Solutions SUSY exam resit 2018

1

1) $\gamma_0 = \begin{pmatrix}0&1\\-1&0\end{pmatrix}$, $\gamma_1 = \begin{pmatrix}0&1\\1&0\end{pmatrix}$ $\gamma_2 = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$

$$\gamma_0^2 = -1$$
$$\gamma_1^2 = 1$$
$$\gamma_2^2 = 1$$

$$\gamma_0\gamma_1 + \gamma_1\gamma_0 = \begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} + \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&1\\-1&0\end{pmatrix} = \begin{pmatrix}1&0\\0&-1\end{pmatrix} + \begin{pmatrix}-1&0\\0&1\end{pmatrix} = 0$$

$$\gamma_0\gamma_2 + \gamma_2\gamma_0 = \begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix} + \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&1\\-1&0\end{pmatrix} = \begin{pmatrix}0&-1\\-1&0\end{pmatrix} + \begin{pmatrix}0&1\\1&0\end{pmatrix} = 0$$

$$\gamma_1\gamma_2 + \gamma_2\gamma_1 = \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix} + \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}0&-1\\1&0\end{pmatrix} + \begin{pmatrix}0&1\\-1&0\end{pmatrix} = 0$$

$$\Rightarrow \{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$$

2)

$$[\gamma_0, \gamma_1] = \begin{pmatrix}1&0\\0&-1\end{pmatrix} - \begin{pmatrix}-1&0\\0&1\end{pmatrix} = 2\begin{pmatrix}1&0\\0&-1\end{pmatrix} = 2\gamma_2$$

$$[\gamma_0, \gamma_2] = \begin{pmatrix}0&-1\\-1&0\end{pmatrix} - \begin{pmatrix}0&1\\1&0\end{pmatrix} = -2\gamma_1 \quad \Rightarrow \quad [\gamma_2, \gamma_0] = 2\gamma_1$$

$$[\gamma_1, \gamma_2] = \begin{pmatrix}0&-1\\1&0\end{pmatrix} - \begin{pmatrix}0&1\\-1&0\end{pmatrix} = -2\gamma_0$$

$\mu = 0, \nu = 1$

$$[\gamma_0, \gamma_1] = 2\varepsilon_{012}\gamma^2 = 2\gamma_2 \quad ✓$$

$\mu = 0, \nu = 2 \Rightarrow$ $[\gamma_0, \gamma_2] = 2\varepsilon_{021}\gamma^1 = -2\gamma_1$ ✓

$\mu = 1, \nu = 2 \Rightarrow$ $[\gamma_1, \gamma_2] = 2\varepsilon_{120}\gamma^0 = 2\gamma^0 = -2\gamma_0$ ✓

3)

$\gamma_0\varepsilon = \begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}0&1\\-1&0\end{pmatrix} = \begin{pmatrix}-1&0\\0&-1\end{pmatrix}$ symmetric

$\gamma_1\varepsilon = \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&1\\-1&0\end{pmatrix} = \begin{pmatrix}-1&0\\0&1\end{pmatrix}$ symmetric

$\gamma_2\varepsilon = \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&1\\-1&0\end{pmatrix} = \begin{pmatrix}0&1\\1&0\end{pmatrix}$ symmetric

4) Choose $\alpha = 1, \beta = 2$, $\gamma = 1$, $\delta = 2$

$$\Rightarrow (\gamma_\mu\varepsilon)_{\alpha\beta}(\gamma^\mu\varepsilon)_{\gamma\delta} = (\gamma_\mu\varepsilon)_{12}(\gamma^\mu\varepsilon)_{12} = (\gamma_2\varepsilon)_{12}(\gamma_2\varepsilon)_{12} = \underline{1}$$

$$RHS = c(\varepsilon_{11}\varepsilon_{22} + \varepsilon_{12}\varepsilon_{21}) = -c \Rightarrow \boxed{c = -1}$$

(2)

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\,\theta^\rho(\gamma^\mu\varepsilon)_{\alpha\rho}\,\partial_\mu$$

1)

$$\{D_\alpha, D_\beta\} = \left\{ \frac{\partial}{\partial\theta^\alpha} + i\,\theta^\gamma(\gamma^\mu\varepsilon)_{\alpha\gamma}\,\partial_\mu \, , \, \frac{\partial}{\partial\theta^\beta} + i\,\theta^\delta(\gamma^\nu\varepsilon)_{\beta\delta}\,\partial_\nu \right\}$$

$$= i(\gamma^\nu\varepsilon)_{\beta\alpha}\,\partial_\nu + i(\gamma^\mu\varepsilon)_{\alpha\beta}\,\partial_\mu$$

$$= 2i\,(\gamma^\mu\varepsilon)_{\alpha\beta}\,\partial_\mu$$

2)

$$[D_\alpha, D_\beta] = \varepsilon_{\alpha\beta}\,T$$ (since LHS is antisymmetric in $(\alpha\beta)$)

$$\varepsilon^{\alpha\beta}[D_\alpha, D_\beta] = \varepsilon^{\alpha\beta}(D_\alpha D_\beta - D_\beta D_\alpha) = 2\,\varepsilon^{\alpha\beta}D_\alpha D_\beta = 2D^2$$

$$\varepsilon^{\alpha\beta}\varepsilon_{\alpha\beta}\,T = -2T$$

$$\Rightarrow \quad 2D^2 = -2T \Rightarrow T = -D^2$$

$$\Rightarrow \quad \boxed{[D_\alpha, D_\beta] = -\varepsilon_{\alpha\beta}\,D^2}$$

3)

$$D_\alpha D_\beta = \frac{1}{2}\{D_\alpha, D_\beta\} + \frac{1}{2}[D_\alpha, D_\beta]$$

$$= i(\gamma^\mu\varepsilon)_{\alpha\beta}\,\partial_\mu - \frac{1}{2}\varepsilon_{\alpha\beta}\,D^2$$

③

1) $\bar{Q}_{\dot\alpha} |s\rangle \in |s+\frac{1}{2}\rangle \oplus |s-\frac{1}{2}\rangle$

$|s\rangle$ has $2s+1$ states and we have 2 creation operators $\bar{Q}_{\dot\alpha}$

$\Rightarrow$ $\bar{Q}_{\alpha}|s\rangle$ contains $2(2s+1)$ states.

This can also be obtained by adding the representations in the RHS

$|s+\frac{1}{2}\rangle$ has $2(s+\frac{1}{2})+1$ states

$|s-\frac{1}{2}\rangle$ has $2(s-\frac{1}{2})+1$ states

In total

$$2s+1+1+2s-1+1 = 4s+2 = 2(2s+1)$$

2) $\bar{Q}_{\dot\alpha}|0\rangle = |\frac{1}{2}\rangle$ there are $2\frac{1}{2}+1 = 2$ states

3) Acting with more than 2 $\bar{Q}$'s yields zero since $\bar{Q}$ operators anticommute and there are only two of them $\bar{Q}_1$, $\bar{Q}_2$. Hence, the maximum number is 2.

4) # states level 0 $= 2s+1$

states level 1 $= 2(2s+1)$

states level 2 $=$ # level 1 $-$ # level 0

$= 2(2s+1) - (2s+1) = 2s+1$

These states fit in a spin s representation.

4

1) The number of states in a $|s,r\rangle$ representation is the same as the number of states in $|s\rangle$ $\times$ number of states in $|r\rangle$

$\Rightarrow$ $(2s+1) \times (2r+1)$

2) $\bar{Q}_{\dot{\alpha} i}$ $\Rightarrow$ $2 \times 2 = 4$ creation operators.

We obtain $4 \times (2s+1) \times (2r+1)$ states at level 1

3) $\bar{Q}_{\dot{\alpha} i}$ transforms as $(\frac{1}{2}, \frac{1}{2})$ representation under $SU(2) \times SU(2)$. Given the composition of angular momentum, we find

$$\bar{Q}_{\dot{\alpha} i} |s,r\rangle \in |s-\tfrac{1}{2}, r-\tfrac{1}{2}\rangle \oplus |s-\tfrac{1}{2}, r+\tfrac{1}{2}\rangle \oplus |s+\tfrac{1}{2}, r-\tfrac{1}{2}\rangle \oplus |s+\tfrac{1}{2}, r+\tfrac{1}{2}\rangle$$

There are $[2(s-\frac{1}{2})+1][2(r-\frac{1}{2})+1] + [2(s-\frac{1}{2})+1][2(r+\frac{1}{2})+1]$

$+ [2(s+\frac{1}{2})+1][2(r-\frac{1}{2})+1] + [2(s+\frac{1}{2})+1][2(r+\frac{1}{2})+1]$

$= 2s(4r+2) + (2s+2)(4r+2) = (4s+2)(4r+2) = 4(2s+1)(2r+1)$

4) $\varepsilon^{\dot{\alpha}\dot{\beta}} \bar{Q}_{\dot{\alpha} i} \bar{Q}_{\dot{\beta} j}$ transforms as $(0,1)$

$\varepsilon^{\dot{\alpha}\dot{\beta}} \bar{Q}_{\dot{\alpha} i} \bar{Q}_{\dot{\beta} j} |s,r\rangle \in |s, r-1\rangle \oplus |s,r\rangle \oplus |s, r+1\rangle$

$(2s+1)(2(r-1)+1 + 2r+1 + 2(r+1)+1)$

$= (2s+1)(6r+3) = 3(2s+1)(2r+1)$

$\varepsilon^{ij} \bar{Q}_{\dot{\alpha} i} \bar{Q}_{\dot{\beta} j} |s,r\rangle \in |s+1, r\rangle \oplus |s,r\rangle \oplus |s-1, r\rangle$

$\Rightarrow$ $3(2s+1)(2r+1)$ states.

5) At level 3 we have $4(2s+1)(2r+1)$ states (just as for level 1)

At level 4 —— $(2s+1)(2r+1)$ (just as for level 0)

In total, we have

$N_0 + N_2 + N_4 = (2s+1)(2r+1) + 6(2s+1)(2r+1) + (2s+1)(2r+1)$

$= 8(2s+1)(2r+1)$

$N_1 + N_3 = 4(2s+1)(2r+1) + 4(2s+1)(2r+1)$

$= 8(2s+1)(2r+1)$

5 1) $\{\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot\alpha}\} = -2i\sigma^\mu_{\alpha\dot\alpha}\mathcal{D}_\mu$

$$\Rightarrow \quad \cancel{\{D_\alpha, \bar{D}_{\dot\alpha}\}} + D_\alpha \bar{A}_{\dot\alpha} + \bar{D}_{\dot\alpha} A_\alpha + \{A_\alpha, \bar{A}_{\dot\alpha}\} = -2i\sigma^\mu_{\alpha\dot\alpha}(\partial_\mu + A_\mu)$$

$$\Rightarrow \quad \boxed{D_\alpha \bar{A}_{\dot\alpha} + \bar{D}_{\dot\alpha} A_\alpha + \{A_\alpha, \bar{A}_{\dot\alpha}\} = -2i A_{\alpha\dot\alpha}}$$

2)

$$\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = \{T^{-1}D_\alpha T, T^{-1}D_\beta T\} = T^{-1}(D_\alpha D_\beta + D_\beta D_\alpha)T = 0$$

$$\{\bar{D}_{\dot\alpha}, \bar{D}_{\dot\beta}\} = \{U^{-1}\bar{D}_{\dot\alpha}U, U^{-1}\bar{D}_{\dot\beta}U\} = U^{-1}(\bar{D}_{\dot\alpha}\bar{D}_{\dot\beta} + \bar{D}_{\dot\beta}\bar{D}_{\dot\alpha})U = 0$$

3)

$$\mathcal{D}_\alpha = T^{-1}D_\alpha T \longrightarrow (\bar{X}T)^{-1}D_\alpha(\bar{X}T) = T^{-1}\bar{X}^{-1}D_\alpha(\bar{X}T)$$

$$= T^{-1}\bar{X}^{-1}\bar{X}D_\alpha T = T^{-1}D_\alpha T = \mathcal{D}_\alpha$$

Similarly, $\bar{\mathcal{D}}_{\dot\alpha}$ is invariant. This implies A_α and $\bar{A}_{\dot\alpha}$ are invariant.

4)

$$\mathcal{D}_\alpha = T^{-1}D_\alpha T \rightarrow (Tg)^{-1}D_\alpha Tg = g^{-1}T^{-1}D_\alpha Tg$$

$$\bar{\mathcal{D}}_{\dot\alpha} = U\bar{D}_{\dot\alpha}U^{-1} \rightarrow (g^{-1}U)\bar{D}_{\dot\alpha}(g^{-1}U)^{-1} = g^{-1}U\bar{D}_{\dot\alpha}U^{-1}g$$

If $g = T^{-1}$

$$\Rightarrow \begin{cases} \mathcal{D}_\alpha \longrightarrow D_\alpha & \text{so } A_\alpha = 0 \\ \bar{\mathcal{D}}_{\dot\alpha} \rightarrow TU\bar{D}_{\dot\alpha}(TU)^{-1} = e^{-2V}\bar{D}_{\dot\alpha}e^{2V} \end{cases}$$

If $g = U$

$$\mathcal{D}_\alpha \rightarrow (TU)^{-1}D_\alpha TU = e^{2V}D_\alpha e^{-2V}$$

$$\bar{\mathcal{D}}_{\dot\alpha} \rightarrow \bar{D}_{\dot\alpha} \Rightarrow \bar{A}_{\dot\alpha} = 0$$

5) choose $g = U(TU)^{-1/2} = T^{-1}(TU)^{1/2}$

$$\mathcal{D}_\alpha \longrightarrow (Tg)^{-1}D_\alpha Tg = (TU)^{-1/2}D_\alpha(TU)^{1/2} = e^V D_\alpha e^{-V}$$

$$\bar{\mathcal{D}}_{\dot\alpha} \rightarrow (g^{-1}U)\bar{D}_{\dot\alpha}U^{-1}g = (TU)^{1/2}\bar{D}_{\dot\alpha}(TU)^{-1/2} = e^{-V}\bar{D}_{\dot\alpha}e^V$$