King's College London

University Of London

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Candidate No: .... **Desk No:** ....

MSc and MSci Examination

7CCMMS40T Supersymmetry (MSc)
7CCMMS40U Supersymmetry (MSci)

Summer 2018 Resit/Replacement Examination

Time Allowed: Two Hours

All questions carry equal marks.
Full marks will be awarded for complete answers to FOUR questions.
If more than four questions are attempted, then only the best FOUR will count.

NO calculators are permitted.

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TURN OVER WHEN INSTRUCTED

Questions

1. (Gamma matrices in three space-time dimensions)

In three space-time dimensions we can take the Dirac gamma matrices to be

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1}$$

1. Show that $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$ where $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1)$.
2. Show that $[\gamma_\mu, \gamma_\nu] = 2\epsilon_{\mu\nu\rho}\gamma^\rho$, where $\epsilon_{\mu\nu\rho}$ is the completely anti-symmetric tensor with $\epsilon_{012} = 1$ and $\gamma^\mu = \eta^{\mu\nu}\gamma_\nu$.
3. Show that the 2×2 matrices $\gamma_\mu\epsilon$ are symmetric for all μ, where $\epsilon = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.
4. Compute the constant c in the formula below

$$(\gamma_\mu\epsilon)_{\alpha\beta}(\gamma^\mu\epsilon)_{\gamma\delta} = c(\epsilon_{\alpha\gamma}\epsilon_{\beta\delta} + \epsilon_{\alpha\delta}\epsilon_{\beta\gamma}). \tag{2}$$

2. (Three-dimensional D-algebra)

The supersymmetry covariant derivatives in three space-time dimensions are defined to be

$$D_\alpha = \frac{\partial}{\partial \theta^\alpha} + i\theta^\beta (\gamma^\mu \epsilon)_{\alpha\beta} \partial_\mu, \tag{3}$$

where $\epsilon_{\alpha\beta} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right)$ and $(\gamma^\mu \epsilon)_{\alpha\beta} = (\gamma^\mu \epsilon)_{\beta\alpha}$.

Note that in three dimensions the Grassmann-odd coordinates θ are real ($\theta^* = \theta$) and therefore there are no $\dot{\alpha}$ indices or $\bar{D}$ covariant derivatives.

1. Write the anti-commutator $\{D_\alpha, D_\beta\}$ in terms of ∂_μ.

2. Show that
$$[D_\alpha, D_\beta] = -\epsilon_{\alpha\beta} D^2, \tag{4}$$
where $D^2 = \epsilon^{\alpha\beta} D_\alpha D_\beta$ and $\epsilon_{\alpha\beta}\epsilon^{\beta\gamma} = \delta_\alpha^\gamma$. Hint: the answer is determined up to a multiplicative constant by the anti-symmetry in α and β. The multiplicative constant can be obtained by contraction.

3. Show that
$$D_\alpha D_\beta = i(\gamma^\mu \epsilon)_{\alpha\beta} \partial_\mu - \frac{1}{2} \epsilon_{\alpha\beta} D^2, \tag{5}$$
by writing the product $D_\alpha D_\beta$ as a combination of commutator and anti-commutator.

3. ($\mathcal{N} = 1$ supersymmetry multiplets)

We have shown in class that a four-dimensional massive supersymmetry multiplet can be built by acting with "creation operators" $\bar{Q}_{\dot{\alpha}}$ on a vacuum $|s\rangle$ of spin s which is annihilated by "annihilation operators" Q_α. There are $2s+1$ states of spin s which can be identified by the eigenvalues of the z projection of the spin operator. These eigenvalues are $-s, -s+1, \ldots, s-1, s$.

Recall from quantum mechanics that by acting with operators O_m transforming under a spin j representation, we obtain all the states with spins $|s-j|, \ldots, s+j$.

1. Take $s \geq \frac{1}{2}$. What spins appear in the states $\bar{Q}_{\dot{\alpha}}|s\rangle$ obtained by the action of one creation operator? How many such states are there?

2. What spins appear in the states $\bar{Q}_{\dot{\alpha}}|s=0\rangle$? How many such states are there?

3. What is the maximum number of $\bar{Q}$'s we can act with and still produce non-trivial states?

4. Given that the number of states obtained by acting with an even number of creation operators should be equal to the number of states obtained by acting with an odd number of creation operators, how many states are obtained by acting with two creation operators? What is the spin content of these states?

4. ($\mathcal{N} = 2$ supersymmetry multiplets)

In this question we build massive representations of four-dimensional $\mathcal{N} = 2$ supersymmetry, following the example of $\mathcal{N} = 1$ supersymmetry. In the $\mathcal{N} = 2$ case the supercharges are Q^i_α and $\bar{Q}_{\dot{\alpha}i}$ where $i = 1, 2$. We have two SU(2) groups; the first acts on the spin index $\dot{\alpha}$ while the second acts on the supercharge label i.

The vacuum is described by two labels $|s, r\rangle$ and is annihilated by Q^i_α. The other states in the multiplet can be obtained by acting with the creation operators on this vacuum. In the following we will find the decomposition under the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ group described above.

1. Show that the degeneracy of the vacuum is $(2s+1)(2r+1)$. Hint: the two SU(2) groups act independently of each other and there are $2j + 1$ states in a spin j multiplet.

2. The states at level one are generated by acting with a single creation operator of the form $\bar{Q}_{\dot{\alpha}i}$. How many creation operators are there? How many states do you expect to find at level one?

3. Taking into account that the vacuum transforms in the (r, s) representation, that $\bar{Q}_{\dot{\alpha}i}$ transforms in the $(\frac{1}{2}, \frac{1}{2})$ representation and taking the angular momentum composition separately for the two SU(2) groups, find the possible values for the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ irreducible representations that appear in this tensor product. (Hint: one of the possibilities is $(s-\frac{1}{2}, r-\frac{1}{2})$. What are the others?)

4. At level two, keeping in mind that the creation operators anti-commute, we find $\epsilon^{\dot{\alpha}\dot{\beta}}\bar{Q}_{\dot{\alpha}i}\bar{Q}_{\dot{\beta}j}$ transforming as $(0, 1)$ and $\epsilon^{ij}\bar{Q}_{\dot{\alpha}i}\bar{Q}_{\dot{\beta}j}$ transforming as $(1, 0)$. Find the number of states and their spin content, as in the previous question.

5. The states at level three can be obtained by CPT conjugation from those at level one while those at level four can be obtained by CPT conjugation of those at level zero. Show that in total we have $8(2s+1)(2r+1)$ bosonic and $8(2s + 1)(2r + 1)$ fermionic states in this multiplet.

5. (Gauge connections)

In a gauge theory the derivatives ∂_μ become gauge covariant derivatives $\mathscr{D}_\mu = \partial_\mu + A_\mu$, where A_μ is the gauge connection (or vector potential). Under gauge transformations the covariant derivative transforms as $\mathscr{D}_\mu \to g^{-1}(x)\mathscr{D}_\mu g(x)$, where $g(x)$ is an element of the gauge group (for an Abelian theory $g(x) = e^{i\phi(x)}$ but we will consider the general case of a non-Abelian gauge theory).

In a supersymmetric theory, we also have the derivatives D_α and $\bar{D}_{\dot\alpha}$. Instead of following the construction in the lecture, we could build a supersymmetric gauge theory by also adding gauge connections for these derivatives. In other words, we define $\mathscr{D}_\alpha = D_\alpha + A_\alpha$, $\bar{\mathscr{D}}_{\dot\alpha} = \bar{D}_{\dot\alpha} + \bar{A}_{\dot\alpha}$. The derivatives ∂_μ, D_α and $\bar{D}_{\dot\alpha}$ satisfy the relations $\{D_\alpha, D_\beta\} = 0$, $\{\bar{D}_{\dot\alpha}, \bar{D}_{\dot\beta}\} = 0$, $\{D_\alpha, \bar{D}_{\dot\alpha}\} = -2i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu$. We impose the constraints that the gauge covariant derivatives satisfy

$$\{\mathscr{D}_\alpha, \mathscr{D}_\beta\} = 0, \qquad \{\bar{\mathscr{D}}_{\dot\alpha}, \bar{\mathscr{D}}_{\dot\beta}\} = 0, \qquad \{\mathscr{D}_\alpha, \bar{\mathscr{D}}_{\dot\alpha}\} = -2i\sigma^\mu_{\alpha\dot\alpha}\mathscr{D}_\mu. \tag{6}$$

1. Using the third constraint above, write the gauge connection $A_{\alpha\dot\alpha} = \sigma^\mu_{\alpha\dot\alpha}A_\mu$ in terms of derivatives D_α, $\bar{D}_{\dot\alpha}$ and connections A_α, $\bar{A}_{\dot\alpha}$.

2. Show that the first two constraints above can be solved by[1]

$$\mathscr{D}_\alpha = T^{-1}D_\alpha T, \qquad \bar{\mathscr{D}}_{\dot\alpha} = U\bar{D}_{\dot\alpha}U^{-1}, \tag{7}$$

   Where T and U are some superfields.

3. The superfields T and U above are called pre-potentials since the gauge potentials A can be found from T and U by taking derivatives. Show that under the pre-gauge transformations

$$T \to \bar{X}T, \qquad U \to UY, \tag{8}$$

   with $\bar{X}$ anti-chiral and Y chiral, the gauge connections A_α, $\bar{A}_{\dot\alpha}$ are invariant.

4. There is a second type of gauge transformations acting on the pre-potentials T and U which *does* change the gauge connections

$$T \to Tg, \qquad U \to g^{-1}U. \tag{9}$$

   Compute the gauge connections A_α and $\bar{A}_{\dot\alpha}$ after a gauge transformation with $g = T^{-1}$ and after a gauge transformation $g = U$. Write the answer in terms of the invariant quantity $TU = e^{-2V}$ (which serves as a definition for V).

[1] In the formulas below the derivatives act on T, U^{-1} and on everything to the right. More explicitly, $\mathscr{D}_\alpha X = T^{-1}D_\alpha(TX)$.

5. Some of the previous gauge choices were not compatible with the reality conditions relating A_α and $\bar{A}_{\dot\alpha}$. Find an expression for A_α and $\bar{A}_{\dot\alpha}$ in terms of the superfield V defined by $e^{-2V} = TU$, by performing a gauge transformation as in the previous question with $g = U(TU)^{-\frac{1}{2}} = T^{-1}(TU)^{\frac{1}{2}}$. Note: when using the superfield V to express the dynamical content of the theory, the pre-gauge transformations become usual gauge transformations.