For example, for the unit disk $|\mu_{\mathbb{D}}(0,z)| = -\frac{\log|z|}{\pi}$. Note that $\mu_D(z,z)$ is well defined and has infinite total mass. The reversibility of the Brownian bridge (which can be infered about the convergence of random functions to Brownian bridge) immediately implies that $[\mu_D(z,w)]^R = \mu_D(w,z)$. The conformal invariance of Brownian motion and of Green's function imply directly the conformal invariance of $\mu_D(z,w)$. It is not hard to see it follows from the conformal invariance that these measures $\mu(z,w)$ are continuous in terms of z, w in the Prohorov metric and a coupling argument (by ... following lemma).

Now we extend the definition of $\mu(\cdot,\cdot)$ to be a measure supported on curves with one or two endpoints on the boundary. Suppose D is a connected domain whose boundary is a finite union of curves (we allow the curves to be in the sphere and for infinity to be a boundary point). We will call ∂D nice is it is piecewise analytic, i.e., if it is a finite union of analytic curves. A nice boundary point will be any point at which the boundary is locally an analytic curve. Let B be a Brownian motion starting at $z \in D$ and stopped at its exit time of D,

i.e., at $\tau_D = \{t \geq 0 : B_t \notin D\}$. Define $\mu_D(z, \partial D)$ to be the law of $(B_t, 0 \leq t\tau_D)$. If D has a nice boundary we can write

$$\mu_D(z, \partial D) = \int_{\partial D} \mu_D(z, w)|dw|,$$

where $\mu_D(z, w)$ for $z \in D$ and $w \in \partial D$ denotes a measure supported on $\mathcal{K}_z^w(D)$ with total mass $H_D(z, w)$, where $H_D(z, w)$ denotes the usual Poisson kernel. The normalized probability measure $\mu_D^{\#}(z, w)$ is the law of Brownian motion conditioned to exit D "at w". Since the conformal covariance of Poisson kernels gives that, is $f : D \to D'$ is a conformal transformation from D onto another nice domain D', then $H_D(z, w) = |f'(w)|H_{D'}(f(z), f(w))$. Combined with the conformal invariance of trajectories of Brownian motion,

$$f_*\mu_D(z, w) = |f'(w)|\mu_{f(D)}(f(z), f(w)).$$

It implies that one can define the probability measure $\mu_D^{\#}(z, w)$ for any simply connected D and any boundary point (i.e. prime end) w by conformal invariance. For instance, it suffices to put

$$\mu_D^{\#}(z, w) = f_*\mu_{\mathbb{D}}^{\#}(0, 1),$$

where $f : \mathbb{D} \to D$ is the conformal transformation with $f(0) = z$ and $f(1) = w$. For a fixed boundary point w, the mapping $z \mapsto \mu_D(z, w)$ is continuous by conformal covariance. (??? see [the brownian loop soup])

In the next step, suppose that D is a nice domain, and that z and w are different nice boundary points of D.

$$\mu_D(z, w) = \lim_{\epsilon \to 0_+} \frac{1}{2\epsilon^2}\mu_D(z + \epsilon \mathbf{n}_z, w + \epsilon \mathbf{n}_w) = \lim_{\epsilon \to 0_+} \frac{1}{\epsilon}\mu_D(z + \epsilon \mathbf{n}_z, w),$$

the limit exists since $|\mu_D(\cdot, \cdot)|$ is harmonic.

If $f : D \to D'$ is a conformal transformation and $z, w, f(z)$ and $f(w)$ are nice boundary points, then

$$f_*\mu_D(z, w) = |f'(z)||f'(w)|\mu_{f(D)}(f(z), f(w)).$$

The probability measures $\mu_D^{\#}(z, w)$ are conformally invariant, i.e.,

$$f_*\mu_D^{\#}(z, w) = \mu_{f(D)}^{\#}(f(z), f(w)).$$

This means that one can define $\mu_D^{\#}(z, w)$ by conformal invariance even if z, w are not nice boundary points.

The following some lemmas are for the regularity of the excursion measures with respect to the domain D.

Lemma 4.2. *For any simply connected domain D and any two distinct points w and w' on the boundary of D, the expected time spent in an open subset U of D by an excursion defined under the probability measure $\mu_D^{\#}(w, w')$ is bounded from above by $2area(U)/\pi$.*

Proof. □

cauchy's formula for derivatives

Lemma 4.3. *There is a constant $c < \infty$ such that the following holds. Suppose that D, D' are simply connected domains and $f : D \to D'$ is a conformal transformation with $|f(z_1) - z_1| \le \delta \le 1$ for all $z_1 \in D$. Then for any path γ in D,*

$$d_{\mathcal{K}}(\gamma, f \circ \gamma) \le c[\delta + \sqrt{\delta} t_\gamma + \int_0^{t_\gamma} 1\{dist(\gamma(s), \partial D) \le \sqrt{\delta}\} ds + \int_0^{t_{f\circ\gamma}} 1\{dist(f \circ \gamma(u), \partial D') \le c\sqrt{\delta}\} du].$$

Proof. For any path γ in D, let

$$\theta_\gamma(s) = \int_0^s |f'(\gamma(r))|^2 dr.$$

Then since $f \circ \gamma(\theta_\gamma(s)) = f(\gamma(s))$,

$$d_{\mathcal{K}}(\gamma, f \circ \gamma) \le \sup_{0 \le s \le t_\gamma} |s - \theta_\gamma(s)| + \sup_{0 \le s \le t_\gamma} |\gamma(s) - f(\gamma(s))|.$$

The second term on the right is bounded above by δ and the first term is bounded above by $\int_0^{t_\gamma} ||f'(\gamma(s))|^2 - 1|$. Let

$$\tilde{D} = \tilde{D}_\delta = \{z \in D : \text{dist}(z, \partial D) \le \sqrt{\delta}\}.$$

For $z \in D \setminus \tilde{D}$, Cauchy's formula for derivatives gives that

$$|f'(z) - 1| \le c\delta / \text{dist}(z, \partial D) \le c\sqrt{\delta}.$$

Hence, $\int_0^{t_\gamma} |f'(\gamma(s))|^2 - 1| 1\{\gamma(s) \notin \tilde{D}\} ds \le c\sqrt{\delta} t_\gamma$. For the other part, write

$$\int_0^{t_\gamma} |f'(\gamma(s))|^2 - 1| 1\{\gamma(s) \in \tilde{D}\} ds \le \int_0^{t_\gamma} 1\{\gamma(s) \in \tilde{D}\} ds + \int_0^{t_\gamma} |f'(\gamma(s))|^2 1\{\gamma(s) \in \tilde{D}\} ds.$$

The first term on the right hand side is the amount of time that γ spends within distance $\sqrt{\delta}$ of the boundary. Since $|f(z) - z| \le \delta$, the second term on the right is less than the amount of time that $f \circ \gamma$ spends with distance $2\sqrt{\delta}$ of $\partial D'$. Combining these estimates gives the lemma. □

Combine the two previous lemmas, it is not hard to deduce the continuitu of $\mu_D^\#(z, w)$ with respect to w when z is fixed.

Proposition 4.4. *Suppose that $D \subset \mathbb{D}_+$ is a simply connected domain with $\mathbb{D}_+ \setminus D \subset \delta \mathbb{D}_+$ for some $\delta > 0$. Let z, z', w on ∂D with $|z| = |z'| = 1, |w| < \delta$ and $|z - z'| \le \delta$. Then, the distance between $\mu_D^\#(z, w)$ and $\mu_{\mathbb{D}_+}^\#(z', 0)$ goes to zero with δ, uniformly with respect to the choice of w, z, z' and D.*

Proof. Let f denote the conformal mapping from D onto $\mathbb{D}_+$ such that $f(w) = 0$, $f(z) = z'$ and $|f'(i) = 1|$.

exo: for some constant c, $|f(x) - x| \le c\delta$ for all $x \in D$ The total area in D or D' of the set of points that is at distance less than

$$\mathbb{E}[d_{\mathcal{K}}(\gamma, f \circ \gamma)] \le c\sqrt{\delta}$$

where the expectation is with respect to $\mu_D^\#(z, w)$ the law of $f \circ \gamma$ is $\mu_{\mathbb{D}_+}^\#(z', 0)$ □

4.4 Bubble soup and loop soup

4.4.1 Brownian bubble measure and bubble soup

The *Brownian bubble measure* in $\mathbb{H}$ at the origin is the $\sigma-$finite measure

$$\mu_{\mathbb{H}}^{\rm bub}(0) = \lim_{z\in\mathbb{H}\to 0} \frac{\pi}{{\rm Im}(z)}\mu_{\mathbb{H}}(z,0) = \lim_{z,w\in\mathbb{H}\to 0} \frac{\pi}{2{\rm Im}(z){\rm Im}(w)}\mu_{\mathbb{H}}(z,w).$$

When we speak of the limit , we mean that for every $r > 0$, if we restrict the measures on the right to loops that intersect the circle of radius r (?? so that this is a finite measure), then the limit exists and equals $\mu_{\mathbb{H}}^{\rm bub}(0;r)$ which is $\mu_{\mathbb{H}}^{\rm bub}(0)$ restricted to loops that intersect $\{|z| = r\}$. This limit exists and the normalization can be chosen so that $|\mu_{\mathbb{H}}^{\rm bub}(0;r)| = 1/r^2$ (by the following transformation formula). We can also define $\mu_D^{\rm bub}(z)$ for other domains, at least if ∂D is smooth near z, using conformal invariance,

$$f\circ\mu_D^{\rm bub}(z) = |f'(z)|^2\mu_{f(D)}^{\rm bub}(f(z)).$$

These measures satisfy the restriction property: if $D'\subset D$ and $z\in\partial D'\cap\partial D$, then $\mu_{D'}^{\rm bub}(z)$ is $\mu_D^{\rm bub}(z)$ restricted to loops that are in D'.

We define a *bubble soup* with intensity $\lambda\geq 0$ to be a Poisson point process with intensity $\lambda\mu_{\mathbb{H}}^{\rm hub}$. We can write a realization of the bubble soup as a countable collection $\mathcal{U} = \{(\gamma_j, s_j)\}$. Recall that the law of $\mathcal{U}$ is characterized by the fact that:

- For any two disjoint measurable subsets U_1 and U_2 of $\mathcal{K}_0^0(\mathbb{H})\times[0,\infty)$, $\mathcal{U}\cap U_1$ and $\mathcal{U}\cap U_2$ are independent.
- The law of the number of elements in $\mathcal{U}\cap U$ is the Poisson law with mean $\lambda\mu_{\mathbb{H}}^{\rm bub}(0)\times({\rm length})[U]$ (when this quantity is finite).

We will think of the bubble γ_j as being created at time s_j. Clearly, with probability one $s_j\neq s_k$ for $j\neq k$.

4.4.2 Brownian loop measure and loop soup

Now we will define the Brownian loop measure $\mu^{\rm loop}$. Let $\tilde{\mathcal{K}}$ be the set of loops, i.e., the set of $\gamma\in\mathcal{K}$ with $\gamma(0) = \gamma(t_\gamma)$. Such a γ can also be considered as a function with domain $(-\infty,\infty)$ satisfying $\gamma(s) = \gamma(s+t_\gamma)$. Define $\theta_r : \tilde{\mathcal{K}}\to\tilde{\mathcal{K}}$ by $t_{\theta_r\gamma} = t_\gamma$ and $\theta_r\gamma(s) = \gamma(s+r)$. We say that two loops γ and γ' are equivalent if for some r, $\gamma' = \theta_r\gamma$. We write $[\gamma]$ for the equivalence class of γ. Let $\tilde{\mathcal{K}}_U$ be the set of unrooted loops, i.e. the equivalence class in $\tilde{\mathcal{K}}$. Note that $\tilde{\mathcal{K}}_U$ is a metric space under the metric

$$d_{p,U}(\gamma,\gamma') = \inf_{r\in[0,t_\gamma]} d_{\mathcal{K}}(\theta_r\gamma,\gamma').$$

Any measure supported on $\tilde{\mathcal{K}}$ gives a measure on $\tilde{\mathcal{K}}_U$ by "forgetting the root", i.e., the image probability measure of the map $\gamma\mapsto[\gamma]$. If D is a domain, we define $\tilde{\mathcal{K}}(D), \tilde{\mathcal{K}}_U(D)$ to be the set of unrooted loops that lies entirely in D, i.e., $\gamma[0,t_\gamma]\subset D$. We define the *Brownian loop measure* $\mu^{\rm loop}$ on $\tilde{\mathcal{K}}_U$ by

$$\mu^{\rm loop} = \int_{\mathbb{C}}\frac{1}{t_\gamma}\mu(z,z)dA(z) = \int_{\mathbb{C}}\int_0^\infty \frac{1}{2\pi t^2}\mu^{\#}(z,z;t)dtdA(z),$$

where dA denotes the Lebesgue measure on $\mathbb{C}$. This is a measure on unrooted loops since the defined measure μ^{loop} does not depend on the choice of roots. , We call a Borel measurable function $T : \tilde{\mathcal{K}} \to [0, \infty)$ a *unit weight* if for every $\gamma \in \tilde{\mathcal{K}}$,

$$\int_0^{t_\gamma} T(\theta_r \gamma) dr = 1.$$

One exemple of a unit weight is $T(\gamma) = \frac{1}{t_\gamma}$. Moreover note that for any unit weight T,

$$\mu^{\text{loop}} = \int_{\mathbb{C}} \int_0^{t_\gamma} \frac{1}{t_\gamma} \theta_{r*} \mu(z, z) dr = \int_{\mathbb{C}} T\mu(z, z) dA(z). \tag{4.1}$$

(by forgetting roots, the measure $\mu(z, z)$ is the same as $\theta_{r*}\mu(z, z)$)

If D is a domain, we define μ^{loop} to be μ^{loop} restricted to the curves in $\tilde{\mathcal{K}}_U(D)$: this is the same as the right-hand side of (4.1) with μ_D replacing μ (and D replacing $\mathbb{C}$). By construction and the restriction property of μ, the family $\{\mu_D^{\text{loop}}\}$ satisfies the restriction property. These measures are also conformally invariant:

Proposition 4.5. *If $f : D \to D'$ is a conformal transformation, then*

$$f_* \mu_D^{loop} = \mu_{f(D)}^{loop}.$$

Proof. This proposition follows from two observations. One, which we have already noted, is the conformal invariance of interior to interior measures,

$$f_* \mu_D(z, z) = \mu_{f(D)}(f(z), f(z)).$$

The other is the fact that we can define a unit weight T_f by $T_f(\gamma) = 1/t_\gamma$ if $\gamma \notin \tilde{\mathcal{K}}(D)$, and if $\gamma \in \tilde{\mathcal{K}}(D)$, $T_f(\gamma) = |f'(\gamma(0))|^2 / t_{f\circ\gamma}$. To check that this is a unit weight, note that

$$\int_0^{t_\gamma} T_f(\theta_r \gamma) dr = \frac{1}{t_{f\circ\gamma}} \int_0^{t_\gamma} |f'(\gamma(r))|^2 dr = 1.$$

Therefore,

$$\begin{aligned} f_* \mu_D^{\text{loop}} &= f_*(\int_D T_f \mu_D(z, z) dA(z)) \\ &= \int_D \frac{1}{t_{f\circ\gamma}} |f'(z)|^2 f_* \mu_D(z, z) dA(z) \\ &= \int_D \frac{1}{t_{f\circ\gamma}} \mu_{D'}(f(z), f(z))[|f'(z)|^2 dA(z)] \\ &= \int_{D'} T\mu_{D'}(w, w) dA(w) = \mu_{D'}^{\text{loop}}, \end{aligned}$$

here T denotes the simple unit weight $T(\gamma) = \frac{1}{t_\gamma}$ in the last equality. □

A *Brownian loop soup* with intensity λ is a Poissonian sample from the measure $\lambda\mu^{\text{loop}}$. We will denote by $\mathcal{L}_{\mathbb{C}}$ a realization of the loop soup. A sample of the Brownian loop soup is a countable collection of (unrooted) Brownian loops in the plane. We will use $\mathcal{L}$ to denote the family of loops in $\mathbb{C}$ that are in $\mathbb{H}$. This is the Brownian loop soup in the half-plane.

If $D \subset \mathbb{H}$ is a domain, then we write

- $\mathcal{L}(D)$ for the family of loops in $\mathcal{L}$ that are in D
- $\mathcal{L}^{\perp}(D)$ for $\mathcal{L} \setminus \mathcal{L}(D)$, i.e., the family of loops that intersect $\mathbb{H} \setminus D$.

By definition, for any fixed D, the two random families $\mathcal{L}(D)$ and $\mathcal{L}^{\perp}(D)$ are independent. Note that the law of families $\mathcal{L}(D)$ inherit the conformal invariance and restriction properties of the Brownian loop measure.

4.5 Decompositions of bubble measure and loop measure

The next proposition relates $\mu_{\mathbb{H}}^{\text{bub}}(0)$ to excursion measures. This expression for $\mu_{\mathbb{H}}^{\text{bub}}(0)$ splits the bubble at the point $se^{i\theta}$ at which its distance to the origin is maximal.

Proposition 4.6.

$$\begin{aligned}\mu_{\mathbb{H}}^{bub}(0) &= \pi \int_0^{\infty} \int_0^{\pi} [\mu_{r\mathbb{D}_+}(0, re^{i\theta}) \oplus \mu_{r\mathbb{D}_+}(re^{i\theta}, 0)] r \; d\theta \; dr \\ &= \int_0^{\infty} \frac{4}{\pi r^3} \int_0^{\pi} [\mu^{\#}_{r\mathbb{D}_+}(0, re^{i\theta}) \oplus \mu^{\#}_{r\mathbb{D}_+}(re^{i\theta}, 0)] \sin^2(\theta) d\theta \; dr\end{aligned}$$

Proof. We denote by $\mu_{\mathbb{H}}^{\text{bub}}(0; r)$ the measure $\mu_{\mathbb{H}}^{\text{bub}}(0)$ restricted to loops that intersect $\{|z| = r\}$. By decompositing loops at the hitting time of $r\mathbb{S}_+$, we have

$$\begin{aligned}\mu_{\mathbb{H}}^{\text{bub}}(0; r) - \mu_{\mathbb{H}}^{\text{bub}}(0; r+\delta) &= \frac{\pi}{\epsilon} \int_0^{\pi} [\mu_{r\mathbb{D}_+}(\epsilon i, re^{i\theta}) \oplus \mu_{(r+\delta)\mathbb{D}_+}(re^{i\theta}, 0)] r d\theta \\ &= \pi \int_0^{\pi} [\mu_{r\mathbb{D}_+}(0, re^{i\theta}) \oplus \mu_{(r+\delta)\mathbb{D}_+}(re^{i\theta}, 0)] r \; d\theta.\end{aligned}$$

But as

$$\lim_{\delta \to 0_+} \delta^{-1} \mu_{(r+\delta)\mathbb{D}_+}(re^{i\theta}, 0) = \mu_{r\mathbb{D}_+}(r \; e^{i\theta}, 0),$$

we get that

$$\frac{d}{dr} \mu_{\mathbb{H}}^{\text{bub}}(0; r) = -\pi \int_0^{\pi} [\mu_{r\mathbb{D}_+}(0, re^{i\theta}) \oplus \mu_{r\mathbb{D}_+}(re^{i\theta}, 0)] r d\theta,$$

which gives the first equality. The second follows from some calculation on total mass

$$|\mu_{r\mathbb{D}_+}(0, re^{i\theta})| = r^{-2} |\mu_{\mathbb{D}_+}(0, e^{i\theta})| = \frac{2 \sin\theta}{\pi r^2},$$

since the Green's function on $\mathbb{D}_+$ is $G(M, M_0) = \frac{1}{2\pi}[\log \frac{1}{r_{MM_0}} - \log \frac{1}{}]$...

$$\mu_{r\mathbb{D}_+}(0, r \; e^{i\theta}) = r^{-2} |\mu_{\mathbb{D}_+}(0, e^{i\theta})|$$

□

The conformal invariance of μ^{loop} makes it independent of the choice of coordinate axes. Il will be however convenient to have expressions for μ^{loop} that do depend on the axes. We will write the measure on unrooted loops $[\gamma]$ as a measure on rooted loops by choosing the representative γ whose initial point is the (unique) point on the loop of minimal imaginary part (the same works of course for the maximal imaginary part).

Proposition 4.7.

$$\mu^{loop} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mu^{bub}_{\mathbb{H}+iy}(x+iy)dxdy$$

Proof. We define a unit weight T_ϵ on $\gamma \in \tilde{\mathcal{K}}$ that will approximate the Dirac mass at the time of the minimal imaginary part of γ. If $t_\gamma < \epsilon$, then $T_\epsilon(\gamma) = \frac{1}{t_\gamma}$. Suppose that $t_\gamma \geq \epsilon$, and there is a unique $r_0 \in [0, t_\gamma)$ such that $\mathrm{Im}[\gamma(r_0)] < \mathrm{Im}[\gamma(t)]$ for $t \in [0, t_\gamma)\backslash\{r_0\}$. Then $T_\epsilon(\theta_r\gamma) = \frac{1}{\epsilon}$ for $r \in [r_0-\epsilon, r_0]$ and $T_\epsilon(\theta_r\gamma) = 0$ for other $t \in (r_0, 2r_0-\epsilon)$. (Here γ is considered as a periodic function of period t_γ.) If no such unique r_0 exists, set $T_\epsilon(\gamma) = 1/t_\gamma$ (the choice here is irrelevant since this is a set of loops of measure zero). Note that the measures $\mu(z,z)$ are supported on loops for which a unique r_0 exists.??? It is easy to see that T_ϵ is a unit weight, and hence for every ϵ,

$$\mu^{\mathrm{loop}} = \lim_{\epsilon\to 0_+}\int_{\mathbb{C}} T_\epsilon\mu(z,z)dA(z) = \lim_{\epsilon\to 0_+}\int_{\mathbb{C}}\epsilon^{-1}\mu(z,z;\geq\epsilon)dA(z),$$

where $\mu(z,z;\geq\epsilon)$ denotes $\mu(z,z)$ restricted to curves γ with $t_\gamma \geq \epsilon$ and

$$\inf\{\mathrm{Im}(\gamma(t)); t\in[0,\epsilon]\} = \inf\{\mathrm{Im}(\gamma(t)); t\in[0,t_\gamma)\}.(norm?)$$

For fixed ϵ, $\int_{\mathbb{C}}\epsilon^{-1}\mu(z,z;\geq\epsilon)dA(z)$ is the same as μ^{loop} restricted to curves with $t_\gamma \geq \epsilon$. Let us consider the measure $\epsilon^{-1}\mu(z,z;\geq\epsilon)$. For ease let $z=0$. Start a Brownian motion B_t at 0 and let it run until time ϵ; let us write $B_\epsilon = \sqrt{\epsilon}w$. We let $-b\sqrt{\epsilon} = \min\{\mathrm{Im}(B_t) : t\in[0,\epsilon]\}$. Then given $B_t, t\in[0,\epsilon]$, the remainder of the curve is obtained from the measure $\epsilon^{-1}\mu_{\mathbb{H}-ib\sqrt{\epsilon}}(0, w\sqrt{\epsilon})$. As $\epsilon\to 0_+$, this looks like $\epsilon^{-1}\mu_{\mathbb{H}}(ib\sqrt{\epsilon}, ib\sqrt{\epsilon}+w\sqrt{\epsilon})$, which (scaling of BM???)in turn has the same limit as $\epsilon^{-1}b[b+\mathrm{Im}(w)]\mu_{\mathbb{H}}(i\sqrt{\epsilon}, i\sqrt{\epsilon})$. Hence $\lim_{\epsilon\to 0_+}\epsilon^{-1}\mu(0,0;\epsilon) = \frac{1}{2\pi}\mu^{\mathrm{bub}}_{\mathbb{H}}(0)$. □

Lemma 4.8.

4 ??

Proof. Suppose that Y_t is a one-dimensional Brownian motion started at the origin. Let t^* be the time in $[0,1]$ at which Y_t is minimal, let $M = Y_{t^*}$, and let $\Psi = (Y_0-M)(Y_1-M)$. The law of t^* ???is the arcsine law with density $\frac{1}{\pi\sqrt{t(1-t)}}$ on $[0,1]$. Given t^*, Y_0-M and Y_1-M are independent random variables with the distribution of Brownian motion "conditioned to stay positive". ??

$$\mathbb{E}[Y_0 - M | t^* = t] = \sqrt{\frac{\pi t}{2}}$$

and hence $\mathbb{E}[\Psi] = \frac{1}{2}$. □

The next proposition gives an expression for $\mu^{\mathrm{loop}}_{\mathbb{H}}$ by association to an unrooted loop the rooted loop whose root has the maximal absolute value.

Proposition 4.9.

$$\mu^{loop}_{\mathbb{H}} = \frac{1}{2\pi}\int_0^\infty\int_0^\pi \mu^{bub}_{r\mathbb{D}_+}(re^{i\theta})d\theta r dr.$$

Proof. Let $\mathcal{R} = \{x+iy : -\infty < x < \infty, 0 < y < \pi\}$ and $\mathcal{R}_b = \{z \in \mathcal{R} : \mathrm{Re}(z) < b\}$ and let $\phi(z) = e^z$. Conformal invariance tells us that $\phi_* \mu_{\mathcal{R}}^{\mathrm{loop}} = \mu_{\mathbb{H}}^{\mathrm{loop}}$. But the proposition above (rotated 90 degrees) and restriction tell us that

$$\mu_{\mathcal{R}}^{\mathrm{loop}} = \frac{1}{2\pi} \int_{\infty}^{\infty} \int_0^{\pi} \mu_{\mathcal{R}_x}^{\mathrm{bub}}(x+iy) dy dx.$$

The conformal covariance for μ^{bub} gives $\phi_* \mu_{\mathcal{R}_x}^{\mathrm{bub}}(x+iy) = e^{2x} \mu_{e^x \mathbb{D}_+}^{\mathrm{bub}}(e^{x+iy})$. Therefore

$$\begin{aligned} \mu_{\mathbb{H}}^{\mathrm{loop}} = \phi_* \mu_{\mathcal{R}}^{\mathrm{loop}} &= \frac{1}{2\pi} \int_{\infty}^{\infty} \int_0^{\pi} e^{2x} \mu_{e^x \mathbb{D}_+}^{\mathrm{bub}}(e^{x+iy}) dy dx \\ &= \frac{1}{2\pi} \int_0^{\infty} \int_0^{\pi} \mu_{r\mathbb{D}_+}^{\mathrm{bub}}(re^{iy}) dy \; r \; dr \end{aligned}$$

□

4.6 Relations between bubbles and loops

The goal of this section is to prove the main theorem, which relates the Poissonian cloud of loops that intersect a given curve to the Poisson point process of bubbles rooted at a boundary point of a domain by chronologically adding loops to simple curves in the plane. In order to prove this, we need a generalization of the previous results to domains of all shapes, and to show that the convergence holds uniformly.

4.6.1 Some estimates on Poisson kernels

Lemma 4.10. *There exists a constant c such that if $r \in (0, 1/2)$ and $\theta, \varphi \in (0, \pi)$,*

$$H_{\mathbb{D}_+}(re^{i\theta}, e^{i\varphi}) - \frac{2}{\pi} r \sin\theta \sin\varphi| \quad \leq cr^2 \sin\theta \sin\varphi \tag{4.2}$$

$$H_{\mathbb{H}\setminus\overline{\mathbb{D}_+}}(r^{-1}e^{i\theta}, e^{i\varphi}) - \frac{2}{\pi} r \sin\theta \sin\varphi| \quad \leq cr^2 \sin\theta \sin\varphi. \tag{4.3}$$

Proof. Take the map $f : z \mapsto -z - \frac{1}{z}$ from $\mathbb{D}_+$ onto $\mathbb{H}$, hence

$$\begin{aligned} H_{\mathbb{D}_+}(re^{i\theta}, e^{i\varphi}) &= |f'(e^{i\varphi})| H_{\mathbb{H}}(f(re^{i\theta}), f(e^{i\varphi})) \\ &= 2\sin\varphi H_{\mathbb{H}}(f(re^{i\theta}), f(e^{i\varphi})). \end{aligned}$$

But if $|z| \geq \frac{5}{2}$, $|x| \leq 2$

$$H_{\mathbb{D}}(z, x') = \frac{\mathrm{Im}(z)}{\pi[(\mathrm{Re}(z) - x')^2 + \mathrm{Im}(z)^2]} = \frac{\mathrm{Im}(z)}{\pi|z|^2}(1 + O(\frac{1}{|z|})),$$

and

$$f(re^{i\theta}) = \frac{1}{r} e^{i(\pi-\theta)} + O(r),$$

$$\mathrm{Im}[f(re^{i\theta})] = \frac{\sin\theta}{r} + \sin\theta \; O(r).$$

This gives the first expression, and the second is not hard to get from the first using the map $z \mapsto -\frac{1}{z}$. □

Lemma 4.11. *There exists a constant c such that if $e^{-s} \in (\frac{3}{4}, 1)$, $r \in (0, \frac{1}{2})$, and $\theta, \varphi \in (0, \pi)$, then*

$$|H_{\mathbb{D}_+,r}(e^{-s+i\theta}, re^{i\varphi}) - \frac{4}{\pi} \sinh s \sin\theta \sin\varphi| \le crs \sin\theta \sin\varphi, \tag{4.4}$$

where $\mathbb{D}_+, r = \{z \in \mathbb{D}_+ : |z| > r\}$.

asymptotics

Proposition 4.12. *Suppose V_n is a sequence of sets in $\mathbb{H}$ with $u_n = rad(v_n) \to 0$ and such that $\mathbb{H} \setminus V_n$ is simply connected. Let m_n be $\mu_{\mathbb{H}}^{loop}$ restricted to loops that intersect both V_n and the unit circle. Set $h_n = hcap(V_n)$. When $n \to \infty$,*

$$m_n = \frac{h_n}{2} \mu_{\mathbb{H}}^{bub}(0,1)(1 + o(1)),$$

where $o(1)$ is uniformly bounded by a function of u_n that goes to zero with u_n. Note that scaling implies the corresponding results for the measures restricted to paths that intersect any given circle $r\partial\mathbb{D}$.

Proof.

$$m_n = \frac{1}{2\pi} \int_1^\infty \int_0^\pi \mu^{\rm bub}_{r\mathbb{D}_+}(re^{i\theta}|V_n) d\theta r dr$$

$$\mu^{\rm bub}_{\mathbb{H}}(0,1) = \int_1^\infty \frac{4}{\pi r^3} \int_0^\pi [\mu^{\#}_{r\mathbb{D}_+}(0, re^{i\theta}) \oplus \mu^{\#}_{r\mathbb{D}_+}(re^{i\theta}, 0)] \sin^2\theta d\theta dr.$$

It is not hard to observe that

$$[\mu^{\rm bub}_{r\mathbb{D}_+}(re^{i\theta}|V_n)]^{\#} \to [\mu^{\#}_{r\mathbb{D}_+}(0, re^{i\theta}) \oplus \mu^{\#}_{r\mathbb{D}_+}(re^{i\theta}, 0)]$$

uniformly on $\{1 \le r \le R\}$ and $\theta \in (0, \pi)$. (????Note that there is a conformal transformation $g : \mathbb{H} \setminus V_n \to \mathbb{H}$ with $\max |g(z) - z| \le cu_n$).

We now focus on the total masses. It suffices to show that

$$\mu^{\rm bub}_{\mathbb{D}_+}(e^{i\theta}|V_n) = 4h_n \sin^2\theta [1 + O(u_n)] \tag{4.5}$$

In fact, by scaling rules for $\mu^{\rm bub}$ and hcap, the formula below implies that for all $r \ge 1$,

$$|\mu^{\rm bub}_{r\mathbb{D}_+}(re^{i\theta}|V_n)| = r^{-2} |\mu^{\rm bub}_{\mathbb{D}_+}(e^{i\theta}|r^{-1}V_n)| 4r^{-4} h_n \sin^2\theta [1 + O(u_n)],$$

and the proposition follows. To prove (4.5), first note that

$$|\mu^{\rm bub}_{\mathbb{D}_+}(e^{i\theta}|V_n)| = \lim_{\epsilon \to 0_+} \frac{\pi}{\epsilon} \mu_{\mathbb{D}_+}(\exp(-\epsilon + i\theta), \exp(i\theta))[B \text{ hits } V_n].$$

The estimates (5.2) and (5.3) and the use of scaling rule imply that the following measures are very close for large R and small u:

- The measure of $1_{\sigma_u < T} \arg(B_{\sigma_u})$ when B is defined under the measure $R\mathbb{P}^{iR}$.
- The measure of $1_{\sigma_u < T, \sigma_u < \sigma_1} \arg(B_{\sigma_u})$ when B is defined under the measure
$$(2 \sin\theta \ \sinh(\epsilon))^{-1} \mathbb{P}^{\exp(-\epsilon + i\theta)}.$$

After hitting $u\mathbb{D}_+$, it is possible to "couple" the two paths up to their first hitting of V_n. After the hitting of V_n, we want to estimate the probability that the path go back to the unit circle without hitting $\mathbb{R}$ and they hit it in the neighborhood of $\exp(i\theta)$. By (5.1), this will occur with probability

$$\frac{2}{\pi}\mathrm{Im}(B_{\rho V_n})\sin\theta d\theta.$$

Therefore, we get finally that (recall that these estimates are uniform in θ, ϵ and R)

$$\begin{aligned}|\mu^{\mathrm{bub}}_{\mathbb{D}_+}(e^{i\theta}|V_n)| &\sim \lim_{\epsilon\to 0, R\to 0}\frac{2\pi}{\epsilon}\sin\theta\sinh\epsilon R\mathbb{E}^{iR}[\frac{2}{\pi}\mathrm{Im}(B_{\rho V_n})\sin\theta]\\ &\sim 4\sin^2\theta\lim_{R\to\infty} R\mathbb{E}^{iR}[\mathrm{Im}(B_{\rho V_n})]\\ &\sim 4h_n\sin^2\theta\end{aligned}$$

□

The following is the main theorem of this section. Now suppose that $\eta : [0,\infty)\to\mathbb{C}$ is a simple curve with $\eta(0,\infty)\subset\mathbb{H}$ and $|\eta(t)|\to\infty$ as $t\to\infty$, parametrized by capacity, i.e. that $\mathrm{hcap}[\eta[0,t]] = 2t$.

Theorem 4.13. *Suppose $H_t = \mathbb{H}\setminus\eta[0,t]$ and let g_t be the unique conformal transformation of H_t onto $\mathbb{H}$ such that $g_t(\eta(t)) = 0$ and $g_t(z)\sim z$ as $z\to\infty$. Let $f_t = g_t^{-1}$ which maps $\mathbb{H}$ conformally onto H_t with $f_t(0) = \eta(t)$. For every $t<\infty$, if $\mathcal{U}$ is a bubble soup with intensity $\lambda>0$, consider the set of loops (preimage of bubbles appearing before time t)*

$$\mathcal{U}_{\eta,t} = \{f_{s_j}\circ\gamma_j : (\gamma_j, s_j)\in\mathcal{U}, s_j\le t\},$$

then $\mathcal{U}_{\eta,t}$, considered as a collection of unrooted loops (by forgetting the root), is a realization of $\mathcal{L}^\perp(H_t)$ with intensity λt.

Lemma 4.14. *Let $\mathcal{L}$ be a realization of the loop soup in $\mathbb{H}$ with intensity λ. We write elements of $\mathcal{L}$ as $[\gamma]$ since they are equivalent classes of loops. For every $[\gamma]\in\mathcal{L}$, let r_j denote the smallest r such that $\gamma(r)\in\eta[0,t_\gamma]$. Then with probability one, there is a unique representative of $[\gamma]$, such that $\gamma(0) = \eta(r)$ and $\gamma(0,t_\gamma)\subset H_r$.*

proof of ... The proof of this theorem follows from ... Let $\mathcal{L}_t^\perp$ denote the set of loops $\gamma_j\in\mathcal{L}$ with $r_j\le t$, i.e. the set of loops that intersect $\eta[0,t]$. For $r>0$, let $\mathcal{L}_t^\perp(r)$ denote the set of loops $\gamma_j\in\mathcal{L}_t$ such that $\mathrm{rad}[g_{r_j}\circ\gamma_j] = \sup\{g_{r_j}\circ\gamma_j(s) : 0\le s\le t_{g_{r_j}\circ\gamma_j}\}\ge r$. Note that with probability one $\mathcal{L}_t(r)$ is finite for each $t<\infty, r>0$???.

It suffices to show that for every $r>0$ the set of loops $\{g_{r_j}\circ\gamma_j : \gamma_j\in\mathcal{L}_t^\perp(r)\}$ is a Poissonian realization of the measure $\lambda t\mu^{\mathrm{bub}}_{\mathbb{H}}(0;r)$. (We only need to do this for the case $r=1$, the other cases are essentially the same). Let $\mathcal{A}_t = \{g_{r_j}\circ\gamma_j : \gamma_j\in\mathcal{L}_t(1)^\perp\}$. We have noted that

$\mathcal{A}_{t+\epsilon}\setminus\mathcal{A}_t$ is independent of $\mathcal{A}_t$ ... conformal invariance implies that the distribution of $g_t\circ[\mathcal{A}_{t+\epsilon}\setminus\mathcal{A}_t]$, derived from the curve η, is the same as the distribution of $\mathcal{A}_\epsilon$ derived from the

It suffices to prove the two conditions above for $t=0$

lemma: (section 5.2) $m_n = \frac{h_n}{2}\mu^{\mathrm{bub}}_{\mathbb{H}}(0,1)(1+o(1))$ □

Corollary 4.15. *The process $(\tilde{\gamma_r}, r \geq 0)$ is a Poisson point process with intensity λ times the Brownian bubble measure in $\mathbb{H}$.*

4.7 Parametrization

Suppose that $\eta : [0, \infty) \to \mathbb{C}$ is a curve as before, and let $\mathcal{L}_t^{\perp}$ denote a realization of the loop soup in $\mathbb{H}$ restricted to curves that intersect $\eta[0, t]$. We are going to show that if the path $\eta[0, \infty)$ has dimension strictly less than two, then the sum of all the time-lengths of the loops in $\mathcal{L}_t^{\perp}$ is almost surely finite. This will imply that one can construct a continuous path by attaching these loops "chronologically" to η.

Lemma 4.16. *Suppose that there exists a positive function g such that*

$$area\{z : dist(z, \eta[0, T]) \leq e^{-s}\} \leq g(s),$$

where $\int_0^{\infty} g(s)ds < \infty$. Then with probability one,

$$\sum_{\gamma \in \mathcal{L}_T^{\perp}} t_{\gamma} < \infty.$$

Proof.

$$\#\{\gamma \in \mathcal{L}_T^{\perp} : t_{\gamma} > 1\} < \infty$$

□

Proposition 4.17. *If D is any nonempty open domain, then*

$$\sum_{\gamma \in \mathcal{L}(D)} t_{\gamma} = \infty$$

5 CLE: construction via loop-soups

5.1 Markovian characterization of CLE

... gives a criterion for conformal loop ensembles, which is stated as follows. We say that $\overline{\Gamma} = (\gamma_j, j \in J)$ is a simple loop configuration in the unit disc $\mathbb{D}$ if it is a collection of simple loops in $\mathbb{D}$ that are *disjoint* and *non-nested* and that is *locally finite* (for each $\epsilon > 0$, only finite many loops γ_j have a diameter greater than ϵ).

We say that (the law of) a non-empty random simple loop configuration $\overline{\Gamma} = (\gamma_j, j \in J)$ in the unit disc $\mathbb{D}$ satisfies the **conformal restriction axioms** if the following hold:

1. Conformal invariance: the law of $\overline{\Gamma}$ is invariant under any conformal transformation from $\mathbb{D}$ onto itself.

2. Restriction: for any simply connected subset U of $\mathbb{D}$, define U^* to be the set obtained by removing from U all the loops and interiors of loops of U that do not stay in U. (?? U^* is an open subset of $\mathbb{D}$ because of the local finiteness condition.) The restriction property is that given U^*, the conditional law of the set of loops in $\overline{\Gamma}$ that *stay* in U (and therefore U^* is $\mathbb{P}_{U^*}$).

Note that conformal invariance enables us to define the law $\mathbb{P}_D$ of the loop-ensemble in any simply connected domain $D \neq \mathbb{C}$. We can also define $\mathbb{P}_D$ if D has several connected components by taking independent samples in each of the connected components.

5.2 Loop-soup percolation

The goal is to prove that almost surely, the set of outer boundaries of outermost clusters of the Brownian loop-soup does give a simple loop configuraton which satisfies the conformal restriction axioms for a range of c, whereas when c is large, the Brownian loop-soup has only one cluster.

Recall the (weak) restriction properties of Brownian loop measure. First we are going to establish some lemmas on properties of the collection $\overline{\mathcal{C}}$ as (closures of) the clusters defined by Γ.

Lemma 5.1. *Suppose that $\mathbb{P}_c(\overline{\mathcal{C}} = \{\overline{\mathbb{U}}\}) < 1$. Let $U \subset \mathbb{U}$ denote some open subset of $\mathbb{U}$ and define $U*$ to be the set obtained by removing from U all the (closures of) loop-soup clusters $\overline{\mathcal{C}}$ that do not stay in U. Then, conditionally on $U*$ (with $U* \neq \emptyset$), the set of loops of Γ that do stay in $U*$ is distributed like a Brownian loop-soup in $U*$.*

Proof. The event $\overline{\mathcal{C}} \neq \{\overline{\mathbb{U}}\}$ implies that $U^* \neq \emptyset$.

We proceed by approximating U^* by a nested sequence of dyadic sets if $U^* \neq \emptyset$. For any $n \geq 1$, let $U_n^* = U_n^*(U^*)$ to be the largest union of dyadic squares of side-lengths 2^{-n} that is contained in $U*$. For each $n \geq 1$, and for each (deterministic) union V_n of such dyadic squares, the loop-soup restricted to V_n is independent of the event $\{U_n^* = V_n\}$, since for a Brownian loop soup, the two random families $\mathcal{L}(U)$ and $\mathcal{L}^{\perp}(U)$ are independent. It immedaitely follows that conditionned on U^*, the set of loops that do stay in U_n^* is distributed like a Brownian loop-soup in U_n^*. Since this holds for all n, it also holds for open set $U^* = \cup_{n \geq 1} U_n^*$. □

Lemma 5.2. *Suppose that $\mathbb{P}_c(\overline{\mathcal{C}} = \{\overline{U}\}) < 1$ and that there is a $\mathbb{P}_c$ positive probability that $\overline{\mathcal{C}}$ contains an element intersecting the boundary of $\mathbb{U}$. Then for all positive c', $\mathbb{P}_{c+c'}(\overline{\mathcal{C}} = \{\overline{\mathbb{U}}\}) = 1$.*

Proof. Let A_1 be the union of all elements of $\overline{\mathcal{C}}$ that intersect some prescribed boundary arc ∂ of $\mathbb{U}$ of positive probability.

If U is a fixed open set such that $\overline{U} \subset \mathbb{U}$ the law of set of loops contained in U is the same as its original law

the law of loops of Γ that do not intersect $\overline{A}_1$ is the image of an independent Γ

the conformal radius ρ_1 of $\overline{\mathbb{U}} \setminus \overline{A}_1$ seen from 0 has a strictly positive probability to be smaller than one

iterate the procedure: the conformal radius of U_n seen from 0 is dominated by a product of i.i.d. copies of ρ_1.

for any positive δ one can almost surely find in $\overline{\Gamma}$ a finite sequence of clusters $\overline{C}_1, \cdots, \overline{C}_k$ such that $d(C_j, C_{j+1}) = 0$ for all $j < k$, such that C_1 touches ∂ and $d(C_k, 0) \leq \delta$.

by conformal invariance, simultaneously for all points z with rational coordinates, for all rational δ, and all boundary arcs of $\partial\mathbb{U}$ of positive length.

each loop of the loop-soup surrounds some point with rational coordinates ... any two clusters in $\overline{\mathbb{C}}$ are "connected" via a finite sequence of adjacent clusters in $\overline{\mathcal{C}}$.

add by an independent loop-soup, the new loops will almost surely join together any two adjacent clusters of $\overline{\mathcal{C}}$ into a single cluster $\square$

Lemma 5.3. *There is a critical constant $c_0 \in [0, \infty]$ such that*

1. *If $c > c_0$, then $\mathbb{P}_c(\overline{\mathcal{C}} = \overline{\mathbb{U}}) = 1$.*
2. *If $c \in (0, c_0)$, then $\mathbb{P}_c$ almost surely*
   (a) *$\overline{\mathcal{C}}$ contains infinitely many elements.*
   (b) *No element of $\overline{\mathcal{C}}$ intersects the boundary of $\mathbb{U}$.*
   (c) *No two elements of $\overline{\mathcal{C}}$ intersect each other.*

Proof. Set $c_0 = \inf\{c > 0 : \mathbb{P}_c(\overline{\mathcal{C}} = \overline{\mathbb{U}}) = 1\}$. The previous lemma implies that $\mathbb{P}_c(\overline{\mathcal{C}} = \overline{\mathbb{U}}) = 1$ for all $c > c_0$ and no element of $\overline{\mathcal{C}}$ intersects the boundary of $\mathbb{U}$ a.s. if $c \in (0, c_0)$.

If two elements of $\overline{\mathcal{C}}$ intersect each other with $\mathbb{P}_c$ positive probability, then applying Lemma 5.1 to some U containing one but not the other with positive probability and by conformal invariance, we find that there is a positive probability that an element of $\overline{\mathcal{C}}$ intersects $\partial\mathbb{U}$. Lemma 5.2 implies $c \geq c_0$.

If $c > 0$ and $\overline{\mathcal{C}}$ has only finitely many elements with positive probability, then with the same probability, at least one of these elements must intersect ∂U since the loops of Γ are dense in $\mathbb{U}$ a.s.. $\square$

Remark 5.4. We say c is *sub-critical* if the (a),(b) and (c) of Lemma 5.3 hold $\mathbb{P}_c$ almost surely. We will show that c_0 is sub-critical at the end of this section. The proof of Lemma 5.3 shows that in order to prove that some c is sub-critical, it suffices to check (b).

Lemma 5.5. *The c_0 of Lemma 5.3 lies on $(0, \infty)$. Moreover, when $c > 0$ is small enough, there are $\mathbb{P}_c$ almost surely continuous paths and loops in D that intersect no element of Γ.*

Proof. We first prove that for small $c > 0$ there exists almost surely simple paths crossing $\mathbb{U}$ but intersect no element of Γ. couple the loop-soup with a well-known fractal percolation model:

consider the unit square $D = (0, 1)^2$ (instead of $\mathbb{U}$). For each n, we will divide it into 4^n dyadic squares of side-length 2^{-n}. To each such square C, associate a Bernoulli random variable $X(C)$ equal to one with probability p. $p < 1$ large enough, M connects the two opposite sides of the unit square with positive probability and M contains loops with positive probability.

associate (in a deterministic, scale-invariant and translation-invariant) (rewrite by the other) manner to each loop $l \subset (0, 1)^2$ a dyadic site site $s = (j2^{-n(l)}, j'2^{-n(l)}) \in (0, 1)^2$ such that l is contained in the square S_l with side-length $2 \times 2^{-n(l)}$ and bottom-left corner at s.

replace all loops l in the loop-soup by the square S_l

for each fixed square

$$S = (j2^{-m}, (j + 1)2^{-m}) \times (j'2^{-m}, (j' + 2)2^{-m}) \subset (0, 2)^2,$$

the probability that there exists no loop l in the loop-soup such that $S_l = S$ is less or equal to $\exp(-bc)$ all these events (when one lets S vary) are independent

the loop-soup percolation process is dominated by a variant of Mandelbrot's fractal percolation model:

$$\mathbb{P}[\tilde{X}(S) = 1] = \exp(-bc)$$

□

Lemma 5.6. *If c is sub-critical, the probability that there are k disjoint chains of loops in $\overline{\mathcal{C}}$ crossing from the inside to the outside of a fixed annulus decays exponentially in k.*

Proof. No element of $\overline{\mathcal{C}}$ intersects the boundary of $\mathbb{U}$ $\mathbb{P}_c$ almost surely if c is sub-critical. So for all $r < 1$, there exists $r_1 \in (r, 1)$ such that the probability that there is a crossing of $\{z : r < |z| < r_1\}$ is strictly less than one.

Now we consider any given annulus $\{z : r < |z - z_0| < r'\}$. By conformal invariance, the probability that the loop-soup restricted to $\{z : |z - z_0| < \frac{r'}{r_1}\}$ (we can choose r_1 such that $\{z : |z - z_0| < \frac{r'}{r_1}\} \subset \overline{\mathbb{U}}$) containes no crossing of the annulus is positive. Moreover, the event that no loop intersects both the circles of radius r' and r'/r_1 around z_0 happens with positive probability and is independent of the previous event. The intersection of the two events is contained in the complement of the crossing event. Therefore, the probability that there exists a crossing of the annulus is strictly smaller than one. Then result follows from the BK inequality for Poisson point process. □

Lemma 5.7. *If c is sub-critical, the set $\overline{\mathcal{C}}$ is almost surely locally finite, and moreover the elements of $\overline{\Gamma}$ are almost surely continuous simple loops.*

Proof. Fix $\epsilon > 0$. For any $z_0 \in \overline{\mathbb{U}}$, $\mathbb{P}_c$ almost surely there exists only finite (closures of) clusters in $\overline{\mathcal{C}}$ whose distance to z is less than $\frac{2}{\epsilon}$, otherwise there would be infinitely many crossings of annulus $\{z : \frac{\epsilon}{2} < |z - z_0| < \epsilon\}$. The same argument holds for any point in an $\frac{\epsilon}{2}-$dense set in $\overline{\mathbb{U}}$, which implies that $\overline{\mathcal{C}}$ is almost surely locally finite.

Recall that the elements of Γ are all simple loops. Conditioned on one loop of Γ, the others in $\overline{\mathbb{U}} \setminus \text{int}\Gamma$ are independent of Γ. When c is sub-critical, almost surely no loop in $\overline{\mathbb{U}} \setminus \text{int}\Gamma$ intersects Γ. Therefore no element of $\overline{\mathcal{C}}$ can have a cut point. If the boundary is a continuous curve, it must be simple. If the boundary has a point of discontinuity, then there must be infinitely many chains of loops in Γ crossing an annulus surrounding that point, which does not happen almost surely. □

Combining these lemmas above, one can conclude that there exists a positive constant c_0 such that for all $c \in (0, c_0)$, the set $\overline{\Gamma}$ is a (random) simple loop configuration that satisfies the conformal restriction axioms, whereas when c is (strictly) greater than c_0, Γ has only one cluster almost surely.

5.3 Relation between c and κ

If c is sub-critical, then the conformal restriction properties of the set $\overline{\Gamma}$ implies that it is a CLE_κ for some $\kappa \in (\frac{8}{3}, 4]$. In other words, for such c, the $\overline{\Gamma}$ is equivalent in law to the loop ensemble constructed via branching $\text{SLE}_{\kappa,\kappa-6}$.

In this section, we are going to describe the conformal loop-ensemble via the pinned measure of branching $\mathrm{SLE}_{\kappa,\kappa-6}$ to identify κ in terms of c and show that c_0 is sub-critical.

Suppose that $\overline{\Gamma}$ is a random loop ensemble that satisfies the conformal restriction axioms

5.4 Identifying the critical intensity c_0

branching SLE, size of loop-soup clusters It is more conveniant to work in the upper half-plane $\mathbb{H}$ instead of the unit disk.

5.5 An intermediate way to measure the size of sets

Suppose that $\alpha \in (0,1]$, and that A is a bounded closed subset of the upper half-plane $\mathbb{H}$. Define

$$M(A) = M_\alpha(A) := \lim_{s\to\infty} s\mathbb{E}[(\mathrm{Im}B^{is}_{\tau(A)})^\alpha],$$

where B^{is} is a Brownian motion started at is and stopped at $\tau(A)$, the first time that it hits $A\cup\mathbb{R}$. The limit exists and is finite by the explicit form of Poisson kernel on the half-plane $\mathbb{H}$ applied by mapping conformally the unbounded component of $\mathbb{H}\setminus A$ onto $\mathbb{H}$. Note that M_1 is the usual half-plane capacity used in the context of chordal Loewner chains. ???

Il will be useful to compare $M(A)$ with some quantities involving dyadic squares and rectangles that A intersects. We will consider the usual hyperbolic tiling of $\mathbb{H}$ by squares of the form $[a2^j,(a+1)2^j]\times[2^j,2^{j+1}]$, for integers a,j. Let $\mathcal{S}$ be the set of all such squares. For each hull A, we define $\mathcal{S}(A)$ to be the set of squares in $\mathcal{S}$ that A intersects, and let

$$\hat{A} = \cup_{S\in\mathcal{S}(A)} S$$

the union of these dyadic squares.

Basic facts:

- $M(aA) = a^{\alpha+1}M(A)$
- for any two bounded closed sets A and A', $M(A\cup A') \le M(A) + M(A')$; if $A\subset A'$, $M(A)\le M(A')$

Lemma 5.8. *There exists a universal positive C such that*

$$CM(\hat{A}) \le M(A) \le M(\hat{A}).$$

Proof. bounding below the probability that (after this hitting time of $\hat{A}$) the Brownian motion makes a loop around S before it hits any square of $\mathcal{S}$ that is not adjacent to S □

Lemma 5.9. *There exists a universal positive constant C' such that for any hull A,*

$$C'\sum_{S\in\mathcal{S}(A)} M(S) \le M(A) \le \sum_{S\in\mathcal{S}(A)} M(s)$$

Proof. it is sufficient to prove the result in the case where $A = \hat{A}$ is the union of squares in $\mathcal{S}$.

for each $j \in \mathbb{Z}$, note that each even square $S \in \mathcal{S}_j$, and for each $z \in S$, the probability that a Brownian motion started from z hits the real line before hitting any other even square in $\mathcal{S}_j$ is bounded from below; it follows from the strong Markov property that the expected number of squares in $\mathcal{S}_{j+k+1}$ that B visits before exiting $\mathbb{H}$ is bounded by $K \times 2^{-k}$

$$\mathbb{E}[\sum_{S\in\mathcal{S}}(\mathrm{Im}B^z_{\tau(S)})^\alpha] \le \sum_{k\ge 0}(2^{j+k+1})^\alpha \times (K2^{-k}) + \sum_{k<0}(2^{j+k+1})^\alpha \times (K2^{-k}) \quad \le C(Imz)^\alpha$$

apply this statement to the Brownian motion B^{is} after its first hitting time $\tau(A)$ of $A \cup \mathbb{R}$, □

Given a hull A, we denote by $\tilde{A} = \tilde{A}(A, \overline{\Gamma})$ the random hull whose complement is the unbounded component of the set obtained by removing from $\mathbb{H} \setminus A$ all the loops of $\overline{\Gamma}$ that intersect A. ?Local finiteness implies that $\tilde{A}$ is a hull almost surely. picture ..... Now define

$$N(A) = N_\kappa(A) := \mathbb{E}[M(\Phi_A(\tilde{A}))].$$

For each square $S = [a2^j, (a+1)2^j] \times [2^j, 2^{j+1}]$, define $R(S)$ to be all the squares that lie strictly under S

Proposition 5.10. *If $c < 1$ and $c \le c_0$, then there is an $\alpha(c) \in (0,1)$ such that,*

$$\mathbb{E}[diam(\tilde{A})^{1+\alpha}] < \infty$$

for all hulls A.

Proof. proof in Markov.... □

If we write $A_S = A \cap S$, then

$$\mathbb{E}[M(\Phi_A(\tilde{A}_S))] \le C(c)M(S).$$

By scaling, it suffices to consider the case where $S = [0,1] \times [1,2]$ and hence $R = R(S)$ is the rectangle $[0,1] \times [0,2]$ We want to prove that $\mathbb{E}[M(\Phi_A(\tilde{A}_S))]$ tends to zero uniformly with respect to A as $c \to 0$. In fact, the probability that there are no loops intersecting $A^\epsilon_S = \{z : \mathrm{dist}(z, A_S) \le \epsilon\}$ goes to 1 as c goes to 0 by the decomposition formula. Therefore $\mathbb{E}[M(\Phi_A(\tilde{A}_S)]$ goes to zero by dominated convergence applied to $\mathrm{diam}(\tilde{A}_S)^{1+\alpha}$.

Lemma 5.11.

$$N(A) \le C_1(c)M(A).$$

$C_1(C)$ tends to zero as c tends to zero.

Proof. Putting together the above estimates, it is not hard to get that

$$\begin{aligned}\mathbb{E}[M(\Phi_A(\tilde{A}))] &\le \sum_{S\in\mathcal{S}(A)} \mathbb{E}[M(\Phi_A(\tilde{A}_S))] \\ &\le \sum_{S\in\mathcal{S}(A)} C(c)\mathbb{E}[M(S)] \\ &\le C_1(c)M(A),\end{aligned}$$

where $C_1(c)$ (which depends only on c) tends to zero as c tends to zero. □

the probability (that the diameter of the set of loops intersecting a small ball on the boundary of $\mathbb{H}$ is large) decays quickly with the size of this ball when $\kappa < 4$. Intuitively one can guess that when $\kappa < 4$, two big clusters will be unlikely to be vary close, i.e., in some sense, there

adding a loop-soup with a very small intensity c' on top of the loop-soup with intensity c_0 might not be sufficient to make all clusters percolate, and this would contradict the fact that c_0 is critical

Proof. Suppose that $c_0 < 1$. We choose c' to be small enough so that the product of the corresponding constants $C_1(c_0)$ and $C_1(c')$ in the above lemma is less than 1, so that one can have an exponential decay on M by iteration. We will view that the loop-soup Γ with intensity $c_0 + c'$ as the superposition of a loop-soup Γ_0 with intensity $c_0\mu$ and a loop-soup Γ' with intensity $c'\mu$ which are mutually independent since they are Poisson point process.

For any given hull A, Γ contains a chain of loops that join A to the line $L_R = \{z \in \mathbb{H} : \mathrm{Im}(z) = R\}$. This implies that one can find a chain $\gamma_0, \gamma_1, \cdots, \gamma_k$ in Γ almost surely such that $\gamma_0 \cap A \neq \emptyset, \gamma_k \cap L_R \neq \emptyset$ and $\gamma_{i-1} \cap \gamma_i \neq \emptyset$ for $i = 1, \cdots, k$. Besides, these loops alternatively belong to $\overline{\Gamma}_0$ and $\overline{\Gamma}'$. But in the following, we will show by construction that this does not holds when $c_0 < 1$.

First, consider the loops of $\overline{\Gamma}_0$ that intersecting A. Let A_1 be the hull generated by the union of these loops and A.Recall that the expected value of $M(A_1)$ is finite because $\alpha = \alpha(c_1)$. Then add to A_1, the loops of $\overline{\Gamma}'$ that intersect A_1 and the union generates a hull B_2. add to B_1 the loops of $\overline{\Gamma}_0$ that intersect B_1.

$\mathrm{hcap}(A) \leq M(A)$ is $A \subset \{z : \mathrm{Im}(z) \leq 1\}$?if A is a hull such that there exists $z \in A$ with $\mathrm{Im}(z) \geq 1$, then $M(A) \geq c\ \mathrm{hcap}(A_n) \leq M(A_n)$, which implies almost surely $\Sigma_n \mathrm{hcap}(A_n) < \infty$

$M(A_n)$ decays exponentially, which implies that $M(A_n)$ almost surely decays eventually faster than some exponential sequence.

□

References

[LSW01] G. F. Lawler, O. Schramm, and W. Werner. Conformal invariance of planar loop-erased random walks and uniform spanning trees. *ArXiv Mathematics e-prints*, December 2001.

[RS03] S. Rohde and O. Schramm. Basic properties of SLE. *Annals of Mathematics*, May 2003.