King's College London

University Of London

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Candidate No: .... **Desk No:** ....

MSc and MSci Examination

7CCMMS40T Supersymmetry (MSc)
7CCMMS40U Supersymmetry (MSci)

Summer 2018

Time Allowed: Two Hours

All questions carry equal marks.

Full marks will be awarded for complete answers to FOUR questions.

If more than four questions are attempted, then only the best FOUR will count.

NO calculators are permitted.

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Questions

1. (Pauli matrices)

The four-dimensional Pauli matrices are

$$\sigma^0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$\bar{\sigma}^0 = \sigma^0, \qquad \bar{\sigma}^1 = -\sigma^1, \qquad \bar{\sigma}^2 = -\sigma^2, \qquad \bar{\sigma}^3 = -\sigma^3.$$

They satisfy the relations

$$\sigma_{\mu,\alpha\dot{\alpha}}\bar{\sigma}_{\nu}^{\dot{\alpha}\beta} + \sigma_{\nu,\alpha\dot{\alpha}}\bar{\sigma}_{\mu}^{\dot{\alpha}\beta} = 2\eta_{\mu\nu}\delta_{\alpha}^{\beta}, \qquad \bar{\sigma}_{\mu}^{\dot{\alpha}\alpha}\sigma_{\nu,\alpha\dot{\beta}} + \bar{\sigma}_{\nu}^{\dot{\alpha}\alpha}\sigma_{\mu,\alpha\dot{\beta}} = 2\eta_{\mu\nu}\delta_{\dot{\beta}}^{\dot{\alpha}} \tag{1}$$

with $\eta_{\mu\nu} = \mathrm{diag}(+,-,-,-)$.

If x^μ is a vector, then we define two 2×2 matrices associated to it: $x_{\alpha\dot{\alpha}} = x_\mu\sigma^\mu_{\alpha\dot{\alpha}}$ and $x^{\dot{\alpha}\alpha} = x_\mu\bar{\sigma}^{\mu,\dot{\alpha}\alpha}$.

Given three vectors x^μ, y^μ and z^μ, we can define another vector $w_\mu = \epsilon_{\mu\nu\rho\sigma}x^\nu y^\rho z^\sigma$. The purpose of this exercise is to find the 2×2 matrix associated to w^μ in terms of the matrices associated to x, y and z.

1. Show that

$$x_{\alpha\dot{\alpha}}y^{\dot{\alpha}\beta} + y_{\alpha\dot{\alpha}}x^{\dot{\alpha}\beta} = 2(x\cdot y)\delta_{\alpha}^{\beta}, \qquad x^{\dot{\alpha}\alpha}y_{\alpha\dot{\beta}} + y^{\dot{\alpha}\alpha}x_{\alpha\dot{\beta}} = 2(x\cdot y)\delta_{\dot{\beta}}^{\dot{\alpha}} \tag{2}$$

   for any vectors x and y.

2. Define

$$t_{\alpha\dot{\beta}}(x,y,z) = x_{\alpha\dot{\alpha}}y^{\dot{\alpha}\beta}z_{\beta\dot{\beta}} - z_{\alpha\dot{\alpha}}y^{\dot{\alpha}\beta}x_{\beta\dot{\beta}}. \tag{3}$$

   It is obvious that $t(x,y,z) = -t(z,y,x)$. Show that

$$t(x,y,z) = -t(y,x,z), \qquad t(x,y,z) = -t(x,z,y). \tag{4}$$

   In other words, $t(x,y,z)$ is completely anti-symmetric in x, y and z.

3. The vector $w^\mu = \epsilon^{\mu\nu\rho\sigma}x_\nu y_\rho z_\sigma$ is also anti-symmetric in x, y and z hence $w_{\alpha\dot{\beta}}$ is proportional to $t_{\alpha\dot{\beta}}(x,y,z)$. Find the proportionality constant by taking $x_\mu = (0,1,0,0)$, $y_\mu = (0,0,1,0)$, $z_\mu = (0,0,0,1)$ and using $\epsilon^{0123} = 1$ to compute both $w_{\alpha\dot{\beta}}$ and $t_{\alpha\dot{\beta}}(x,y,z)$.

2. (Superspace equations of motion)

Consider the most general supersymmetric theory of chiral superfields Φ_i with action

$$S = \int d^4x d^2\theta d^2\bar{\theta} K(\bar{\Phi}^i, \Phi_j) + \int d^4x d^2\theta W(\Phi_i) + \int d^4x d^2\bar{\theta} \bar{W}(\bar{\Phi}^i). \tag{5}$$

You are given the following identities

$$\int d^4x d^2\theta (D^2 X) Y = \int d^4x d^2\theta X (D^2 Y), \tag{6}$$

$$\int d^4x d^2\bar{\theta} D^2 X = \frac{1}{4} \int d^4x d^2\theta d^2\bar{\theta} X, \tag{7}$$

where X, Y are superfields and $D^2 = D^\alpha D_\alpha$.

1. Show that the chirality constraint $D_\alpha \bar{\Phi}^i = 0$ is solved by $\bar{\Phi}^i = D^2 Y^i$.
2. Take the variation of the superfield $\bar{\Phi}^i$ to be $\delta\bar{\Phi}^i = D^2 \delta Y^i$. Using the identities above show that the superspace equation of motion can be written

$$D^2 \partial_i K + \frac{1}{4} \partial_i \bar{W} = 0. \tag{8}$$

3. Show that the equation of motion above can be rewritten

$$D^2 \Phi_l = (D_\alpha \Phi_j)(D^\alpha \Phi_k)(K^{-1})_l{}^i K_i{}^{jk} - \frac{1}{4}(K^{-1})_l{}^i \bar{W}_i, \tag{9}$$

where

$$\bar{W}_i = \frac{\partial \bar{W}}{\partial \bar{\Phi}^i}, \qquad K_i{}^j = \frac{\partial K}{\partial \bar{\Phi}^i \partial \Phi_j}, \qquad K_i{}^{jk} = \frac{\partial K}{\partial \bar{\Phi}^i \partial \Phi_j \partial \Phi_k}, \tag{10}$$

and we have assumed that $K_i{}^j$ is invertible as a matrix.

3. (Soft supersymmetry breaking)

A supersymmetric theory has a number of desirable features, such as cancellation of quadratic UV divergences. An important question is if we can break supersymmetry explicitly while still preserving these properties. It turns out that this is possible if we break supersymmetry as described below.

We will study a theory of chiral superfields in which we introduce some supersymmetry-breaking terms, with coupling constants of positive mass dimension, so as to preserve power-counting renormalizability.

1. Consider a number of chiral superfields Φ_i and a Kähler potential $\bar{\Phi}^i Z_i{}^j \Phi_j$, where Z is a dimensionless hermitian matrix. In a renormalizable supersymmetric theory Z is taken to be constant, however in this question we take
$$Z_i{}^j = \delta_i{}^j + A_i{}^j \theta^2 + \bar{A}_i{}^j \bar{\theta}^2 + B_i{}^j \theta^2 \bar{\theta}^2, \tag{11}$$
where $A_i{}^j$, $\bar{A}_i{}^j$ and $B_i{}^j$ are coupling constants, i.e. they do not depend on the coordinates $(x, \theta, \bar{\theta})$. Note that all the coefficients are Lorentz scalars since we do not want to explicitly break Lorentz symmetry.

   Show that after a field redefinition
$$\bar{X}^i = \bar{\Phi}^i(\delta_i^j + \bar{A}_j{}^i \bar{\theta}^2), \qquad X_i = (\delta_i^j + A_i{}^j \theta^2)\Phi_j \tag{12}$$
we get
$$\bar{X}^i \hat{Z}_i{}^j X_j = \bar{\Phi}^i Z_i{}^j \Phi_j. \tag{13}$$
Compute $\hat{Z}$ in terms of the components of Z.

2. What is the effect of the components of $\hat{Z}$ on the mass matrix of scalar fields $x_i = X_i|$? Is the mass matrix of the fermions affected by the components of $\hat{Z}$?

3. The superpotential for a renormalizable theory is
$$W(X) = \frac{1}{2} C_{ij} X^i X^j + \frac{1}{3!} C_{ijk} X^i X^j X^k. \tag{14}$$
Just like for $Z_i{}^j$, write the Grassmann expansion of C_{ij} and C_{ijk} consistent with the requirement of holomorphy and power-counting renormalizability (coupling constants of positive mass dimension).

4. It turns out that there is another possible term in the Lagrangian $\int d^2\theta d^2\bar{\theta} D^k_{ij} X^i X^j \bar{X}_k$ and its complex conjugate. What is the dimensions of D^k_{ij}? Find the terms in the Grassmann expansion of D^k_{ij} which are compatible with Lorentz symmetry and power-counting renormalizability.

4. ($\mathcal{N} = 4$ super-Yang-Mills, Kähler potential)

Take V to be a gauge multiplet and X_1, X_2, X_3 chiral multiplets with values in the Lie algebra of some gauge group G. We denote the hermitian conjugate multiplets by $\bar{X}^1$, $\bar{X}^2$, $\bar{X}^3$. These superfields have the following gauge transformations

$$X_i \to e^{i\Lambda} X_i e^{-i\Lambda}, \tag{15}$$

$$\bar{X}^i \to e^{i\bar{\Lambda}} \bar{X}^i e^{-i\bar{\Lambda}}, \tag{16}$$

$$e^{-2V} \to e^{i\bar{\Lambda}} e^{-2V} e^{-i\Lambda}, \tag{17}$$

where Λ is a chiral superfield ($\bar{D}_{\dot{\alpha}}\Lambda = 0$) and $\bar{\Lambda}$ is its hermitian conjugate.

1. Show that the gauge transformations above are compatible with the chirality constraints $D_\alpha \bar{X}^i = 0$ and $\bar{D}_{\dot{\alpha}} X_i = 0$.

2. Show that the gauge transformations are compatible with hermitian conjugation (that is, the hermitian conjugation of the gauge transformation is equal to the gauge transformation of the hermitian conjugation).

3. We take the Kähler potential of a theory with this matter content to be

$$K = \sum_{i=1}^{3} \mathrm{tr}(e^{-2V} X_i e^{2V} \bar{X}^i). \tag{18}$$

   Show that this Kähler potential is real and gauge invariant.

4. Show that the Kähler potential is invariant under a global symmetry, acting as

$$X_i \to U_i{}^j X_j, \qquad \bar{X}^i \to \bar{X}^j (U^{-1})_j{}^i, \tag{19}$$

   where $U_i{}^j$ are complex Grassmann-even quantities. What constraint is imposed on U by the condition that $\bar{X}^i$ is the complex-conjugate of X_i, $\bar{X}^i = (X_i)^*$?

5. ($\mathcal{N} = 4$ super-Yang-Mills, superpotential)

To fully specify a gauge theory interacting with matter fields, we need to give the superpotential as well. In this problem the superpotential is a holomorphic function of the chiral superfields X_1, X_2 and X_3.

The purpose of this exercise is to show that the component fields transform under a $\mathrm{SU}(3) \times \mathrm{U}(1)$ group in the same way as the decomposition of an irreducible representation of a larger symmetry $\mathrm{SU}(4)$. The extra generators inside $\mathrm{SU}(4)$ can be used to produce four supercharges by acting on the supercharges which generates the $\mathcal{N} = 1$ superspace.

1. Using the gauge transformations in the previous question, i.e. $X_i \to e^{i\Lambda} X_i e^{-i\Lambda}$, show that $\mathrm{tr}(P(X_1, X_2, X_3))$ is gauge invariant for any polynomial P.

2. Take the superpotential to be
$$W(X_1, X_2, X_3) = \frac{1}{3!} \epsilon^{ijk} \,\mathrm{tr}(X_i [X_j, X_k]), \tag{20}$$
where ϵ^{ijk} is the completely antisymmetric Levi-Civita tensor.

   Using the fact that $M_i{}^l M_j{}^m M_k{}^n \epsilon^{ijk} = \epsilon^{lmn} \det M$, show that the superpotential is invariant under a $\mathrm{SU}(3)$ group acting as $X_i \to U_i{}^j X_j$, with $U^{-1} = U^\dagger$ and $\det U = 1$.

3. Show that the $\int d^2\theta W(X_1, X_2, X_3)$ is also invariant under a $\mathrm{U}(1)$ R-symmetry acting as
$$\theta'^{,\alpha} = \exp(i\psi)\theta^\alpha, \qquad X_i'(\theta') = \exp\Big(\frac{2i\psi}{3}\Big) X_i(\theta), \tag{21}$$
for a constant ψ.

4. In the previous question we showed that the scalar fields in the X_i multiplets have an R-charge equal to $\frac{2}{3}$. Given that the fermions in these multiplets are proportional to $D_\alpha X_i$, find their R-charge.

5. There is also a fermion in the gauge multiplet, the gaugino. This fermion does not transform under $\mathrm{SU}(3)$ transformations, but it does transform under the $\mathrm{U}(1)$ R-symmetry. Given the transformation of the vector multiplet $V'(\theta', \bar\theta') = V(\theta, \bar\theta)$ and the fact that the gaugino is proportional to $\bar D^2 (e^{2V} D_\alpha e^{-2V})$, show that its R-charge is 1.