Lecture 8. 10:00 7/3/17

RECORDED.

Actions for Chiral Multiplets

$$S = \int d^4x\, d^2\theta\, d^2\bar\theta\, \underbrace{K(\bar\Phi, \Phi)}_{\text{Kähler potl}} + \left(\int d^4x\, d^2\theta\, \underbrace{W(\Phi)}_{\text{Superpotential}} + c.c. \right)$$

$K(\bar\Phi, \Phi)$ real

$W(\Phi)$ is holomorphic.

- 2 types of integral
  - full superspace int.
  - chiral " int.

- There are no explicit derivatives.

Actions for several chiral fields Φ^i $i = 1, \dots, n$

$$S = \int d^4x\, d^2\theta\, d^2\bar\theta\, \underset{\text{real}}{K(\bar\Phi^{\bar i}, \Phi^i)} + \left(\int d^4x\, d^2\theta\, \underset{\text{holomorphic in } \Phi^i}{W(\Phi^i, \dots \Phi^n)} + c.c. \right).$$

$$\Phi^i(\cancel{x}\, y, \theta) = \varphi^i(y)$$

Compute the space-time action

$$S \ni \int d^4x\, K^{(\bar\varphi\varphi)}_{i\bar{i}}\, \varphi^i \varphi^{\bar{i}}$$

(metric on a complex manifold)

"Stiffly" connection
Geometrical picture.

Such manifolds are called <u>Kähler manifolds</u>.

There are other interesting terms in the space-time action such as.

$$\underbrace{R_{i\bar{i}j\bar{j}}}_{\text{Riemann tensor}} (\psi^i \psi^j)(\bar\psi^{\bar{i}} \bar\psi^{\bar{j}})$$

$\mathcal{N} = 2 \Rightarrow$ hyper Kähler.

Focus of lecture $\Rightarrow$ How to make Gauge fields (spin 1) supersymmetric.

<u>Real Superfields</u>.

Superfield $\rightarrow V(x, \theta, \bar\theta)$.

Impose a reality condition : $V(x,\theta,\bar\theta) = V^+(x,\theta,\bar\theta) = \bar{V}(x,\theta,\bar\theta)$

- $(\theta)^0(\bar\theta)^0 \rightarrow$ real scalar
- $(\theta)^1(\bar\theta)^0 \rightarrow$ left spinor
- $(\theta)^0(\bar\theta)^1 \rightarrow$ right spinor — c.c.

$(\theta)^2(\bar\theta)^0 \rightarrow$ complex scalar
$(\theta)^0(\bar\theta)^2 \rightarrow$ complex scalar — c.c.
$(\theta^1)(\bar\theta^1) \rightarrow$ <u>real ~~scalar~~ vector.</u>

Gauge field (spin 1)

what we are after

Odd order $\Rightarrow$ spinor Fermion.

$$\theta^\alpha \bar\theta^{\dot\alpha} V_{\alpha\dot\alpha} = (\theta\sigma^\mu\bar\theta) V_\mu$$

$$V_\mu^+ = V_\mu \quad (\theta\sigma^\mu\bar\theta)^+ = (\theta\sigma^\mu\bar\theta)$$

At level 3. $\Rightarrow$ $(\theta)^2(\bar\theta)^1 \rightarrow$ spinor.

& c.c.

$(\theta)^1(\bar\theta)^2 \rightarrow$ spinor

Level 4 $(\theta)^2(\bar\theta)^2 \rightarrow$ real scalar.

Gauge fields have Gauge transformation.

$$V_\mu \overset{\sim}{\Rightarrow} V_\mu + \partial_\mu \lambda$$ (Equivalence class). λ = real scalar..

$$\delta V_\mu = \partial_\mu \lambda.$$

Gauge transformation for superfield V?

$$V \sim V + \Phi + \bar\Phi$$ (superfield gauge transformation / equivalence)

Expanding Φ, $\Rightarrow$.

$$\Phi = \ldots + i(\theta\sigma^\mu\bar\theta)\partial_\mu\varphi$$

$$\bar\Phi = \ldots + -i(\theta\sigma^\mu\bar\theta)\partial_\mu\bar\varphi$$

Focus on. $(\theta\sigma^\mu\bar\theta)$

$\Rightarrow$. $V_\mu \sim V_\mu + i\partial_\mu(\varphi-\bar\varphi)$

ie. Gauge parameter is $i(\varphi-\bar\varphi)$, which is real.

<u>Wess - Zumino Gauge</u>

The multiplet contains

- Auxiliary fields — necessary to close susy algebra off-shell
- Pure gauge — to ensure ~~commute~~ susy Gauge transformations supersymmetric
- Physical d.o.f

Gauge where we only have auxiliary fields and physical fields

(partial gauge fixing).

$$V_{WZ}^{(x,\theta,\bar\theta)} = \underbrace{(\theta\sigma^\mu\bar\theta)V_\mu(x)}_{\text{Gauge field.}} + \underbrace{i\theta^2(\bar\theta\bar\psi)(x) - i\bar\theta^2(\theta\psi)(x)}_{\text{from. Gauge fixing}} + \underbrace{\tfrac{1}{2}\theta^2\bar\theta^2 D(x)}_{\text{Real auxiliary field, cf } F}$$

<u>WARNING:</u> The Wess-Zumino Gauge is not supersymmetric!
(though the theory is still supersymmetric.)
cf QED + choice of non Lorentz invariant Gauge.

Now, we want an Action, which is Gauge-invariant.

$$\delta V = \Phi + \bar\Phi \quad \text{- how to get gauge-invariant quantities?}$$

$$W_\alpha = -\frac{1}{4}\bar D^2 D_\alpha V \quad \text{is gauge invariant}$$

$$V = \ldots \, (i\theta^2(\bar\theta\bar\psi) - i(\bar\theta^2(\theta\psi))$$

$\longrightarrow$ has terms ...?

<u>Proof of Gauge invariance.</u>

$$\delta W_\alpha = \frac{1}{4}\bar D^2 D_\alpha \delta V$$

$$= -\frac{1}{4}\bar D^2 D_\alpha(\Phi + \cancel{\bar\Phi}) \qquad (\bar\Phi \text{ is antichiral})$$

$$= -\frac{1}{4}\Big(\underbrace{[\bar D^2, D_\alpha]}_{\sim\, \partial_{\alpha\dot\alpha}\bar D^{\dot\alpha}} + \cancel{D_\alpha \bar D^2}\Big)\Phi = 0.$$

W_α is a chiral superfield

How do we know $\bar D_{\dot\alpha} W_\alpha = -\frac{1}{4}\underbrace{(\bar D_{\dot\alpha}\bar D^2)}_{\sim\, \bar D^3 = 0} D_\alpha V = 0$

Expansion of W_α.

$W_\alpha(y, \theta)$, where $y^\mu = x^\mu + i(\theta\sigma^\mu\bar\theta)$

· <u>Procedure</u>

- Start with the definition
  $$W = -\frac{1}{4}\bar D^2 D_\alpha V$$
- The expansion of W_α should be in variables (y, θ) but the expansion of V is in variables $(x, \theta, \bar\theta)$, so do change of variables
- Work in the WZ gauge (since W_α is gauge invariant)

$$V_{WZ} = (\theta\sigma^\mu\bar\theta)V_\mu + i\theta^2(\bar\theta\bar\lambda) - i\bar\theta^2(\theta\lambda) + \frac{1}{2}\theta^2\bar\theta^2 D$$

Shift $x^\mu \to y^\mu = x^\mu + i(\theta\sigma^\mu\bar\theta)$

$$V_\mu(x) = V_\mu(y - i(\theta\sigma^\mu\bar\theta))$$

$$= V_\mu(y) - i(\theta\sigma^\nu\bar\theta)\partial_\nu V_\mu(y) + (\dots)$$

$$\left((\quad) \cdot (\theta\sigma^\mu\bar\theta)^3 = 0 \right)$$

$\boxed{\text{First term}}$

$$(\theta\sigma^\mu\bar\theta)V_\mu(x) = (\theta\sigma^\mu\bar\theta)V_\mu(y) - i(\theta\sigma^\mu\bar\theta)(\theta\sigma^\nu\bar\theta)\partial_\nu V_\mu$$

$$(\theta\sigma^\mu\bar\theta)(\theta\sigma^\nu\bar\theta) = \theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\theta^\beta\sigma^\nu_{\beta\dot\beta}\bar\theta^{\dot\beta} = -\underbrace{(\theta^\alpha\theta^\beta)}_{\overset{?}{=}\frac{1}{2}\varepsilon^{\alpha\beta}\theta^2}(\bar\theta^{\dot\alpha}\bar\theta^{\dot\beta})\sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\beta}$$

$$= \frac{1}{4}\theta^2\bar\theta^2\,\varepsilon^{\alpha\beta}\underbrace{\varepsilon^{\dot\alpha\dot\beta}\sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\beta}}_{\bar\sigma^{\nu,\dot\alpha\alpha}}$$

$$= \frac{1}{4}\theta^2\bar\theta^2(\sigma^\mu\bar\sigma^\nu)_\alpha{}^\alpha \qquad (\mathrm{tr}(\sigma^\mu\bar\sigma^\nu) = 2\eta^{\mu\nu})$$

$$= \frac{1}{4}\theta^2\bar\theta^2\,\mathrm{tr}(\sigma^\mu\bar\sigma^\nu) = \frac{1}{2}\theta^2\bar\theta^2\eta^{\mu\nu}$$

$$\Rightarrow \boxed{(\theta\sigma^\mu\bar\theta)V_\mu(x) = (\theta\sigma^\mu\bar\theta)V_\mu(y) - \frac{i}{2}\theta^2\bar\theta^2(\partial V)(y)}$$

Next term.

$$i\theta^2(\bar\theta\bar\psi)(x) = i\theta^2(\bar\theta\bar\psi)(y)$$

$$= i\theta^2(\bar\theta\bar\psi)(x + i\theta\sigma^\mu\bar\theta)$$

Taylor expand.

$$= i\theta^2(\bar\theta\bar\psi)(x) + i(\theta\sigma^\nu\bar\theta)\partial_\nu(\theta^2(\bar\theta\psi)(x))$$

Similarly the 3rd term. and so we obtain

$$\Rightarrow \boxed{V_{WZ}(x,\theta,\bar\theta) = (\theta\sigma^\mu\bar\theta)(y) + i\theta^2(\bar\theta\bar\psi)(y) - i\bar\theta^2(\theta\psi)(y) + \tfrac{1}{2}\theta^2\bar\theta^2(D - i\partial V)(y)}$$

Recall that in the coordinate system $(y,\theta,\bar\theta)$ the expressions for supersymmetry covariant derivative are:-

$$\begin{cases} D_\alpha = \partial_\alpha + 2i(\sigma^\mu\bar\theta)_\alpha \dfrac{\partial}{\partial y^\mu} \\ \bar D_{\dot\alpha} = \partial_{\dot\alpha} \end{cases}$$

$$D_\alpha V_{WZ} = (\sigma^\mu\bar\theta)_\alpha V_\mu(y) + 2i\theta_\alpha(\bar\theta\bar\psi)(y) + \theta_\alpha\bar\theta^2(D - i\partial V)(y) + 2i(\sigma^\nu\bar\theta)_\alpha\underbrace{\left((\theta\sigma^\mu\bar\theta)\partial_\nu V_\mu(y) + i\theta^2(\bar\theta\partial_\nu\bar\psi)(y)\right)}$$

$(\lambda \equiv \psi)$ in my notation

∂_α. part

$\partial_\alpha(\theta\sigma^\mu\bar\theta) = (\sigma^\mu\bar\theta)_\alpha$

$\partial_\alpha\theta^2 = 2\theta_\alpha$

$\partial_\alpha(\theta\psi) = \psi_\alpha$.

from $2i(\sigma^\mu\bar\theta)_\alpha\dfrac{\partial}{\partial y^\mu}$

only 2 terms as. other terms have $\theta^3, \bar\theta^3$ etc dependency.

$$2i\,(\sigma^\nu\bar\theta)_\alpha\,(\theta\sigma^\mu\bar\theta)\,\partial_\nu V_\mu(y) = 2i\,\sigma^\nu_{\alpha\dot\alpha}\,(\theta\sigma^\mu)_{\dot\beta}\,\bar\theta^{\dot\beta}\,\partial_\nu V_\mu(y)$$

$$= -2i\,\underbrace{\bar\theta^{\dot\alpha}\bar\theta^{\dot\beta}}_{\frac{1}{2}\varepsilon^{\dot\alpha\dot\beta}\bar\theta^2}\,\sigma^\nu_{\alpha\dot\alpha}\,(\theta\sigma^\mu)_{\dot\beta}\,\partial_\nu V_\mu(y)$$

$$= -i\,\bar\theta^2\,\underbrace{\sigma^\nu_{\alpha\dot\alpha}\,\varepsilon^{\dot\alpha\dot\beta}\,(\theta\sigma^\mu)_{\dot\beta}}_{(\theta\sigma^\mu(-\varepsilon)\sigma^{\nu T})_\alpha\ ??}\,\partial_\nu V_\mu(y).$$

$$\bar\sigma^{\mu\dot\alpha\alpha} = \varepsilon^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta}\sigma^\mu_{\beta\dot\beta}$$

$$\sigma^\mu_{\beta\dot\beta} = \varepsilon_{\beta\alpha}\varepsilon_{\dot\beta\dot\alpha}\bar\sigma^{\mu,\dot\alpha\alpha}$$

$$\sigma^\mu_{\beta\dot\beta}\,\varepsilon^{\dot\beta\dot\gamma} = \varepsilon_{\beta\alpha}\varepsilon_{\dot\beta\dot\alpha}\varepsilon^{\dot\beta\dot\gamma}\bar\sigma^{\mu,\dot\alpha\alpha}.$$

$$= -\varepsilon_{\beta\alpha}\bar\sigma^{\mu,\dot\gamma\alpha}$$

$$= \bar\sigma^{\mu,\dot\gamma\alpha}\varepsilon_{\alpha\beta}.$$

$$= 2i\,(\sigma^\nu\bar\theta)_\alpha\,(\theta\sigma^\mu\bar\theta)\,\partial_\nu V_\mu(y)$$

$$= i\bar\theta^2\left(\bar\sigma^{\nu,\dot\beta\gamma}\varepsilon_{\gamma\alpha}\right)(\theta\sigma^\mu)_{\dot\beta}\,\partial_\nu V_\mu(y)$$

$$= -i\bar\theta^2\,(\theta\sigma^\mu\bar\sigma^\nu\varepsilon)_\alpha\,\partial_\nu V_\mu(y)$$

Recall $\quad \sigma^\mu\bar\sigma^\nu = \eta^{\mu\nu} + 2\sigma^{\mu\nu}$

$$= -i\bar\theta^2\,\theta_\alpha\,(\partial V)(y) - 2i\bar\theta^2\,(\theta\sigma^{\mu\nu}\varepsilon)\,\underbrace{\partial_\nu V_\mu(y)}.$$

can replace with

$\sigma^{\mu\nu} = -\sigma^{\nu\mu} \Rightarrow$. $\quad \frac{1}{2}\underbrace{(\partial_\nu V_\mu - \partial_\mu V_\nu)}(y)$

$= -\frac{1}{2}f_{\mu\nu}(y) \quad f_{\nu\mu} = -f_{\mu\nu}$

Final part. $\Rightarrow \quad -2(\sigma^\mu\bar\theta)_\alpha\theta^2\bar\theta(\partial_\mu\bar\psi)(y) = -\theta^2\bar\theta^2(\sigma^\mu\partial_\mu\bar\psi)$

Putting everything together $\Rightarrow$.

$$\therefore\quad D_\alpha V_{WZ} = (\sigma^\mu\bar\theta)_\alpha V_\mu(y) + 2i\theta_\alpha(\theta\bar\psi)(y) - \underbrace{i\bar\theta^2\psi_\alpha}$$

$$+ \underbrace{\theta_\alpha\bar\theta^2}D + \underbrace{i\bar\theta^2(\sigma^{\mu\nu}\theta)_\alpha}f_{\mu\nu} - \underbrace{\theta^2\bar\theta^2(\sigma^\mu\partial_\mu\bar\psi)}$$

$$W_\alpha = -\tfrac{1}{4}\bar D^2 D_\alpha V_{WZ}$$

$\llcorner$ due to $\bar D^2$, only terms with $\bar\theta^2 \Rightarrow$ non-zero terms. (highlighted above).

$$= -i\psi_\alpha + \theta_\alpha D^{\text{real}} + i(\sigma^{\mu\nu}\theta)_\alpha f_{\mu\nu} - \theta^2(\sigma^\mu\partial_\mu\bar\psi)_\alpha.$$

GAUGE INVARIANT, CHIRAL, SUPERSYMMETRIC INVARIANT, ETC.

Can also define $\bar{W}_{\dot\alpha} = -\frac{1}{4} D^2 \bar{D}_{\dot\alpha} V$

antichiral $\quad D_\alpha \bar{W}_{\dot\alpha} = 0$

$$\bar{W}_{\dot\alpha} = \dots + \bar\theta_{\dot\alpha} D + \dots$$

should contain

$$D^\alpha W_\alpha = -\varepsilon^{\alpha\beta} D_\beta W_\alpha = 2D + \dots$$

$$(D^\beta \theta_\alpha = \delta^\beta_\alpha , \quad D_\alpha \theta^\beta = \delta^\beta_\alpha$$

$$\bar{D}^{\dot\alpha} \bar{W}_{\dot\alpha} = 2D + \dots$$

$$D^\alpha W_\alpha = \bar{D}^{\dot\alpha} W_{\dot\alpha}$$

$$D^\alpha W_\alpha + \bar{D}_{\dot\alpha} \bar{W}^{\dot\alpha} = 0 \quad \text{(Supersymmetric Bianchi Identity)}$$

$$\cancel{\partial_{[\rho} F_{\mu\nu]}} \;\; \partial_{[\mu} F_{\nu\rho]} = 0 \Leftrightarrow F_{\mu\nu} = \partial_{[\mu} V_{\nu]}$$

$$D^\alpha W_\alpha + \bar{D}_{\dot\alpha} \bar{W}^{\dot\alpha} = 0 \longleftrightarrow W_\alpha = -\frac{1}{4} \bar{D}^2 D_\alpha V$$

$$(D^\alpha \bar{D}^2 D_\alpha = \bar{D}_{\dot\alpha} D^2 \bar{D}^{\dot\alpha})$$

? sign Lecture notes