$\mathcal{N}=2$ tensor multiplet in six dimensions

In this problem we will study the Abelian $\mathcal{N}=2$ tensor multiplet in six dimensions. Finding a non-Abelian version of this multiplet is a famous unsolved problem. We denote Lorentz vectors indices by $\mu,\nu=0,\dots,5$, R-symmetry indices by $i,j=1,\dots,4$ and Lorentz spinor indices $\alpha,\beta=1,\dots,4$.

We will use an invariant matrix Ω_{ij}, $\Omega=-\Omega^{\intercal}$ and $\Omega_{ij}\Omega^{jk}=-\delta_i^k$. The theory has supercharges $\mathrm{Q}_{\alpha i}$ which satisfy the algebra $\{\mathrm{Q}_{\alpha i},\mathrm{Q}_{\beta j}\}=\Omega_{ij}\mathrm{P}_{\alpha\beta}$.

Answer the following questions (the questions marked by a ♣ are more difficult).

1. Given the symmetry of the $\{\mathrm{Q},\mathrm{Q}\}$ anti-commutator, find the symmetry property of $\mathrm{P}_{\alpha\beta}$. How many independent entries does $\mathrm{P}_{\alpha\beta}$ have? Compare with the number of entries of a six-dimensional vector.

2. We will build a multiplet by acting on a field Φ_{ij} with $\Phi_{ij}=-\Phi_{ji}$ and $\Phi_{ij}\Omega^{ij}=0$. Count the number of degrees of freedom of Φ. If Φ are scalar fields in six dimensions, what is the mass dimension of Φ? What is the mass dimension of P and Q?

3. We take the action of Q on Φ to be schematically $[\mathrm{Q},\Phi]\sim\Omega\Psi$. Fill in the missing indices, making sure that the symmetries and constraints on Φ are satisfied. What is the mass dimension of Ψ? Compare to the mass dimension of a fermion in six dimensions.

4. We take the action of Q on Ψ to be
$$\{Q_{\alpha i},\Psi_{\beta j}\}=\Omega_{ij}H_{\alpha\beta}+c\partial_{\alpha\beta}\Phi_{ij},\tag{1}$$
where c is a constant. Write H in terms of $\{\mathrm{Q},\Psi\}$ by using the constraint satisfied by Φ. Determine the symmetry of $H_{\alpha\beta}$ under the permutation $\alpha\leftrightarrow\beta$.

5. ♣ Determine the value of c for which the action of Q is compatible with the algebra. Hint: you need insure that $[\{\mathrm{Q}_{\alpha i},\mathrm{Q}_{\beta j}\},\Phi_{kl}]=\Omega_{ij}\partial_{\alpha\beta}\Phi_{kl}$ holds. Also, you will need the super-Jacobi identity $\{\mathrm{F},[\mathrm{F}',\mathrm{B}]\}=[\{\mathrm{F},\mathrm{F}'\},\mathrm{B}]-\{\mathrm{F}',[\mathrm{F},\mathrm{B}]\}$ where F, F′ are Grassmann odd generators and B is Grassmann even.

6. Given the identities $\sigma^{\mu}_{\alpha\beta}=-\sigma^{\mu}_{\beta\alpha}$ and $\sigma^{\mu\nu\rho}_{\alpha\beta}=\sigma^{\mu\nu\rho}_{\beta\alpha}$ and $\sigma^{\mu\nu\rho}_{\alpha\beta}=\frac{1}{3!}\epsilon^{\mu\nu\rho}{}_{\mu'\nu'\rho'}\sigma^{\mu'\nu'\rho'}_{\alpha\beta}$, we have that a symmetric bi-spinor $G_{\alpha\beta}$ can be put in correspondence with a three-form $G_{\mu\nu\rho}$ by $G_{\alpha\beta}=\frac{1}{3!}\sigma^{\mu\nu\rho}_{\alpha\beta}G_{\mu\nu\rho}$. The three-form G satisfies another constraint. What is it?

7. How many independent components are there in a three-form in six dimensions? How many independent components are there in H with the symmetry property discussed above? Do these numbers match? Do they match after imposing the constraint discussed in the previous question?

8. ♣ Compute the Jacobi relation for $\mathrm{Q}_{\alpha i}$, $\mathrm{Q}_{\beta j}$ and $\Psi_{\gamma k}$ and find a relation between $[\mathrm{Q}, H]$ and $\partial\Psi$. Using the fact that $\bar{\sigma}^{\mu,\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\alpha'\beta'}\sigma^{\mu}_{\alpha'\beta'}$, show that Ψ satisfies the Dirac equation $\partial^{\alpha\beta}\Psi_{\beta k} = 0$.

9. ♣ Apply a supersymmetry transformation to the Dirac equation and find the equations satisfied by Φ and H. Use the identity $\bar{\sigma}^{\mu}\sigma^{\nu} + \bar{\sigma}^{\nu}\sigma^{\mu} = 2\eta^{\mu\nu}$ to show that the scalar fields satisfy the free scalar equations of motion.