(21) Real superfields

Consider a real superfield $V = \bar{V} = V^{\dagger}$

$V(x, \theta, \bar{\theta})$. The general expansion has many components

$\theta^0 \bar{\theta}^0 \rightarrow$ scalar.

$\theta^1 \bar{\theta}^0 \rightarrow$ left Weyl spinor.

$\theta^0 \bar{\theta}^1 \rightarrow$ right Weyl spinor.

$\theta^1 \bar{\theta}^1 \rightarrow$ vector. It's the coefficient of $\theta \sigma^{\mu} \bar{\theta}$

$\theta^2 \bar{\theta}^0 \rightarrow$ scalar

$\theta^0 \bar{\theta}^2 \rightarrow$ another scalar

$\theta^2 \bar{\theta}^1 \rightarrow$ right Weyl spinor

$\theta^1 \bar{\theta}^2 \rightarrow$ left Weyl spinor.

$\theta^2 \bar{\theta}^2 \rightarrow$ scalar. One might also guess there is a tensor of type $(\theta \sigma^{\mu} \bar{\theta})(\theta \sigma^{\nu} \bar{\theta})$
Show that this is proportional to $\eta^{\mu\nu}$

There are too many fields in this superfield, even after imposing the reality condition.

We impose an equivalence relation

$$V \sim V + \phi + \bar{\phi}$$

where ϕ is a chiral superfield.

How does the vector component of V transform under this equivalence?

$$V \ni \theta \sigma^{\mu} \bar{\theta}\, v_{\mu}$$

$$\phi \ni i(\theta \sigma^{\mu} \bar{\theta})\, \partial_{\mu} \varphi \qquad , \qquad \bar{\phi} \ni -i(\theta \sigma^{\mu} \bar{\theta})\, \partial_{\mu} \bar{\varphi}$$

$$\Rightarrow \quad v_{\mu} \sim v_{\mu} + i \partial_{\mu} (\varphi - \bar{\varphi})$$

This is the same as a gauge transformation.

It is a bit tedious but it can be shown that using this equivalence we can gauge away the terms of degrees zero and one in the $\theta, \bar{\theta}$ expansion. Therefore, there is a gauge, called Wess-Zumino gauge where the expansion of V starts at order two

$$V_{WZ} = \theta \sigma^{\mu} \bar{\theta}\, v_{\mu} + i \theta^2 \bar{\theta} \bar{\lambda} - i \bar{\theta}^2 \theta \lambda + \frac{1}{2} \theta^2 \bar{\theta}^2 D$$

D is an auxiliary field, just like F for a chiral multiplet

v_{μ} is a gauge field

$\lambda, \bar{\lambda}$ are Weyl fermions.

Warning! The Wess-Zumino gauge breaks supersymmetry. When working out the supersymmetry transformations of the component fields, one needs to add compensating gauge transformations to restore the WZ gauge, before identifying the supersymmetry transformations of the component fields.

2.2 Fermionic Superfield

We constructed a superfield V containing a vector (gauge) field. One can also ask for a superfield whose bottom component is the fermion λ. This appears at order $\bar\theta^2\theta$ in the expansion of V. In order to extract it in a SUSY covariant way, we need to act with $2\,\bar D$ and one D supersymmetry covariant derivatives. We can order these derivatives in different ways, and we would obtain superfields whose expansion is slightly different. We choose the ordering where

- the resulting superfield is chiral
- the resulting superfield is gauge invariant.

Therefore, we define

$$W_\alpha = -\frac{1}{4}\bar D^2 D_\alpha V \qquad \left(\bar W_{\dot\alpha} = -\frac{1}{4} D^2 \bar D_{\dot\alpha} V\right)$$

Clearly $\bar D_{\dot\alpha} W_\alpha = 0$

(one can go to a coordinate system where $\bar D_{\dot\alpha} = \frac{\partial}{\partial\bar\theta^{\dot\alpha}}$, so then it's obvious $\bar D^3 = 0$.
Another method is to use the fact that $\bar D^3$ is of rank 3 in the spinor indices and is completely antisymmetric)

Under gauge transformations,

$$\delta W_\alpha = -\frac{1}{4}\bar D^2 D_\alpha \delta V = -\frac{1}{4}\bar D^2 D_\alpha(\Phi + \cancel{\bar\Phi}) = -\frac{1}{4}\bar D^2 D_\alpha\Phi$$

$$= -\frac{1}{4}\bar D_{\dot\alpha}\bar D^{\dot\alpha} D_\alpha \Phi = \frac{1}{4}\bar D^{\dot\alpha}\bar D_{\dot\alpha} D_\alpha\Phi = \frac{1}{4}\bar D^{\dot\alpha}\left(\{\bar D_{\dot\alpha}, D_\alpha\} - D_\alpha\cancel{\bar D_{\dot\alpha}}\right)\Phi$$

$$= \frac{1}{4}\bar D^{\dot\alpha}\left(-2\sigma^\mu_{\alpha\dot\alpha}P_\mu\Phi\right) = -\frac{1}{2}\sigma^\mu_{\alpha\dot\alpha}P_\mu\left(\bar D^{\dot\alpha}\Phi\right) = 0$$

Since W_α is gauge invariant, we can compute its expansion in the WZ gauge, where

$$V_{WZ} = (\theta\sigma^\mu\bar\theta)\,v_\mu(x) + i\,\theta^2(\bar\theta\bar\lambda)(x) - i\,\bar\theta^2(\theta\lambda)(x) + \frac{1}{2}\theta^2\bar\theta^2 D(x)$$

Shift the argument $x \longrightarrow y$. Since $y^\mu = x^\mu + i\theta\sigma^\mu\bar\theta$, we can do this for free in λ, $\bar\lambda$ and D. For the term containing v we have

$$(\theta\sigma^\mu\bar\theta)\,v_\mu(x) = (\theta\sigma^\mu\bar\theta)\left(v_\mu(y) - i(\theta\sigma^\nu\bar\theta)\partial_\nu v_\mu(y)\right)$$

Next,
$$(\theta\sigma^\mu\bar\theta)(\theta\sigma^\nu\bar\theta) = \theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\,\theta^\beta\sigma^\nu_{\beta\dot\beta}\bar\theta^{\dot\beta}$$

$$= -(\theta^\alpha\theta^\beta)(\bar\theta^{\dot\alpha}\bar\theta^{\dot\beta})\,\sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\beta}$$

$$= \frac{1}{4}\theta^2\bar\theta^2\,\varepsilon^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta}\,\sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\beta}$$

$$= \frac{1}{4}\theta^2\bar\theta^2\,\sigma^\mu_{\alpha\dot\alpha}\bar\sigma^{\nu,\dot\alpha\alpha} = \frac{1}{4}\theta^2\bar\theta^2\,\mathrm{tr}(\sigma^\mu\bar\sigma^\nu) = \frac{1}{2}\theta^2\bar\theta^2\,\eta^{\mu\nu}$$

$$\Rightarrow\ (\theta\sigma^\mu\bar\theta)\,v_\mu(x) = (\theta\sigma^\mu\bar\theta)\,v_\mu(y) - \frac{i}{2}\theta^2\bar\theta^2(\partial\cdot v)(y)$$

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$$\Rightarrow \quad V_{WZ} = (\theta\sigma^\mu\bar\theta)\, v_\mu(y) + i\,\theta^2(\bar\theta\bar\lambda)(y) - i\,\bar\theta^2(\theta\lambda)(y)$$
$$+ \frac{1}{2}\,\theta^2\bar\theta^2\,(D(y) - i\,\partial\cdot v(y))$$

For this coordinate system,

$$\begin{cases} D_\alpha = \partial_\alpha + 2i\,(\sigma^\mu\bar\theta)_\alpha \dfrac{\partial}{\partial y^\mu} \\ \bar D_{\dot\alpha} = \bar\partial_{\dot\alpha} \end{cases}$$

$$D_\alpha V_{WZ} = (\sigma^\mu\bar\theta)_\alpha\, v_\mu(y) + 2i\,\theta_\alpha(\bar\theta\bar\lambda)(y) - i\,\bar\theta^2\lambda_\alpha(y) + \theta_\alpha\bar\theta^2(D - i\partial v)$$
$$+ 2i(\sigma^\mu\bar\theta)_\alpha\left[(\theta\sigma^\nu\bar\theta)\,\partial_\mu v_\nu(y) + i\,\theta^2\bar\theta\,(\partial_\mu\bar\lambda)(y)\right]$$

Using $\quad \bar\theta^{\dot\alpha}\bar\theta^{\dot\beta} = \frac{1}{2}\,\varepsilon^{\dot\alpha\dot\beta}\,\bar\theta^2$

$\qquad \sigma^\mu\bar\sigma^\nu = \eta^{\mu\nu} + 2\sigma^{\mu\nu}$

we find

$$2i\,(\sigma^\mu\bar\theta)_\alpha\,(\theta\sigma^\nu\bar\theta)\,\partial_\mu v_\nu(y) = i\,\bar\theta^2\left[(\eta^{\mu\nu} + \sigma^{\mu\nu})\theta\right]_\alpha \partial_\mu v_\nu$$
$$= i\,\bar\theta^2\,\theta_\alpha(\partial\cdot v) + i\,\bar\theta^2\,(\sigma^{\mu\nu}\theta)_\alpha\, f_{\mu\nu},$$

where $\quad f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$

Also,

$$-2\,(\sigma^\mu\bar\theta)_\alpha\,\theta^2\,\bar\theta\,(\partial_\mu\bar\lambda) = -\,\theta^2\bar\theta^2\,(\sigma_\mu\,\partial_\mu\bar\lambda)_\alpha$$

$$\Rightarrow \quad D_\alpha V_{WZ} = (\sigma^\mu\bar\theta)_\alpha\, v_\mu + 2i\,\theta_\alpha(\bar\theta\bar\lambda) - i\,\bar\theta^2\lambda_\alpha$$
$$+ \theta_\alpha\bar\theta^2 D + i\,\bar\theta^2(\sigma^{\mu\nu}\theta)_\alpha f_{\mu\nu} - \theta^2\bar\theta^2(\sigma_\mu\partial_\mu\bar\lambda)$$

$$\Rightarrow \quad W_\alpha = -\frac{1}{4}\,\bar D^2 D_\alpha V = -\frac{1}{4}\,\bar D^2 D_\alpha V_{WZ}$$
$$= -i\,\lambda_\alpha + \theta_\alpha D + i\,(\sigma^{\mu\nu}\theta)_\alpha f_{\mu\nu} - \theta^2\,(\sigma_\mu\partial_\mu\bar\lambda)_\alpha$$