Statistical mechanics models via the lens of potential theory

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September 26th, 2022

Concepts in (discrete) potential theory

- Random walks and the Laplace operator
- Green's function and capacity

The capacity of BRWs at and above the critical dimension 6

Dimension 2: conformal invariance and beyond

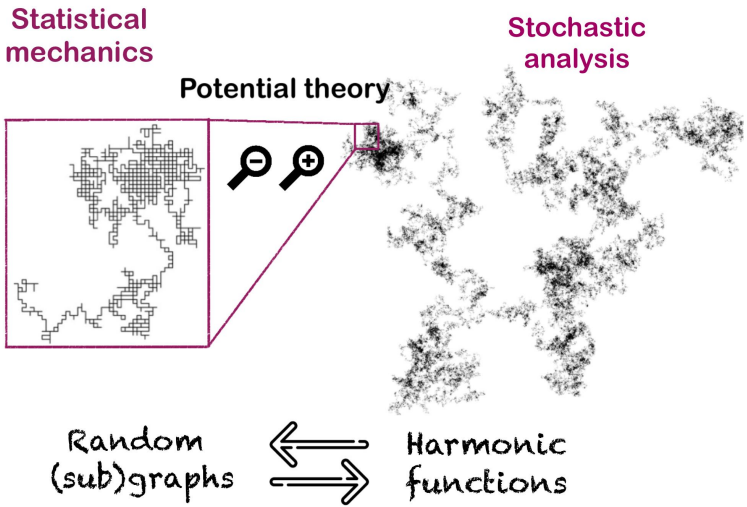
- Schramm-Loewner evolution (SLE)
- Conformal loop ensemble (CLE)

Crossing estimates of simple CLEs

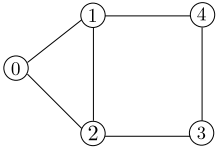
Convergence of massive LERWs to massive SLE(2) in dimension 2

Convergence of near-critical Ising energy densities and dimer height fluctuations

Concepts in (discrete) potential theory


Statistical mechanics
Stochastic analysis
Potential theory
Random (sub)graphs
Harmonic functions

Labeled graph



Adjacency matrix

	0	1	2	3	4
0	0	1	1	0	0
1	1	0	1	0	1
2	1	1	0	1	0
3	0	0	1	0	1
4	0	1	0	1	0

Degree matrix

	0	1	2	3	4
0	2	0	0	0	0
1	0	3	0	0	0
2	0	0	3	0	0
3	0	0	0	2	0
4	0	0	0	0	2

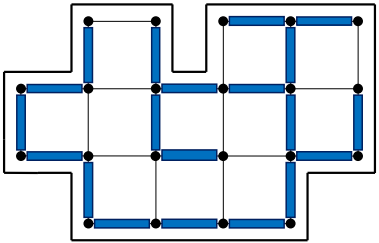
Laplacian

	0	1	2	3	4
0	2	−1	−1	0	0
1	−1	3	−1	0	−1
2	−1	−1	3	−1	0
3	0	0	−1	2	−1
4	0	−1	0	−1	2

Simple Random walks	Laplacian
$\mathbb{P}[X_{t+1} = y \mid X_t = x] = \frac{1_{x\sim y}}{\deg(x)}$	$\Delta h(x) = \deg(x)\,(h(x) - \mathbb{E}[h(X_{t+1}) \mid X_t = x])$

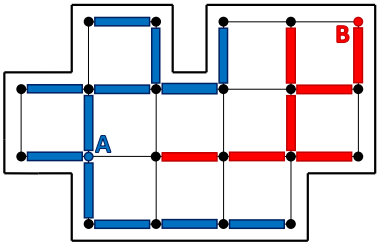
Green's function
$G(x,y) = \mathbb{E}\left[\sum_{n=0}^{\tau_\partial} 1_{\{X_t = y\}}\right]$ satisfies $\Delta_y G(x,y) = \begin{cases} \deg(x) & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$

Uniform Spanning Tree (UST) model



Choose a *tree* $\mathrm{ST}(\emptyset)$ uniformly at random

A variant:



Given **A** and **B**, choose a pair of trees (rooted at A and B) uniformly at random.

Kirchhoff: resistance $R(A,B) = \dfrac{\#\mathrm{SF}(A,B)}{\#\mathrm{ST}(\emptyset)}$

The **harmonic (electrostatic) capacity**: measure a set's ability to hold electrical charge. For $d \geq 3$, let the **potential** v be harmonic ($\Delta V = 0$), taking values 1 on ∂K and 0 at infinity,

$$\mathrm{cap}(K) = \int_{\mathbb{R}^n \setminus K} |\nabla v(x)|^2 \mathrm{dV}(x), \quad K \subset \mathbb{R}^d.$$

When ∂K is smooth, Green's formula implies that

$$\mathrm{cap}(K) = \int_{\partial K} \partial_n \mathbb{P}_x(\tau_K^+ = \infty) \mathrm{dS}(x),$$

where $\mathbb{P}_x$ denotes the probability of a Brownian motion started at x, τ_K^+ denotes its return time, ∂_n stands for the normal derivative.

For $K \subset \mathbb{R}^d$, $d \geq 3$

$$\mathrm{cap}(K) = \int_{\mathbb{R}^n \setminus K} |\nabla v(x)|^2 \mathrm{dV}(x) = \int_{\partial K} \partial_n \mathbb{P}_x(\tau_K^+ = \infty) \mathrm{dS}(x)$$

Definition ((Discrete) capacity)

Given a distribution η on $\mathbb{Z}^d$, for any finite $K \subset \mathbb{Z}^d$, $d \geq 3$,

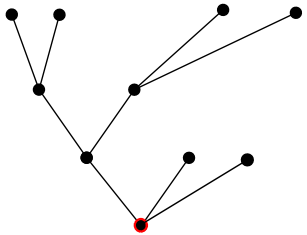
$$\mathrm{cap}_\eta(K) = \sum_{x \in K} \mathbf{P}_x^\eta(\tau_K = \infty),$$

where $\mathbf{P}_x^\eta$ denotes the law of a random walk with transition probability η started at x and τ_K^+ denotes its first-return time.

The capacity of BRWs at and above the critical dimension 6

Definition (Galton-Watson (GW) tree)

A random **rooted** tree such that the number of children of each node is i.i.d. acccording to μ. It is called **critical** if $\sum_{n\in\mathbb{N}} n\mu(n) = 1$.

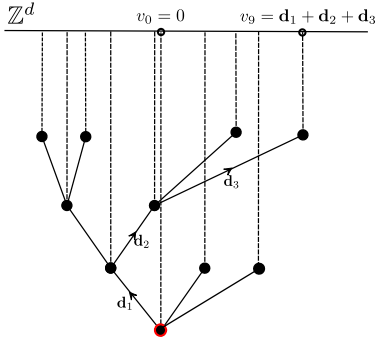


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Definition (Branching random walk (BRW))

Attaching i.i.d. displacement in $\mathbb{Z}^d$ to edges of a GW-tree according to θ.



R_n : vertices visited by a BRW indexed by a GW-tree of size n.

Theorem (Le Gall-Lin, '14)

Denote the covariance matrix of θ by M_θ,

- *if $d \geq 5$, $\frac{1}{n}\#R_n \xrightarrow[n\to\infty]{(\mathrm{P})} C(d,\mu,\theta)$ in probability;*
- *if $d = 4$, $\frac{\log n}{n}\#R_n \xrightarrow[n\to\infty]{L^2} 8\pi^2\sigma^4$, where $\sigma^2 = (\det M_\theta)^{\frac{1}{4}}$;*
- *if $d \leq 3$, $n^{-d/4}\#R_n \xrightarrow[n\to\infty]{(\mathrm{d})} a_\theta \lambda_d(\mathrm{supp}(\mathcal{I}))$, where $a_\theta = 2^{d/4}(\det M_\theta)^{1/2}$, and $\lambda_d(\mathrm{supp}(\mathcal{I}))$ stands for the Lebesgue measure of the support of the random measure on $\mathbb{R}^d$ known as Integrated Super-Brownian Excursion.*

Theorem (Bai-W', AAP '20)

- *if* $d \geq 7$, $\frac{\mathrm{cap} R_n}{n} \xrightarrow[n\to\infty]{\mathrm{P}} C(d, \theta, \mu)$, *with* $C(d, \ldots) > 0$ *inexplicit;*
- *if* $d = 6$, $\frac{\log n}{n} \mathrm{cap} R_n \xrightarrow[n\to\infty]{\mathrm{P}} C_G$, *with* C_G *explicit.*

Idea of the proof.

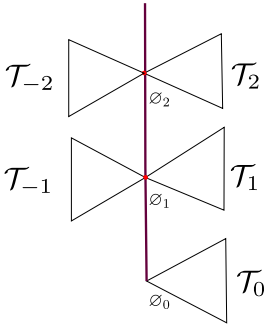
Step 1: compare with a double-infinite critical branching random walk;

Step 2: establish Green's function estimates on infinite BRWs

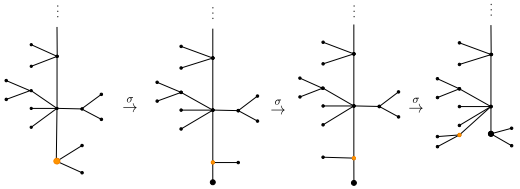
$$\sum_{i=-n}^{n} G(v_i) \sim \begin{cases} (C_G)^{-1} \cdot \log n & \text{if } d = 6 \\ C & \text{if } d > 6. \end{cases}$$

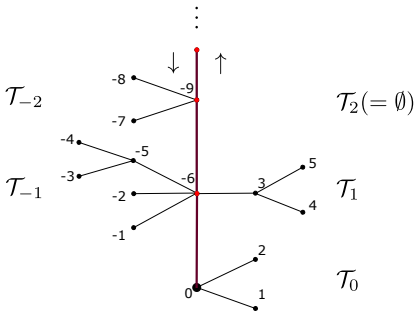
□

- Let the vertices on the spine **except the root** have i children on the left and j children on the right with probability $\mu(i+j+1)$ $(\sum_{i,j\geq 0}\mu(i+j+1)=1)$;



- Let the vertices on the spine **except the root** have i children on the left and j children on the right with probability $\mu(i+j+1)$;
- Invariant and ergodic under translation;





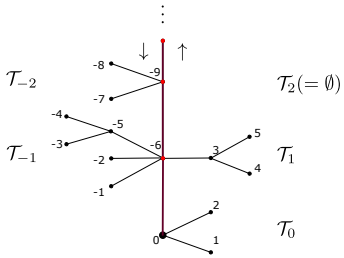
- Let the vertices on the spine **except the root** have i children on the left and j children on the right with probability $\mu(i+j+1)$;
- Invariant and ergodic under translation;

Let v_i be the spatial position of the i-th node of the GW-tree; $R_n := \{v_0, \dots, v_n\}$.

- Subadditivity of the capacity:

$$\lim_{n\to\infty} \frac{\mathrm{cap}(\{v_0, \dots, v_n\})}{n} \text{ exists a.s..}$$

Q: The limit vanishes or not?



For $x, y \in \mathbb{Z}^d$, the Green's function for simple random walks

$$G(x,y) = G(x-y) = \sum_{k=0}^{\infty} \mathbf{P}_0(S_k = x-y).$$

- The capacity of $R_n := \{v_0, \ldots, v_n\}$ can be rewritten as

$$\mathrm{cap}(R_n) = \sum_{i=1}^{n} \mathbf{1}_{\{v_i \notin v_{i+1}, \ldots, v_n\}} \mathbf{P}_{v_i}(\tau^+_{R_n} = \infty).$$

- Moreover

$$\sum_{i=1}^{n} \mathbf{1}_{v_i \notin \{v_{i+1}, \ldots, v_n\}} \mathbf{P}_{v_i}(\tau^+_{R_n} = \infty) \sum_{j=1}^{n} G(v_i, v_j) = n.$$

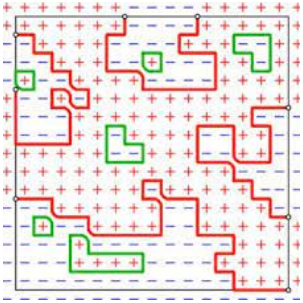
- The range of branching random walk converges with scaling $n^{-1/4}$ to the support of the integrated super Brownian excursion (ISE).
- Below dimension 6, the capacity of the support of ISE does not vanish.

Theorem (Bai and Hu, '22)

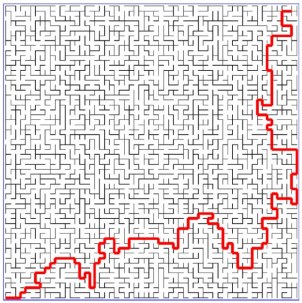
Let $\mathcal{R}$ be the range of ISE. Under mild conditions on μ and θ, there exists C_d explicit depending on the dimension n, such that

$$n^{-\frac{d-2}{4}} \operatorname{cap} R_n \to C_d \cdot \operatorname{cap} \mathcal{R} \text{ in distribution.}$$

Dimension 2: conformal invariance and beyond



The domain walls in the Ising model © L. Coquille.



Unique path between any two given boundary vertices has the law of a LERW © O. Schramm.

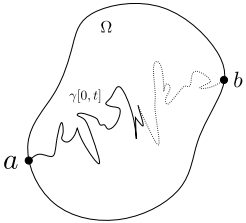
Question: how to prove the convergence and what is the limit?

First step: construct a plausible candidate for the limit in continuum.

Ω: simply connected planar domain, $a, b \in \partial\Omega$,

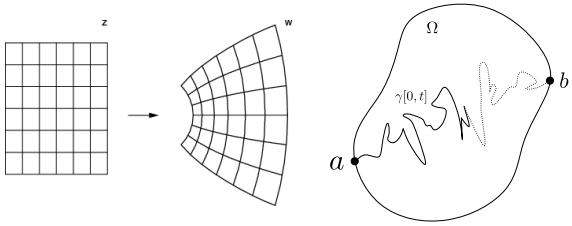
- random **slits** in Ω from a to b
- candidates for the scaling limit of 2D statistical physics models at critical temperature





Ω: simply connected planar domain, $a, b \in \partial\Omega$,

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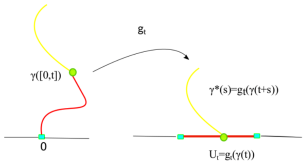
For a SLE γ in Ω from a to b:

- **Conformal invariance**: if φ is a conformal map, then $\varphi(\gamma)$ is a SLE in $\varphi(\Omega)$ from $\varphi(a)$ to $\varphi(b)$.
- **Domain Markov property**: the conditional law of $\gamma[t, \infty)$ given $\gamma[0, t]$ is a SLE in $\Omega \setminus \gamma[0, t]$ from $\gamma(t)$ to b.

- $(\gamma(t))_{t\geq 0}$: a simple slit growing from $\partial\mathbb{H} = \mathbb{R}$ parametrized by half-plane capacity;
- g_t : mapping out function of the slit at time t,

$$g_t(z) \overset{z\to\infty}{\sim} z + \frac{2t}{z};$$

- ξ_t : $g_t(\gamma(t))$ the image of the tip on $\mathbb{R}$.



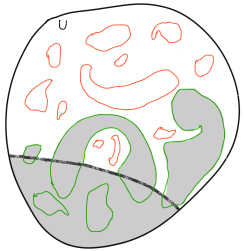
Definition (Loewner's equation)

If $\tau_z = \inf\{t \geq 0 : z \in \gamma(t)\}$, then

$$\dot{g}_t(z) = \frac{2}{g_t(z) - \xi_t} \text{ for } t \in [0, \tau_z),\ g_0(z) = z \in \mathbb{H}.$$

Schramm Loewner's evolution (SLE_κ): $\xi_t = \sqrt{\kappa}B_t$.

- **Conformal invariance:** $\varphi(\mathrm{CLE}(\Omega_1)) \overset{(d)}{\sim} \mathrm{CLE}(\Omega_2)$ if $\varphi : \Omega_1 \to \Omega_2$ is conformal.
- **Restriction:** if U is a simply connected subset of Ω
  1. obtain $\tilde{U}$ by **removing** from Ω all the $\mathrm{CLE}(\Omega)$ loops (and their interior) that do not entirely stay in U;
  2. in each **connected component** C of $\tilde{U}$, the **conditional law** of the set of loops that lie entirely in C is distributed as $\mathrm{CLE}(C)$.

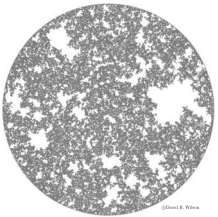


Conformal loop ensemble satisfies

- **Conformal invariance:** ...
- **Restriction:** ...







Theorem (Sheffield-Werner '10)

Each conformal loop ensemble is the CLE_κ *with* $\kappa \in (8/3, 4]$*, which has* SLE_κ*-type loops.*

The construction follows.

Definition (2D Brownian loop (bridge) with time length t)

A probability measure $\mathbb{P}_{z,t}$ on

$$(B_s^1 - \frac{s}{t}B_t^1, B_s^2 - \frac{s}{t}B_t^2)_{0\le s\le t},$$

where $B_\cdot^1$ and $B_\cdot^2$ are two independent 1D Brownian motions started from 0.



Definition (Brownian loop measure μ)

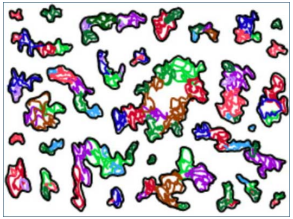
A measure on **unrooted** loops with **random time lengths**:

$$\mu = \int_{\mathbb{C}} \int_0^\infty \frac{1}{4\pi t} \mathbb{P}_{z,t}\, dt\, d^2z.$$

Restriction: for domain Ω, μ_Ω is the restriction of μ to the set of loops in Ω.

Definition (Brownian loop soup in Ω, Lawler-Werner 06')

A Poisson point process of intensity $c\mu_\Omega$ for $c > 0$.



Consider **clusters**: pathwise connected components.

Definition (Sheffield-Werner '10)

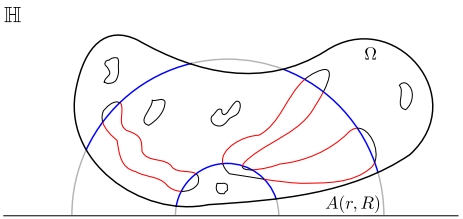
For $c \in (0, 1]$, the outermost boundary of a Brownian loop-soup of intensity c is a CLE_κ, where $\kappa \in (8/3, 4]$ satisfies the relation $c = (3\kappa - 8)(6 - \kappa)/2\kappa$.

Crossing estimates of simple CLEs

Theorem (Bai, W., IMRN '21)

For all simply connected subdomains Ω of the upper half-plane, let $\mathrm{CLE}_\kappa(\Omega)$, $\kappa \leq 4$, be the **non-nested** CLE *in Ω. Then for all $s > 0$,*

$$\sup_{\Omega \subseteq \mathbb{H}} \mathbb{P}[\mathrm{Cross}_{A(r,R)}(\mathrm{CLE}_\kappa(\Omega)) \geq n] = O(s^n).$$



[!] Not the same asymptotic regime as the arm exponents: sending $n \to \infty$ rather than $r/R \to 0$: $\mathbb{P}[\mathrm{Cross}_{A(r,R)}(\mathrm{CLE}_\kappa(\Omega)) \geq n] = (r/R)^{\Omega(n^2)}$ **[Wu-Zhan 16']**.

Corollary (Bai, W., IMRN '21)

- *For a collection of punctures inside Ω, fix a triangulation;*
- *Let $\mathcal{L}_\Omega^{\mathrm{CLE}_\kappa}$ be the* **nested** $\mathrm{CLE}_\kappa(\Omega)$ *obtained after* **removing** *loops surrounding no more than one puncture;*
- *Then for any* **macroscopic lamination** Γ *and $s > 0$,*

$$\mathbb{P}[\mathcal{L}_\Omega^{\mathrm{CLE}_\kappa} \text{ is homotopic to } \Gamma] = O(s^{|\Gamma|}).$$

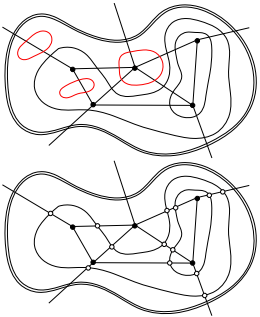
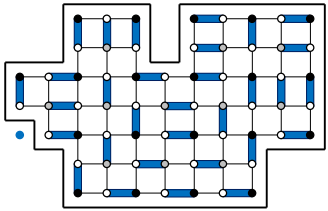
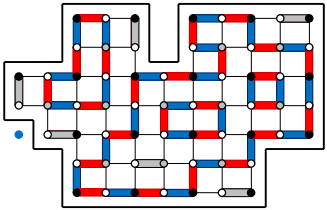


Figure: Complexity $|\Gamma|$

- **Temperleyan discretizations** on $\mathbb{Z}^2$: simply connected domains s.t. all corners have even coordinates.
- **Dimer (= Domino) model**: uniformly chosen perfect matching.



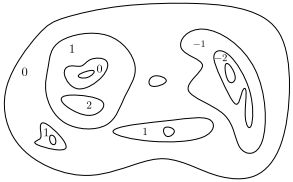
- **Temperleyan discretizations**: simply connected domains s.t. all corners have even coordinates with one removed.
- **Dimer (= Domino) model**: uniformly chosen perfect matching.



• **Double-dimer model**: superposition of two independent dimer configurations on the same domain, which is a fully-packed collection of loops and double-edges.

$\mathcal{L}_{\Omega^\delta}^{\mathrm{dbl-d}}$: the set of loops after removing double edges.

Height function $h^{\mathrm{dbl-d}}$: height profile with contour lines $\mathcal{L}^{\mathrm{dbl-d}}$, such that the orientation of loop $\gamma \in \mathcal{L}$ is chosen randomly.



It is a famous prediction (supported by many strong results) that

double-dimer loop configuration converges to the nested $\mathrm{CLE}_4(\Omega)$**.**

Kenyon '00: $h^{\mathrm{dbl-d}} \to \mathrm{GFF}(\Omega)$ as $\delta \to 0$ (in Temperleyan domains)
$+$ a natural coupling between CLE_4 and GFF.

[!] The convergence of $h^{\mathrm{dbl-d}}$ is not strong enough for the level lines $\mathcal{L}^{\mathrm{dbl-d}}$ of $h^{\mathrm{dbl-d}}$. The precompactness for the dbl-dimer loop ensemble is not proven yet.

Fix $\lambda_1, \dots, \lambda_n \in \Omega$. On $X_{\text{unip}} := \{\rho : \pi_1(\Omega \setminus \{\text{punct.}\}) \to \mathrm{SL}_2 \mid \mathrm{Tr}(\rho[\lambda_i]) = 2\}$, define

$$\tau^\delta(\rho) = \mathbb{E}\left[\prod_{\gamma \in \mathcal{L}_{\Omega^\delta}^{\mathrm{dbl-d}}} \left(\frac{1}{2}\mathrm{Tr}\,\rho(\gamma)\right)\right]$$

- Kenyon'11+ Dubédat'14: **convergence and conformal invariance of τ^δ**;
- Basok-Chelkak'18: **conformal invariance of probabilities of macroscopic topological events** for the double-dimer loop ensemble.

Dubédat '14 + Basok-Chelkak '18 + Bai-W. '21: for any mascroscopic lamination Γ,

$$\mathbb{P}[\mathcal{L}_{\Omega^\delta}^{\mathrm{dbl-d}} \sim \Gamma] \to \mathbb{P}[\mathcal{L}_{\Omega}^{\mathrm{CLE}_4} \sim \Gamma] \quad \text{as } \delta \to 0.$$

- L : macroscopic lamination; f_L : basis of functions on X_{unip} indexed by L.
- $p_L^\delta = \mathbb{P}[\mathcal{L}_{\Omega^\delta}^{\mathrm{dbl-d}} \sim \Gamma]$; $\quad p_L^{\mathrm{CLE}_4} = \mathbb{P}[\mathcal{L}_\Omega^{\mathrm{CLE}_4} \sim \Gamma]$.

Based on [Dubédat '14] and [Bai-W '21], the limit of $\tau^\delta(\rho)$ can be written as follows on the **whole** X_{unip}

$$\mathbb{E}\left[\prod_{\gamma\in\mathcal{L}_\Omega^{\mathrm{CLE}_4}} \frac{1}{2}\rho(\gamma)\right] = \sum_{L-\text{macroscopic}} p_L^{\mathrm{CLE}_4} f_L(\rho). \qquad \rightsquigarrow$$

$$\begin{array}{ccc} \sum p_L^\delta f_L & \xrightarrow{\text{[Dubédat '14]}} & \sum p_L^{\mathrm{CLE}_4} f_L \\ \Big\Downarrow & \text{[Basok, Chelkak '18]} & \\ p_L^\delta & \longrightarrow & p_L^{\mathrm{CLE}_4} \end{array}$$

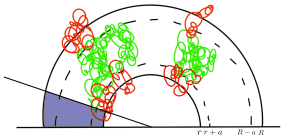
Remark (An example on $\mathbb{C}$)

Take $r > 0$ and $\{1, r-z, rz-z^2, \ldots, rz^n - z^{n+1}, \ldots\}$ as a basis. If $|a_n| < (r+\epsilon)^{-n}$,

$$\left(\sum_{n=0}^{N^\delta} a_n^\delta (rz^n - z^{n+1}) \to \sum_{n=0}^{\infty} a_n (rz^n - z^{n+1})\right) \Rightarrow (a_n^\delta \to a_n).$$

- **Decomposition:**

$$\begin{aligned}&\mathsf{Cross}_{A(r,R)}(\mathrm{CLE}) \le \mathsf{Clus}_{A(r+a,R-a)}(\mathcal{B}_{<a})\\ &+\mathsf{Cross}_{A(R-a,R)}(\mathcal{B}_{>a}) + \mathsf{Cross}_{A(r,r+a)}(\mathcal{B}_{>a})\end{aligned}$$



- **Iteration:** $f(n) := \sup_{U\subseteq\mathbb{H}} \mathbb{P}\left[\#\{\text{crossing clusters of } A(r,R)\} \ge n\right]$, for any $s > 0$,

$$f(n+1) \le \frac{s}{2} f(n) + cq^n \cdot f((1-\epsilon)n) + O(s^{2n}).$$

1. whether the n-th crossing cluster hits the shaded region
2. for $r < r' < \rho' < R' < R$ and any $s > 0$,

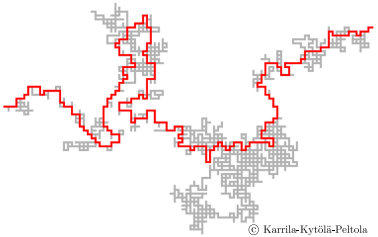
$$\begin{aligned}\mathbb{P}[\mathsf{Clus}_{A(r,R)}(\mathcal{B}(\rho,P)) \ge n] \le\ & \mathbb{P}[\mathsf{Clus}_{A(r,r')}(\mathcal{B}(\rho,\rho')) \ge (1-\epsilon)n]\\ & \times \mathbb{P}[\mathsf{Clus}_{A(R',R)}(\mathcal{B}(\rho',P)) \ge (1-\epsilon)n] + O(s^n).\end{aligned}$$

Convergence of massive LERWs to massive SLE(2) in dimension 2

$(\Omega^\delta; a^\delta, b^\delta)$ subgraph of $\delta\mathbb{Z}^2$,

- Loop erasure of a simple random walk started from a^δ on Ω^δ conditioned on hitting $\partial\Omega^\delta$ at b^δ.



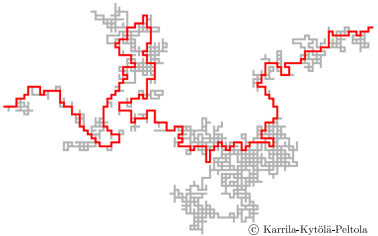


γ: simple path from a^δ to b^δ in Ω^δ,

$$w(\gamma) = \sum_{S:\mathrm{LE}(S)=\gamma} \left(\frac{1}{4}\right)^{|S|}$$

Fix $m > 0$, $(\Omega^\delta; a^\delta, b^\delta)$ subgraph of $\delta\mathbb{Z}^2$,

- Loop erasure of a simple random walk started from a^δ on Ω^δ **with probability $m^2\delta^2$ to be killed at each step**;
- **Conditioned to hit the boundary at b^δ before dying**: simple path from a^δ to b^δ in Ω^δ.



γ: simple path from a^δ to b^δ in Ω^δ,

$$w^{(m)}(\gamma) = \sum_{S:\mathrm{LE}(S)=\gamma} \left(\frac{1-m^2\delta^2}{4}\right)^{|S|}$$

Theorem (Lawler, Schramm and Werner, '03)

Approximate $(\Omega; a, b)$ *with* C^1*-smooth* $\partial\Omega$ *by a subgraph* $(\Omega^\delta; a^\delta, b^\delta)$ *of* $\delta\mathbb{Z}^2$*. As* $\delta \to 0$*,*

$$\textit{LERWs from } a^\delta \textit{ to } b^\delta \textit{ in } \Omega^\delta \xrightarrow{(d)} \mathrm{SLE}_2 \textit{ from } a \textit{ to } b \textit{ in } \Omega,$$

the uniform distance modulo parametrization $d(\gamma, \tilde{\gamma}) = \inf_\varphi \sup_{t \geq 0} |\gamma(t) - \tilde{\gamma}(\varphi(t))|$.

Remark

- As $\dim_H(\mathrm{SLE}_2) = 5/4$, SLE_2 be parametrized by the $5/4$-dimensional Minkowski content (the natural parametrization)
- **[Lawler-Viklund '16]**: discrete curves parametrized so that each edge is traversed in the time duration $\delta^{-5/4}$ converge in probability

$Z^{(m)}$: partition function of mLERW = partition function of mSRW

Lemma (Chelkak and W., '19)

$$\frac{Z^{(m)}_{\Omega^\delta}(a^\delta, b^\delta)}{Z^{(0)}_{\Omega^\delta}(a^\delta, b^\delta)} \geq \exp(-cm^2).$$

⋆ **Robust estimates which survive in continuum.**

By Jensen's inequality,

$$\frac{Z^{(m)}_{\Omega^\delta}(a^\delta, b^\delta)}{Z^{(0)}_{\Omega^\delta}(a^\delta, b^\delta)} \geq \mathbb{E}[(1-m^2\delta^2)^{|S^\delta|}] \geq (1-m^2\delta^2)^{\mathbb{E}[\#S^\delta]}.$$

Random walk estimates give $\mathbb{E}[\#S^\delta] \leq \text{const} \cdot \delta^{-2}$ uniformly on Ω^δ. □

Theorem (Chelkak and W., EJP '19)

Approximate **any simply connected** $(\Omega; a, b)$ *by a subgraph* $(\Omega^\delta; a^\delta, b^\delta)$ *of* $\delta\mathbb{Z}^2$,

$$\textit{mLERWs from } a^\delta \textit{ to } b^\delta \textit{ in } \Omega^\delta \xrightarrow[\delta\to 0]{(d)} \mathrm{mSLE}_2 \textit{ from } a \textit{ to } b \textit{ in } \Omega,$$

with driving function of Loewner's equation being $\mathrm{d}\xi_t = \sqrt{2}\mathrm{d}B_t + 2\lambda_t \mathrm{d}t$, *where* $\lambda_t = \frac{\partial}{\partial g(a_t)} \log \frac{P^{(m)}_{\Omega_t}(a_t,b)}{P_{\Omega_t}(a_t,b)}$.

Remark

- A similar convergence holds for the radial case.
- Not just a result of Girsanov transform? The density of mSLE(2) with respect to the classical SLE(2) does not coincide with $\frac{P^{(m)}_{\Omega_t}(a_t,b)}{P_{\Omega_t}(a_t,b)}$.

1. **Precompactness**: a weak crossing estimate implies that the random curve will have **subsequential scaling limits** and those will be **well-described by Loewner evolutions** with random driving forces **[Kemppainen and Smirnov, '12]**.
2. Under some perturbations of the critical statistical physics model, a family of discrete massive martingale observables converge to its continuum limit $(M_t^{(m)}(z))_{t\geq 0}$ for all $z \in \Omega$: **massive harmonic or massive holomorphic**;

$$M_n^{(m)}(\gamma[0,n]) = \frac{\mathbb{P}^{(m)}[\mathrm{LE}(S^v[0,n]) = \gamma[0,n]]}{\mathbb{P}^{(m)}[\mathrm{LE}(S^b[0,n]) = \gamma[0,n]]} = \mathrm{const} \cdot \frac{Z^{(m)}(\gamma(n), v)}{Z^{(m)}(\gamma(n), b)}.$$

3. **Identity the driving function** of Loewner's evolution such that the drift of $(M_t^{(m)}(z))_{t\geq 0}$ for all z vanishes in Itô's sense.

$(\Omega^\delta; a^\delta, b^\delta, v^\delta) \to (\Omega; a, b, v)$, $\gamma^\delta[0, n(t)] \to \gamma(t)$ almost surely,

- As $\delta \to 0$,

$$M^\delta_{n(t)}(v^\delta) \to P_t(v) := -\frac{1}{\pi}\mathrm{Im}\frac{1}{g_t(v) - \xi_t}, \text{ martingale for } \gamma;$$

- The drift of

$$\begin{aligned} \mathrm{d}P_t(v) &= -\frac{1}{\pi}\mathrm{dIm}\frac{1}{g_t(v) - \xi_t} \\ &= -\frac{1}{\pi}\mathrm{Im}\left[\frac{\mathrm{d}\xi_t}{(g_t(v) - \xi_t)^2} + \frac{\mathrm{d}\langle\xi, \xi\rangle_t - 2\mathrm{d}t}{(g_t(v) - \xi_t)^3}\right]; \end{aligned}$$

  vanishes for all v.

- Both processes ξ_t and $\langle\xi, \xi\rangle_t - 2\mathrm{d}t$ are (local) martingales. By Lévy's characterization of Brownian motion, $\xi_t = \sqrt{2}B_t$.

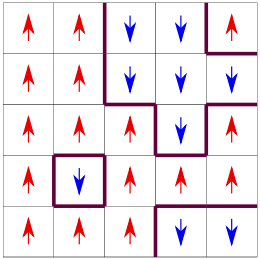
Convergence of near-critical Ising energy densities and dimer height fluctuations

Ising model: random assignment of interacting spins according to

$$\prod_{e\in\mathcal{E}(\Omega^\delta)} \exp(\beta J_e \sigma_{e_+}\sigma_{e_-}).$$



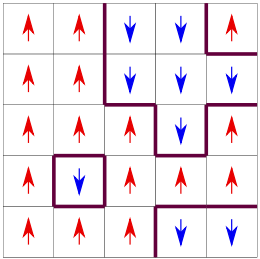
Invented by the physicist **Wilhelm Lenz** (1920), who gave it as a problem to his student **Ernst Ising**; The **1D** Ising model was solved by Ising that it has **no phase transition**.



Ising model: random assignment of interacting spins according to $\prod_{e\in\mathcal{E}(\Omega^\delta)} \exp(-\beta J_e \sigma_{e_+}\sigma_{e_-})$.

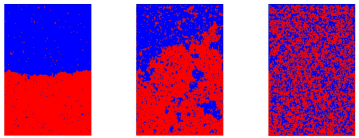


Invented by the physicist **Wilhelm Lenz** (1920), who gave it as a problem to his student **Ernst Ising**; The **1D** Ising model was solved by Ising that it has **no phase transition**.



In the following: 2D square grid, $J_e = J$, $\beta_{\text{crit}}\, J = \frac{1}{2}\log(\sqrt{2}+1)$.

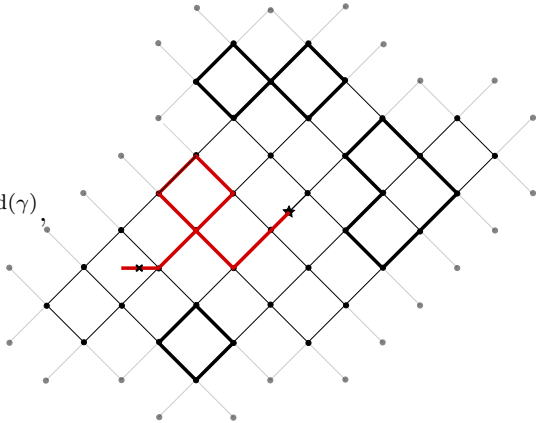
- $J > 0$: **ferromagnetic**, which favors the alignment of neighboring spins;
- **Phase transition**: the correlation changes drastically at a particular temperature



- **Smirnov's two-point fermion correlators**

$$F_a^\delta(z) := \frac{1}{\mathcal{Z}_{\Omega^\delta}} \sum_{\gamma \in \mathcal{C}_{\Omega^\delta}(a,z)} \alpha^{|\gamma|} e^{-\frac{i}{2}\mathrm{wind}(\gamma)},$$

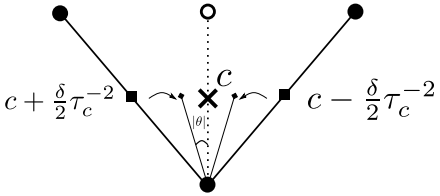
with $\alpha = \exp(-2\beta J)$.



- **Two-point fermion correlators**

$$F_a^\delta(z) := \frac{1}{\mathcal{Z}_{\Omega^\delta}} \sum_{\gamma \in \mathcal{C}_{\Omega^\delta}(a,z)} \alpha^{|\gamma|} e^{-\frac{i}{2}\mathrm{wind}(\gamma)},$$

with $\alpha = \exp(-2\beta J)$.



- **Massive s-holomorphicity**

$$e^{-i\Theta}\mathrm{Proj}_{\tau_c e^{i\Theta}}\left[F_a^\delta\left(c - \frac{\delta}{2}\tau_c^{-2}\right)\right] = e^{i\Theta}\mathrm{Proj}_{\tau_c e^{-i\Theta}}\left[F_a^\delta\left(c + \frac{\delta}{2}\tau_c^{-2}\right)\right].$$

- **Integral of the square**

$$H^\delta := \int^z \mathrm{Im}[(F^\delta(z))^2\, dz];$$

- **Riemann boundary condition** $F_a^\delta(z) \parallel \nu_{\mathsf{tan}}(z)^{-1/2}$ for $z \in \partial\Omega^\delta$.

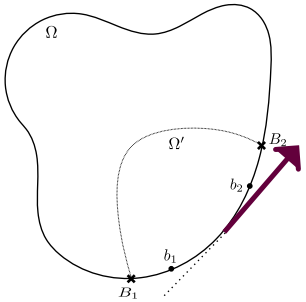
$f(z) \parallel \nu_{\mathsf{tan}}(z)^{-1/2}$ admits **two equivalent interpretations**:

(h) $h := \mathrm{Im} \int^z f^2 dz$ extends continuously to $(B_1 B_2)$ as a constant; for each $[b_1 b_2] \subset (B_1 B_2)$, $g := \Im \int^z f f_{\mathsf{FK}} dz$ extends continuously to $(b_1 b_2)$ as a constant.

(f) $f^{\mathbb{D}} := (f \circ \varphi) \cdot (\varphi')^{\frac{1}{2}}$ with some fixed $\varphi : \mathbb{D} \to \Omega'$

$$\sup_{r<1} \int_{r[\varphi^{-1}(b_1)\varphi^{-1}(b_2)]} \left| f^{\mathbb{D}} \right|^2 |dz| < \infty.$$

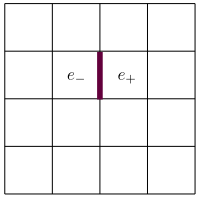
Moreover $\int_{r[\varphi^{-1}(b_1)\varphi^{-1}(b_2)]} \left| \mathrm{Im} \left[(iz)^{1/2} f^{\mathbb{D}} \right] \right|^2 |dz| \xrightarrow{r \to 0} 0$



Remark

(h) is satisfied due to a priori estimates of fermionic observables.

$\epsilon(e) = \sqrt{2}\sigma_{e_+}\sigma_{e_-} - 1$: whether or not to see or not see an edge in contour configuration.



Behavior at and near T_c:

Theorem (Convergence of energy densities in the massive Ising model)

$$\delta^{-n} \cdot \mathbb{E}[\varepsilon_{z_1} \dots \varepsilon_{z_n}] \xrightarrow{\delta \to 0} \left(\tfrac{i}{\pi}\right)^n \langle \psi_{z_1} \psi^\star_{z_1} \dots \psi_{z_n} \psi^\star_{z_n} \rangle = \left(\tfrac{i}{\pi}\right)^n \cdot \mathrm{Pf}[A_{ij}]_{i,j=1}^{2n},$$

where the antisymmetric matrix A is formed by correlators of $\psi_\cdot, \psi^*_\cdot$ such that

$$F_a^\delta(z) \to \overline{\eta}_a \langle \psi_z \psi_a \rangle + \eta_a \langle \psi_z \psi^\star_a \rangle \text{ as } \delta \to 0.$$

the coulping function = the fermionic observable

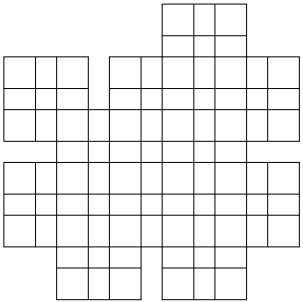
Theorem (Convergence of height gradient field in the massive dimer model on hedgehog domains)

$\mathbb{E}[\mathrm{d}h^\delta(z_1)\dots\mathrm{d}h^\delta(z_n)]$ *converge uniformly on all compact subsets of* Ω *to* $(1/4i\pi)^n$ *times*

$$\sum_{s_1,\dots,s_n\in\{\pm\}} \det[\,\mathbf{1}_{j\neq k} f^{[s_j,s_k]}(z_j,z_k)\,]_{j,k=1}^n \prod_{k=1}^n \mathrm{d}z_k^{[s_k]},$$

where $f^{[++]}(z_j,z_k)=\overline{f^{[--]}(z_j,z_k)}:=\langle\psi_{z_k}\psi_{z_j}\rangle$ *and* $f^{[-+]}(z_j,z_k)=\overline{f^{[+-]}(z_j,z_k)}:=\langle\psi_{z_k}\psi^*_{z_j}\rangle$.

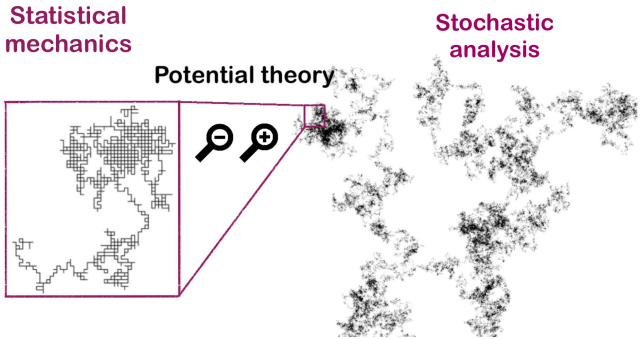
Dimer graph: weights and boundary shape



- **The limiting height function in the massive dimer model**: under which microscopic structure of the boundary, the dimer height profile would correspond to the sine-Gordon model?
- **Interface in the massive Ising model**:
  (a) convergence of martingle observables

$$f^{(m)}_{\Omega_t}(z) = f_{\Omega_t}(z) - mi \iint_{\Omega_t} \langle \epsilon_z \epsilon_w \rangle^{(m)}_{\Omega_t} \diamond f_{\Omega_t}(w)\, \mathrm{d}^2 w,$$

  where $f_{\Omega_t}(z) = \frac{\sqrt{g_t'(z)}}{g_t(z)-\xi_t}$ is the martingale observable for the critical Ising model;
  (b) identification of the limit of interfaces similarly to mLERW?



Thank you for your attention!