(1.) $\{Q_{\alpha i}, Q_{\beta j}\} = \Omega_{ij} P_{\alpha\beta}$

$$= \{Q_{\beta j}, Q_{\alpha i}\} = \Omega_{ji} P_{\beta\alpha} = -\Omega_{ij} P_{\beta\alpha}$$

$$\Rightarrow \quad \boxed{P_{\alpha\beta} = -P_{\beta\alpha}}$$

The indices α, β take 4 values. The number of independent components in $P_{\alpha\beta}$ is $\binom{4}{2} = 6$. This is the same number as in six-dimensional vector.

(2.) ϕ_{ij}, with $\phi_{ij} = -\phi_{ji}$, $\phi_{ij}\Omega^{ij} = 0$

The number of components in an antisymmetric rank 2 tensor is $\binom{4}{2} = 6$. If we subtract one constraint from $\phi_{ij}\Omega^{ij} = 0$ we obtain 5 degrees of freedom.

In a d-dimensional field theory the action for a scalar is

$$S = \int d^d x \, (\partial\phi)^2$$

and it's dimensionless. Counting in mass dimension, this means that

$$-d + (1 + [\phi])2 = 0 \quad \Rightarrow \quad [\phi] = \frac{d}{2} - 1$$

If $d = 6 \Rightarrow \boxed{[\phi] = 2}$

The momentum P is represented by ∂ so $\boxed{[P] = 1}$. The anti-commutation relation $\{Q, Q\} = P$ implies $\boxed{[Q] = \frac{1}{2}}$

(3.)

$$[Q_{\alpha i}, \phi_{jk}] = \Omega_{ij}\Psi_{\alpha k} - \Omega_{ik}\Psi_{\alpha j} + c\,\Omega_{jk}\Psi_{\alpha i}$$

where c needs to be adjusted so that

$$[Q_{\alpha i}, \phi_{jk}\Omega^{jk}] = 0$$

This implies

$$0 = \Omega_{ij}\Omega^{jk}\Psi_{\alpha k} - \Omega_{ik}\Omega^{jk}\Psi_{\alpha j} + c\,\Omega_{jk}\Omega^{jk}\Psi_{\alpha i}$$

$$= -\delta_i^k \Psi_{\alpha k} - \delta_i^j \Psi_{\alpha j} + 4c\,\Psi_{\alpha i} = -2\Psi_{\alpha i} + 4c\,\Psi_{\alpha i} \Rightarrow c = \frac{1}{2}$$

In conclusion,

$$\boxed{[Q_{\alpha i}, \phi_{jk}] = \Omega_{ij}\Psi_{\alpha k} - \Omega_{ik}\Psi_{\alpha j} + \frac{1}{2}\Omega_{jk}\Psi_{\alpha i}}$$

The mass dimension of Ψ is

$$[\Psi] = [Q] + [\phi] = \frac{1}{2} + 2 = \frac{5}{2}$$

The action for a fermion in d dimensions is

$$S = \int d^d x \, \bar{\Psi}\not{\partial}\Psi \quad \Rightarrow \quad -d + 2[\Psi] + 1 = 0 \Rightarrow [\Psi] = \frac{d-1}{2}$$

If $d = 6 \Rightarrow [\Psi] = 5/2$, canonical dimension.

(4.) If

$$\{Q_{\alpha i}, \Psi_{\beta j}\} = \Omega_{ij} H_{\alpha\beta} + c\,\partial_{\alpha\beta}\,\phi_{ij}$$

Contract with Ω^{ij}

$$\Rightarrow \quad \{Q_{\alpha i}, \Psi_{\beta j}\}\,\Omega^{ij} = 4 H_{\alpha\beta}, \quad \text{since} \quad \phi_{ij}\Omega^{ij} = 0$$

$$\Rightarrow \quad \boxed{H_{\alpha\beta} = \frac{1}{4}\{Q_{\alpha i}, \Psi_{\beta j}\}\,\Omega^{ij}}$$

In order to determine the symmetry of $H_{\alpha\beta}$, express Ψ in terms of $[Q, \phi]$.

$$[Q_{\alpha i}, \Psi_{jk}]\,\Omega^{ij} = 4\Psi_{\alpha k} - \Psi_{\alpha k} - \frac{1}{2}\Psi_{\alpha k} = \frac{5}{2}\Psi_{\alpha k}$$

$$\Rightarrow \quad \boxed{\Psi_{\alpha k} = \frac{2}{5}[Q_{\alpha i}, \phi_{jk}]\,\Omega^{ij}}$$

Now replace in the expression for H

$$H_{\alpha\beta} = \frac{1}{10}\{Q_{\alpha i}, [Q_{\beta l}, \phi_{mj}]\}\,\Omega^{ij}\Omega^{lm}$$

$$= \frac{1}{10}\Big([\{Q_{\alpha i}, Q_{\beta l}\}, \phi_{mj}] - \{Q_{\beta l}, [Q_{\alpha i}, \phi_{mj}]\}\Big)\Omega^{ij}\Omega^{lm}$$

$$= \frac{1}{10}\Big(\Omega_{il}\,[\cancel{P_{\alpha\beta}}, \phi_{mj}] - \{Q_{\beta l}, [Q_{\alpha i}, \phi_{mj}]\}\Big)\Omega^{ij}\Omega^{lm}$$

$$= \frac{1}{10}\{Q_{\beta l}, [Q_{\alpha i}, \phi_{jm}]\}\,\Omega^{ij}\Omega^{lm}$$

$$= H_{\beta\alpha}$$

$$\Rightarrow \quad \boxed{H_{\alpha\beta} = H_{\beta\alpha}}$$

We want

$$[\{Q_{\alpha i}, Q_{\beta j}\}, \phi_{kl}] = \Omega_{ij}\,\partial_{\alpha\beta}\,\phi_{kl}$$

$$\{Q_{\alpha i}, [Q_{\beta j}, \phi_{kl}]\} = \{Q_{\alpha i}, \Omega_{jk}\Psi_{\beta l} - \Omega_{jl}\Psi_{\beta k} + \frac{1}{2}\Omega_{kl}\Psi_{\beta j}\}$$

$$= \Omega_{jk}(\Omega_{il}H_{\alpha\beta} + c\,\partial_{\alpha\beta}\phi_{il})$$

$$- \Omega_{jl}(\Omega_{ik}H_{\alpha\beta} + c\,\partial_{\alpha\beta}\phi_{ik})$$

$$+ \frac{1}{2}\Omega_{kl}(\Omega_{ij}H_{\alpha\beta} + c\,\partial_{\alpha\beta}\phi_{ij})$$

Also,

$$\{Q_{\alpha i}, [Q_{\beta j}, \phi_{kl}]\} = [\{Q_{\alpha i}, Q_{\beta j}\}, \phi_{kl}] - \{Q_{\beta j}, [Q_{\alpha i}, \phi_{kl}$$

$$\Rightarrow \quad [\{Q_{\alpha i}, Q_{\beta j}\}, \phi_{kl}] = \{Q_{\alpha i}, [Q_{\beta j}, \phi_{kl}]\} + \{Q_{\beta j}, [Q_{\alpha i}, \phi_{kl}]\}$$

$$= \Omega_{jk}(\cancel{\Omega_{il}} H_{\alpha\beta} + c\,\partial_{\alpha\beta}\phi_{il})$$
$$- \Omega_{jl}(\cancel{\Omega_{ik}} H_{\alpha\beta} + c\,\partial_{\alpha\beta}\phi_{ik})$$
$$+ \frac{1}{2}\Omega_{kl}(\cancel{\Omega_{ij} H_{\alpha\beta}} + c\,\cancel{\partial_{\alpha\beta}\phi_{ij}})$$
$$+ \Omega_{ik}(\cancel{\Omega_{jl}} H_{\beta\alpha} + c\,\partial_{\beta\alpha}\phi_{jl})$$
$$- \Omega_{il}(\cancel{\Omega_{jk}} H_{\beta\alpha} + c\,\partial_{\beta\alpha}\phi_{jk})$$
$$+ \frac{1}{2}\Omega_{kl}(+\cancel{\Omega_{ji}} H_{\beta\alpha} + c\,\cancel{\partial_{\beta\alpha}\phi_{ji}})$$

$$= c\,(\Omega_{jk}\partial_{\alpha\beta}\phi_{il} - \Omega_{jl}\partial_{\alpha\beta}\phi_{ik} - \Omega_{ik}\partial_{\alpha\beta}\phi_{jl} + \Omega_{il}\partial_{\alpha\beta}\phi_{jk} + \Omega_{kl}\partial_{\alpha\beta}\phi_{ij})$$

$$= c\,\partial_{\alpha\beta}(\Omega_{jk}\phi_{il} - \Omega_{jl}\phi_{ik} - \Omega_{ik}\phi_{jl} + \Omega_{il}\phi_{jk} + \Omega_{kl}\phi_{ij} + \Omega_{ij}\phi_{kl})$$
$$- c\,\Omega_{ij}\partial_{\alpha\beta}\phi_{kl}.$$

The combination

$$T_{ijkl} = \Omega_{jk}\phi_{il} - \Omega_{jl}\phi_{ik} - \Omega_{ik}\phi_{jl} + \Omega_{il}\phi_{jk} + \Omega_{kl}\phi_{ij} + \Omega_{ij}\phi_{kl}$$

is totally antisymmetric in $ijkl$. Therefore, we have $T_{ijkl} = \varepsilon_{ijkl} T$.
In order to find the proportionality constant, contract with Ω^{ij}

$$\Rightarrow \quad \varepsilon_{ijkl}\Omega^{ij} T = -\phi_{kl} - \phi_{kl} - \phi_{kl} - \phi_{kl} + 0 + 4\phi_{kl} = 0$$

$$\Rightarrow \quad T = 0 \quad \Rightarrow \quad T_{ijkl} = 0$$

In conclusion,

$$[\{Q_{\alpha i}, Q_{\beta j}\}, \phi_{kl}] = -c\,\Omega_{ij}\partial_{\alpha\beta}\phi_{kl}$$

$$\Rightarrow \quad \boxed{c = -1}$$

6. We have $G_{\alpha\beta} = \frac{1}{3!}\sigma^{\mu\nu\rho}{}_{\alpha\beta}G_{\mu\nu\rho} = \frac{1}{3!}\left(\frac{1}{3!}\varepsilon^{\mu\nu\rho}{}_{\mu'\nu'\rho'}\sigma^{\mu'\nu'\rho'}{}_{\alpha\beta}G_{\mu\nu\rho}\right)$

$$= \frac{1}{3!}\sigma^{\mu'\nu'\rho'}{}_{\alpha\beta}\left(\frac{1}{3!}G_{\mu\nu\rho}\varepsilon^{\mu\nu\rho}{}_{\mu'\nu'\rho'}\right)$$

If we set

$$\frac{1}{3!}G_{\mu\nu\rho}\varepsilon^{\mu\nu\rho}{}_{\mu'\nu'\rho'} = G_{\mu'\nu'\rho'}$$

then we obtain an equality.

We can decompose a rank 3 antisymmetric tensor G into two parts G^+, G^- where $G^{\pm}_{\mu'\nu'\rho'} = \pm\frac{1}{3}G^{\pm}_{\mu\nu\rho}\varepsilon^{\mu\nu\rho}{}_{\mu'\nu'\rho'}$. The G^- part gets projected out.

Only the G^+ part survives. Therefore, we have

$$\boxed{G_{\mu'\nu'\rho'} = \frac{1}{3!} G_{\mu\nu\rho}\, \varepsilon^{\mu\nu\rho}{}_{\mu'\nu'\rho'}}$$

(7) A three-form G in six dimensions has $\binom{6}{3} = \frac{6!}{3!\,3!} = \frac{\cancel{2\cdot 3}\cdot 4\cdot 5\cdot \cancel{6}}{\cancel{2\cdot 3}\cdot \cancel{2\cdot 3}} = 20$ components. They split in 10 components for G^+ and 10 components for G^-.

A symmetric tensor $G_{\alpha\beta}$ has $\frac{4\cdot 5}{2} = 10$ components. We have a match between the number of degrees of freedom of $G_{\alpha\beta}$ and $G^+_{\mu\nu\rho}$.

(8)

$$[Q_{\alpha i}, \{Q_{\beta j}, \Psi_{\gamma k}\}] = [\{Q_{\alpha i}, Q_{\beta j}\}, \Psi_{\gamma k}] - [Q_{\beta j}, \{Q_{\alpha i}, \Psi_{\gamma k}\}]$$

$$[Q_{\alpha i}, \{Q_{\beta j}, \Psi_{\gamma k}\}] = [Q_{\alpha i}, \Omega_{jk} H_{\beta\gamma} - \partial_{\beta\gamma}\phi_{jk}]$$

$$= \Omega_{jk}[Q_{\alpha i}, H_{\beta\gamma}] - \partial_{\beta\gamma}(\Omega_{ij}\Psi_{\alpha k} - \Omega_{ik}\Psi_{\alpha j} + \tfrac{1}{2}\Omega_{jk}\Psi_{\alpha 0})$$

$$\Rightarrow \quad \Omega_{jk}[Q_{\alpha i}, H_{\beta\gamma}] - \partial_{\beta\gamma}(\Omega_{ij}\Psi_{\alpha k} - \Omega_{ik}\Psi_{\alpha j} + \tfrac{1}{2}\Omega_{jk}\Psi_{\alpha i})$$
$$+ \Omega_{ik}[Q_{\beta j}, H_{\alpha\gamma}] - \partial_{\alpha\gamma}(\Omega_{ji}\Psi_{\beta k} - \Omega_{jk}\Psi_{\beta i} + \tfrac{1}{2}\Omega_{ik}\Psi_{\beta j})$$
$$- \Omega_{ij}\partial_{\alpha\beta}\Psi_{\gamma k} = 0$$

Contract with Ω^{ij}

$$\Rightarrow \quad -[Q_{\alpha k}, H_{\beta\gamma}] - \partial_{\beta\gamma}(4\Psi_{\alpha k} - \Psi_{\alpha k} - \tfrac{1}{2}\Psi_{\alpha k})$$
$$+ [Q_{\beta k}, H_{\alpha\gamma}] - \partial_{\alpha\gamma}(-4\Psi_{\beta k} + \Psi_{\beta k} + \tfrac{1}{2}\Psi_{\beta k}) - 4\partial_{\alpha\beta}\Psi_{\gamma k} = 0$$

$$\Rightarrow \quad -[Q_{\alpha k}, H_{\beta\gamma}] - \tfrac{5}{2}\partial_{\beta\gamma}\Psi_{\alpha k} + [Q_{\beta k}, H_{\alpha\gamma}] + \tfrac{5}{2}\partial_{\alpha\gamma}\Psi_{\beta k} - 4\partial_{\alpha\beta}\Psi_{\gamma k} = 0$$

Subtract the $\beta \leftrightarrow \gamma$ permutation

$$-2\tfrac{5}{2}\partial_{\beta\gamma}\Psi_{\alpha k} + [Q_{\beta k}, H_{\alpha\gamma}] - [Q_{\gamma k}, H_{\alpha\beta}] + \tfrac{5}{2}\partial_{\alpha\gamma}\Psi_{\beta k} - \tfrac{5}{2}\partial_{\alpha\beta}\Psi_{\gamma k}$$
$$- 4\partial_{\alpha\beta}\Psi_{\gamma k} + 4\partial_{\alpha\gamma}\Psi_{\beta k} = 0$$

If we contract with $\varepsilon^{\alpha\beta\gamma\delta}$, then this projects out H.

$$\Rightarrow \quad -\tfrac{5}{2}\bar{\partial}^{\alpha\delta}\Psi_{\alpha k} - \tfrac{5}{2}\bar{\partial}^{\beta\delta}\Psi_{\beta k} - 4\bar{\partial}^{\gamma\delta}\Psi_{\gamma k} = 0$$

$$\Rightarrow \quad \boxed{\bar{\partial}^{\alpha\beta}\Psi_{\beta k} = 0}$$

(5.) $0 = \{ Q_{\alpha i}, \bar{\partial}^{\beta\gamma} \psi_{\gamma j} \} = \bar{\partial}^{\beta\gamma} (\Omega_{ij} H_{\alpha\gamma} - \partial_{\alpha\gamma} \phi_{ij})$

$= \Omega_{ij} \bar{\partial}^{\beta\gamma} H_{\alpha\gamma} + \delta^{\beta}_{\gamma} \Box \phi_{ij}$

Contract with Ω^{ij} $\Rightarrow$ $\bar{\partial}^{\beta\gamma} H_{\alpha\gamma} = 0$

$\Rightarrow$ $\Box \phi_{ij} = 0$