2018 Exam Solutions

①

1.) $$x_{\alpha\dot\alpha} y^{\dot\alpha\beta} + y_{\alpha\dot\alpha} x^{\dot\alpha\beta} = x_\mu y_\nu \left(\sigma^\mu_{\alpha\dot\alpha}\bar\sigma^{\nu,\dot\alpha\beta} + \sigma^\nu_{\alpha\dot\alpha}\bar\sigma^{\mu,\dot\alpha\beta}\right)$$
$$= x_\mu y_\nu\, 2\eta^{\mu\nu}\delta_\alpha^\beta = 2(x\cdot y)\,\delta_\alpha^\beta$$

$$x^{\dot\alpha\alpha} y_{\alpha\dot\beta} + y^{\dot\alpha\alpha}x_{\alpha\dot\beta} = x_\mu y_\nu\left(\bar\sigma^{\mu,\dot\alpha\alpha}\sigma^\nu_{\alpha\dot\beta} + \bar\sigma^{\nu,\dot\alpha\alpha}\sigma^\mu_{\alpha\dot\beta}\right) =$$
$$= x_\mu y_\nu\, 2\eta^{\mu\nu}\delta^{\dot\alpha}_{\dot\beta} = 2(x\cdot y)\,\delta^{\dot\alpha}_{\dot\beta}$$

2) $$t_{\alpha\dot\beta}(x,y,z) = x_{\alpha\dot\alpha}y^{\dot\alpha\beta}z_{\beta\dot\beta} - z_{\alpha\dot\alpha}y^{\dot\alpha\beta}x_{\beta\dot\beta}$$
$$= \left(\cancel{2(x\cdot y)\delta_\alpha^\beta} - y_{\alpha\dot\alpha}x^{\dot\alpha\beta}\right)z_{\beta\dot\beta}$$
$$- z_{\alpha\dot\alpha}\left(\cancel{2(x\cdot y)}\,\delta^{\dot\alpha}_{\dot\beta} - x^{\dot\alpha\beta}y_{\beta\dot\beta}\right)$$
$$= -\left(y_{\alpha\dot\alpha}x^{\dot\alpha\beta}z_{\beta\dot\beta} - z_{\alpha\dot\alpha}x^{\dot\alpha\beta}y_{\beta\dot\beta}\right)$$
$$= -t_{\alpha\dot\beta}(y,x,z) \Rightarrow t_{\alpha\dot\beta}(x,y,z) = -t_{\alpha\dot\beta}(y,x,z)$$

$$t_{\alpha\dot\beta}(x,y,z) = x_{\alpha\dot\alpha}y^{\dot\alpha\beta}z_{\beta\dot\beta} - z_{\alpha\dot\alpha}y^{\dot\alpha\beta}x_{\beta\dot\beta}$$
$$= x_{\alpha\dot\alpha}\left(2(y\cdot z)\cancel{\delta^{\dot\alpha}_{\dot\beta}} - z^{\dot\alpha\beta}y_{\beta\dot\beta}\right) - \left(2(y\cdot z)\cancel{\delta^\beta_\alpha} - y_{\alpha\dot\alpha}z^{\dot\alpha\beta}\right)x_{\beta\dot\beta}$$
$$= -\left(x_{\alpha\dot\alpha}z^{\dot\alpha\beta}y_{\beta\dot\beta} - y_{\alpha\dot\alpha}z^{\dot\alpha\beta}x_{\beta\dot\beta}\right)$$
$$= -t_{\alpha\dot\beta}(x,z,y) \Rightarrow t_{\alpha\dot\beta}(x,y,z) = -t_{\alpha\dot\beta}(x,z,y)$$

3) $x_\mu = (0,1,0,0)$, $y_\mu = (0,0,1,0)$, $z_\mu = (0,0,0,1)$

$$\Rightarrow w^\mu = \varepsilon^{\mu\nu\rho\sigma}x_\nu y_\rho z_\sigma = \varepsilon^{\mu 123} = \begin{cases}1 & \text{if } \mu = 0\\ 0 & \text{if } \mu \neq 0\end{cases}$$
$$\Rightarrow w^0 = w_0 = 1 \Rightarrow \boxed{w_{\alpha\dot\alpha} = -\begin{pmatrix}1&0\\0&1\end{pmatrix}_{\alpha\dot\alpha}}$$

Next, $$x_{\alpha\dot\alpha} = x_\mu\sigma^\mu_{\alpha\dot\alpha} = \sigma^1_{\alpha\dot\alpha} = \begin{pmatrix}0&1\\1&0\end{pmatrix}$$
$$\bar y^{\dot\alpha\alpha} = y_\mu\bar\sigma^{\mu,\dot\alpha\alpha} = \bar\sigma^{2,\dot\alpha\alpha} = \begin{pmatrix}0&i\\-i&0\end{pmatrix}$$
$$z_{\alpha\dot\alpha} = z_\mu\sigma^\mu_{\alpha\dot\alpha} = \sigma^3_{\alpha\dot\alpha} = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$$
$$t_{\alpha\dot\alpha}(x,y,z) = (x\cdot\bar y\cdot z - z\,\bar y\,x)_{\alpha\dot\alpha} = \left[\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&i\\-i&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix} - \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&i\\-i&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\right]_{\alpha\dot\alpha}$$
$$= \left[\begin{pmatrix}-i&0\\0&i\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix} - \begin{pmatrix}0&i\\i&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\right]_{\alpha\dot\alpha} = -2i\begin{pmatrix}1&0\\0&1\end{pmatrix}_{\alpha\dot\alpha}$$
$$\Rightarrow \boxed{t_{\alpha\dot\alpha}(x,y,z) = -2i\begin{pmatrix}1&0\\0&1\end{pmatrix}_{\alpha\dot\alpha}}$$
$$\Rightarrow \boxed{2i\,w_{\alpha\dot\alpha} = t_{\alpha\dot\alpha}(x,y,z)}$$

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1.) $D_\alpha D^2 Y^i = 0$ since $D_\alpha D_\beta = -D_\beta D_\alpha$ ($\Rightarrow D_1 D_1 = 0$, $D_2 D_2 = 0$)

$D^2 \sim D_1 D_2$ so $D_\alpha D^2 \sim D_\alpha D_1 D_2$ and for any value of α we can permute terms so that we encounter a square of D.

2.)

$$\delta_\Phi S = \int d^4x\, d^2\theta\, d^2\bar\theta\; \delta\bar\Phi^i \partial_i K + \int d^4x\, d^2\bar\theta\; \delta\bar\Phi^i \partial_i \bar W$$

$$= \int d^4x\, d^2\theta\, d^2\bar\theta\, (D^2 \delta Y^i)\, \partial_i K + \int d^4x\, d^2\bar\theta\, (D^2 \delta Y^i)\, \partial_i \bar W$$

$$= \int d^4x\, d^2\theta\, d^2\bar\theta\; \delta Y^i (D^2 \partial_i K) + \underbrace{\int d^4x\, d^2\bar\theta\; D^2 (\delta Y^i \partial_i \bar W)}_{\frac{1}{4}\int d^4x\, d^2\theta\, d^2\bar\theta\, \delta Y^i \partial_i \bar W}$$

$$= \int d^4x\, d^2\theta\, d^2\bar\theta\; \delta Y^i \left(D^2 \partial_i K + \frac{1}{4} \partial_i \bar W \right)$$

$$\Rightarrow \boxed{D^2 \partial_i K + \frac{1}{4} \partial_i \bar W = 0}$$

3)

$$D^2 \partial_i K = D^\alpha D_\alpha \partial_i K = D^\alpha \left(D_\alpha \phi_j\, \partial^j \partial_i K \right)$$

$$= (D^2 \phi_j)\, \partial^j \partial_i K - (D_\alpha \phi_j)(D^\alpha \phi_k)\, \partial^j \partial^k \partial_i K$$

$$= (D^2 \phi_j)\, K_i{}^j - (D_\alpha \phi_j)(D^\alpha \phi_k)\, \partial^j \partial^k \partial_i K$$

$$\cdot\ (D^2 \phi_j)\, K_i{}^j = (D_\alpha \phi_j)(D^\alpha \phi_k) \underbrace{\partial^j \partial^k \partial_i K}_{K_i{}^{jk}} - \frac{1}{4} \partial_i \bar W$$

Multiply by $(K^{-1})_l{}^i$

$$(D^2 \phi_l) = (D_\alpha \phi_j)(D^\alpha \phi_k)\, (K^{-1})_l{}^i K_i{}^{jk} - \frac{1}{4} (K^{-1})_l{}^i \partial_i \bar W$$

(3)

1) $$Z_i{}^j = \delta_i{}^j + A_i{}^j \theta^2 + \bar{A}_i{}^j \bar{\theta}^2 + B_i{}^j \theta^2 \bar{\theta}^2$$

$$\bar{\Phi}^i Z_i{}^j \Phi_j = \bar{\Phi}^i \Big[\left(\delta_i{}^k + \bar{A}_i{}^k \bar{\theta}^2\right)\left(\delta_k{}^l + \left(B_k{}^l - (\bar{A}A)_k{}^l\right)\theta^2\bar{\theta}^2\right)\left(\delta_l{}^m + A_l{}^m \theta^2\right)\Phi_m$$

since

$$\left(1 + \bar{A}\bar{\theta}^2\right)\left(1 + (B - \bar{A}A)\theta^2\bar{\theta}^2\right)\left(1 + A\theta^2\right)$$

$$= \left(1 + \bar{A}\bar{\theta}^2 + (B - \bar{A}A)\theta^2\bar{\theta}^2\right)\left(1 + A\theta^2\right)$$

$$= 1 + \bar{A}\bar{\theta}^2 + A\theta^2 + \bar{A}A\bar{\theta}^2\theta^2 + (B - \bar{A}A)\theta^2\bar{\theta}^2 = Z$$

Hence, $$\bar{\Phi}^i Z_i{}^j \Phi_j = \bar{X}^i \left(\delta_i{}^j + (B - \bar{A}A)_i{}^j \theta^2\bar{\theta}^2\right) X_j$$

2) $$\int d^4x\, d^2\theta\, d^2\bar{\theta}\; \bar{\Phi}^i Z_i{}^j \Phi_j = \int d^4x\, d^2\theta\, d^2\bar{\theta}\; \bar{X}^i\left(\delta_i{}^j + (B - \bar{A}A)_i{}^j \theta^2\bar{\theta}^2\right) X_j$$

$$= \int d^4x\, d^2\theta\, d^2\bar{\theta}\; \bar{X}^i X_i + \frac{1}{4}\times\frac{1}{4}\,\bar{x}^i (B - \bar{A}A)_i{}^j x_j \quad \text{where} \quad x_i = X_i|$$

This produces a complex mass term $\frac{1}{16}(B - \bar{A}A)$ for the scalars.
The fermionic mass terms are unchanged by this contribution.

3) $$W(X) = \frac{1}{2} C_{ij} X^i X^j + \frac{1}{3!} C_{ijk} X^i X^j X^k$$

$[C_{ijk}] = 0$, $[C_{ij}] = 1$

Hence, we can not include any higher terms in the expansion of C_{ijk}

$$\boxed{C_{ij} = C^{(0)}_{ij} + \theta^2 C^{(1)}_{ij}} \quad \text{where} \quad [C^{(1)}_{ij}] = 0$$

4) $$\int d^4x\, d^2\theta\, d^2\bar{\theta}\; D_{ij}{}^k X^i X^j \bar{X}_k \Rightarrow [D_{ij}{}^k] = -1$$

Keeping only the terms of dimension zero or larger in the expansion, we have

$$D_{ij}{}^k = \underbrace{(D^{(0)})_{ij}{}^k}_{\text{dim } 0}\theta^2 + \underbrace{(D^{(1)})_{ij}{}^k}_{\text{dim } 0}\bar{\theta}^2 + \underbrace{(D^{(2)})_{ij}{}^k}_{\text{dim } 1}\theta^2\bar{\theta}^2$$

4

1) $\bar{D} X_i = 0 \longrightarrow \bar{D}\left(e^{i\Lambda} X_i e^{-i\Lambda}\right) = 0$ since $\bar{D}\Lambda = 0$, $\bar{D} X_i = 0$

$D\bar{X}^i = 0 \longrightarrow D\left(e^{i\bar{\Lambda}} \bar{X}^i e^{-i\bar{\Lambda}}\right) = 0$ since $D\bar{\Lambda} = 0$, $D\bar{X}^i = 0$

2) $$\left(X_i \rightarrow e^{i\Lambda} X_i e^{-i\Lambda}\right) \xrightarrow{+} \left(\bar{X}^i \rightarrow e^{i\bar{\Lambda}} \bar{X}^i e^{-i\bar{\Lambda}}\right)$$

$$\left(e^{-2V} \rightarrow e^{i\bar{\Lambda}} e^{-2V} e^{-i\Lambda}\right) \xrightarrow{+} \left(e^{-2V} \rightarrow e^{i\bar{\Lambda}} e^{-2V} e^{-i\Lambda}\right)$$

3) $$K = \sum_i \mathrm{tr}\left(e^{-2V} X_i e^{2V} \bar{X}^i\right)$$

$$K^* = \sum_i \mathrm{tr}\left(\left(e^{-2V} X_i e^{2V} \bar{X}^i\right)^+\right)$$

$$= \sum_i \mathrm{tr}\left(X_i e^{2V} \bar{X}^i e^{-2V}\right) = \sum_i \mathrm{tr}\left(e^{-2V} X_i e^{2V} \bar{X}^i\right)$$

$$\Rightarrow \boxed{K = K^*}$$

$$K \rightarrow \sum_i \mathrm{tr}\left(\left(e^{i\bar{\Lambda}} e^{-2V} \cancel{e^{-i\Lambda}}\right)\left(\cancel{e^{i\Lambda}} X_i \cancel{e^{-i\Lambda}}\right)\left(\cancel{e^{i\Lambda}} e^{2V} \cancel{e^{-i\bar{\Lambda}}}\right)\left(\cancel{e^{i\bar{\Lambda}}} \bar{X}^i e^{-i\bar{\Lambda}}\right)\right)$$

$$= \sum_i \mathrm{tr}\left(\cancel{e^{i\bar{\Lambda}}} e^{-2V} X_i e^{2V} \bar{X}^i \cancel{e^{-i\bar{\Lambda}}}\right) = K$$

4) $$K \rightarrow \sum_{i,j,j'} \mathrm{tr}\left(e^{-2V} \left(U_i{}^j X_j\right) e^{2V} \left(\bar{X}^{j'} (U^{-1})_{j'}{}^i\right)\right)$$

$$= \sum_{i,j} \underbrace{U_i{}^j (U^{-1})_{j'}{}^i}_{\delta^j_{j'}} \mathrm{tr}\left(e^{-2V} X_j e^{2V} \bar{X}^{j'}\right)$$

$$= \sum_j \mathrm{tr}\left(e^{-2V} X_j e^{2V} \bar{X}^j\right) = K$$

(5)

1) if $X_i \to e^{i\Lambda} X_i e^{-i\Lambda} \Rightarrow X_i X_j \to e^{i\Lambda} X_i X_j e^{-i\Lambda}$

and similarly for any monomial $X_{i_1} \cdots X_{i_n}$

Then, $tr(X_{i_1} \cdots X_{i_n}) \to tr(\cancel{e^{i\Lambda}} X_{i_1} \cdots X_{i_n} \cancel{e^{-i\Lambda}}) = tr(X_{i_1} \cdots X_{i_n})$ invariant

Any polynomial P is a linear combination of invariant monomials so the answer is that $tr(P)$ is invariant.

2) $W(X_1, X_2, X_3) = \frac{1}{3!} \varepsilon^{ijk} tr(X_i [X_j, X_k]) \to$

$$\frac{1}{3!} \varepsilon^{ijk} tr(U_i{}^l X_l [U_j{}^m X_m, U_k{}^n X_n])$$

$$= \frac{1}{3!} (\varepsilon^{ijk} U_i{}^l U_j{}^m U_k{}^n) \, tr(X_l [X_m, X_n])$$

$$= \frac{1}{3!} \det U \, \varepsilon^{lmn} tr(X_l [X_m, X_n])$$

$\Rightarrow W$ is invariant if $\det U = 1$

3) $\theta'_\alpha = e^{i\varphi} \theta_\alpha$, $D'_\alpha = e^{-i\varphi} D_\alpha$, $d^2\theta' = e^{-2i\varphi} d^2\theta$

$$W(X'_1, X'_2, X'_3) = e^{2i\varphi} W(X_1, X_2, X_3)$$

$$\Rightarrow d^2\theta' \, W(X'_1, X'_2, X'_3) = d^2\theta \, W(X_1, X_2, X_3)$$

4) X_i have charge $\frac{2}{3}$, D_α has charge -1

$\Rightarrow D_\alpha X_i$ has charge $-\frac{1}{3}$

5) D_α has charge -1 , $\bar{D}_{\dot\alpha}$ has charge $+1$

$\bar{D}^2 D_\alpha$ has charge $+1$