Lecture 7

10.00 28/2/17

Dimensional analysis

$$S = \int d^4x \, (\partial_\mu \bar{\varphi})(\partial^\mu \varphi)$$ free complex scalars.

In units $c = \hbar = 1$ the action is dimensionless
$\quad$ natural units

$$[S] = 0$$

$$[d^4x] = -4 \quad \text{(mass dimensions } L^{-1}\text{)}$$
$$[\partial_\mu] = 1$$

$$[S] = 0 = \underbrace{-4}_{d^4x} \underbrace{+2}_{\partial_\mu} + 2[\varphi] \Rightarrow [\varphi] = 1$$

For a Fermion. $S = \int d^4x \, \bar{\psi} \not{\partial} \psi$

$$[S] = 0 = \underbrace{-4}_{[d^4x]} \underbrace{+1}_{[\not{\partial}]} + 2[\psi] \Rightarrow [\psi] = 3/2.$$

$[\theta]$

10.20 am $\quad x + \theta\sigma^\mu\bar{\theta}$ - Expression must have same dimensions.

$$[x] = -1 = 2[\theta] \Rightarrow [\theta] = -1/2.$$

$[\varphi] = 1$
$[\psi] = 3/2$
$[\theta] = -1/2$
$[F] = 2$
$[D_\alpha] = [\bar{D}_{\dot{\alpha}}] = +1/2$
(contains derivative of θ

Dimension

$$\Phi(y,\theta) = \overbrace{\varphi(y)}^{1} + \sqrt{2}\,\overbrace{\theta}^{-1/2}\,\overbrace{\psi(y)}^{3/2} - \overbrace{\theta^2}^{-1}\,\overbrace{F(y)}^{2}$$

scalar field $\quad$ spinor field $\quad$ auxiliary field.

$\Phi(y,\theta)$ automatically satisfies constraints.

Building Action / Lagrangian.

$$S = \int d^4x \, d^2\theta \, d^2\bar\theta \; ? \, (\Phi, \bar\Phi)$$

- The action is real (Φ and $\bar\Phi$ have to appear symmetrically)

  $\bar\Phi$ complex conjugate of Φ

- The action is dimension 0

$$0 = \underbrace{-4}_{\int d^4x} \; \underbrace{+1}_{\int d^2\theta = \frac{1}{4}D^2} \; \underbrace{+1}_{\int d^2\bar\theta = \frac{1}{4}\bar D^2} \; + [?].$$

$\Rightarrow \quad [?] = 2. \quad \Rightarrow$ Suggestion $? = \bar\Phi \Phi$

$$\Rightarrow \quad S = \int d^4x \, d^2\theta \, d^2\bar\theta \; \bar\Phi \Phi$$

$$= \frac{1}{16} \int d^4x \, D^2 \bar D^2 \left(\bar\Phi \Phi \right)$$

the spacetime derivatives arise from integrating using the odd coordinates θ and $\bar\theta$

$$\Rightarrow \quad = \int d^4x \Big[\underbrace{(\partial_\mu \bar\varphi)(\partial^\mu \varphi)}_{\text{Free complex scalar}} + \underbrace{\frac{i}{2}\left(\partial_\mu \psi \sigma^\mu \bar\psi - \psi \sigma^\mu \partial_\mu \bar\psi \right)}_{\text{Free fermion.}} + \underbrace{F\bar F}_{\text{auxiliary field}} + \underbrace{\text{total derivative}}_{\text{Ignore}}$$

E of Motion for F is $F = 0$. - has no dynamics.

- but has a role "on shell"
- has to be transformed.

E of M for φ : $\Box \varphi = 0$ $\quad (\Box \bar{\varphi} = 0)$

" " ψ : $\sigma^\mu \partial_\mu \bar{\psi} = 0$ $\quad (\bar{\sigma}^\mu \partial_\mu \psi = 0)$

E of M in superspace.

Need to be able to take the variation of Φ and $\bar{\Phi}$, but they are constrained.

$$S = \int d^4x\, d^2\theta \underbrace{d^2\bar{\theta}}_{\frac{1}{4}\bar{D}^2} \bar{\Phi}\Phi \qquad (\ast\ast)$$

shown in tutorial.

$\int d^2\bar{\theta} = \frac{1}{4}\bar{\partial}^2$

$\bar{D}_{\dot{\alpha}} = \bar{\partial}_{\dot{\alpha}} + i(\theta\sigma^\mu)_{\dot{\alpha}}\partial_\mu,$

$\bar{\partial}_{\dot{\alpha}} = \bar{D}_{\dot{\alpha}} - i(\theta\sigma^\mu)_{\dot{\alpha}}\partial_\mu$

$= \bar{D}_{\dot{\alpha}} - \partial_\mu(i\theta\sigma^\mu)_{\dot{\alpha}})$

$\Rightarrow \bar{\partial}^2 = \bar{D}^2$ + total derivatives.

$$= \int d^4x\, d^2\theta\, \frac{1}{4}\bar{\partial}^2(\bar{\Phi}\Phi)$$

$$= \frac{1}{4}\int d^4x\, d^2\theta\, \bar{D}^2(\bar{\Phi}\Phi).$$

$\underbrace{\bar{D}_{\dot{\alpha}}\Phi = 0}$, $\bar{D}^2 = \bar{D}_{\dot{\alpha}}\bar{D}^{\dot{\alpha}} = \bar{D}_{\dot{\alpha}}(-\varepsilon^{\dot{\alpha}\dot{\beta}}\bar{D}_{\dot{\beta}})$

$\Rightarrow \underbrace{\bar{D}^{\dot{\alpha}}}\Phi = 0$

$(-\varepsilon^{\dot{\alpha}\dot{\beta}})(\bar{D}_{\dot{\beta}}\Phi) = 0.$

$$\bar{D}^2(\bar{\Phi}\Phi) = \bar{D}_{\dot{\alpha}}\bar{D}^{\dot{\alpha}}(\bar{\Phi}\Phi)$$

$$= \bar{D}_{\dot{\alpha}}\left((\bar{D}^{\dot{\alpha}}\bar{\Phi})\Phi + \bar{\Phi}\underbrace{(\bar{D}^{\dot{\alpha}}\Phi)}_{=0}\right)$$

$+$ since $\bar{\Phi}$ = even.

So $\bar{D}^2(\bar{\Phi}\Phi) = \bar{D}_{\dot{\alpha}}\left((\bar{D}^{\dot{\alpha}}\bar{\Phi})\Phi\right)$

Leibnitz rule

$$= (\bar{D}_{\dot{\alpha}}\bar{D}^{\dot{\alpha}}\bar{\Phi})\Phi - (\bar{D}^{\dot{\alpha}}\bar{\Phi})\underbrace{(\bar{D}_{\dot{\alpha}}\Phi)}_{=0 \text{ as above}}$$

since $(\bar{D}^{\dot{\alpha}}\bar{\Phi})$ is odd

D contains $\frac{\partial}{\partial\theta}$ (I think!).

$$= (\bar{D}^2\bar{\Phi})\Phi$$

$$\Rightarrow S = \frac{1}{4}\int d^4x\, d^2\theta\, (\bar{D}^2\bar{\Phi})\Phi$$

When take variation w.r.t Φ $\Rightarrow$ E of M

$$\Rightarrow \quad \bar{D}^2 \bar{\Phi} = 0$$

Note. Could naively take variation wrt Φ from (**). $\Rightarrow$ E of M $\bar{\Phi} = 0$. but this would be wrong!

Two terms in Integrand $(\bar{D}^2\bar{\Phi})$ and Φ

- Φ chiral $(\bar{D}_{\dot\alpha} \Phi = 0)$.
- $\bar{D}^2 \bar{\Phi}$ also chiral since $\bar{D}_{\dot\alpha} \bar{D}^2 = 0$. (3 Ds $\Rightarrow$ 0. (Grassmann odd, only 2 indices available)

Th.

$$\left.\begin{array}{l} \delta S = \int d^4x\, d^2\theta\, \delta Y\, G = 0. \\ Y, G \text{ are chiral} \end{array}\right\} \quad G = 0.$$

$$\left.\begin{array}{l} \delta_{\Phi} S = \int d^4x\, d^2\theta\, (\bar{D}^2 \bar{\Phi})\, \delta \Phi = 0 \\ \Phi,\ \bar{D}^2\bar{\Phi} \text{ chiral} \end{array}\right\} \Rightarrow (\bar{D}^2\bar{\Phi}) = 0. \quad \xrightarrow{\text{conjugate}} D^2 \Phi = 0.$$

Next

11.10 am listen.

$$D^2 \Phi = 0 \quad , \quad \Phi = \Phi(Y, \theta)$$

- Write D in Y, θ coordinates
- Plug in the $\Phi(Y,\theta)$ expansion $\Phi(Y,\theta) = \varphi(Y) + \sqrt{2}\,\theta\,\psi(Y) - \theta^2 F(Y)$
- Identify the components

$$S = \int d^4x\, d^2\theta\, d^2\bar{\theta}\, \bar{\Phi} \Phi$$

Is it the most general action?

Consider

$$S = \int d^4x\, d^2\theta\, d^2\bar{\theta}\, K(\bar{\Phi}, \Phi)$$

Constraints on K.

- $K(\bar{\Phi}, \Phi)$ should be real
- $[K] = 2$
- K does not depend on $D_\alpha \Phi$, $\bar{D}_{\dot\alpha} \bar{\Phi}$ (if it did, higher order derivatives in the Action).

- At lower energies, higher order derivative terms are suppressed.
- Also difficult to work with.

• Redefining $K(\bar{\Phi}, \Phi) \rightarrow K(\bar{\Phi}, \Phi) + \Lambda(\Phi) + \bar{\Lambda}(\bar{\Phi})$

does not change the Action.

holomorphic function and $\bar{\Lambda}$ is its c.c.

check.

$$\int d^4x \, d^2\theta \underbrace{d^2\bar{\theta}}_{\bar{D}^2} \Lambda(\Phi) = 0$$

as.

$$\Rightarrow \bar{D}^2 \Lambda(\Phi) = \bar{D}_{\dot{\alpha}} \Big(\underbrace{(\bar{D}^{\dot{\alpha}} \Phi)}_{0} \Lambda'(\Phi) \Big) = 0$$

Take K as a power series

$$K(\bar{\Phi}, \Phi) = \sum_{m,n=1}^{\infty} C_{mn} \bar{\Phi}^m \Phi^n \qquad C^*_{mn} = C_{nm}$$

from reality

$$[K] = 2 = [C_{mn}] + m + n$$

$$\Rightarrow [C_{mn}] = 2 - m - n$$

In a renormalisable theory, the couplings cannot have -ve mass dimensions

$$\Rightarrow 2 - m - n \geq 0$$

If $2-m-n < 0$, then $C_{mn} = 0$

Possible terms for a renormalisable theory

11.27/8am (?) can split into anti-hol holomorphic $\{ m=0, n=0 \qquad m=1 \quad n=0$

anti holomorphic $\begin{cases} m=0, n=1 \\ m=0, n=2 \end{cases}$ $\quad m=2 \quad n=0 \}$ holomorphic

$m=1 \quad n=1$ | only term that has an effect.

$$\text{ie. } K(\bar{\Phi}, \Phi) = C_{11} \bar{\Phi} \Phi$$

So far only free fields

There are renormalisable supersymmetric field theories. How to get them?

Chiral integrals.

If Σ is a chiral superfield, then

$$\int d^4x \, d^2\theta \, \Sigma \quad \text{is SUSY invariant.}$$

Chiral integrals are more general than non-chiral full superspace integrals since

$$\int d^4x \, d^2\theta \, d^2\bar{\theta} \, \Lambda = \int d^4x \, d^2\theta \, \underbrace{\frac{1}{4}\bar{D}^2 \Lambda}_{\text{chiral}}$$

can also rewrite as an anti-chiral integral.

i.e can transform into a chiral integral.

However, not always possible to rewrite a chiral (anti-chiral) superspace integral as a non-chiral full superspace integral.

Interactions for chiral (anti-chiral) superfields are chiral (anti-chiral) superspace integrals

$$S_{int} = \int d^4x \, d^2\theta \, \underbrace{W(\Phi)}_{\substack{\text{holomorphic} \\ \text{function}}} + \int d^4x \, d^2\bar{\theta} \, \bar{W}(\bar{\Phi})$$

CALLED SUPERPOTENTIAL.

Constraints on W

- W is holomorphic — a powerful contraint
- $[W] = 3$
- As a power series, $W(\Phi) = \sum_{n=0}^{\infty} c_n \Phi^n \qquad c_n \in \mathbb{C}$

Taking dimensions :

$$[W] = 3 = [c_n] + n$$

$$\therefore \quad [c_n] = 3 - n$$

Renormalisability $\Rightarrow$ if $3 - n < 0$, then $c_n = 0$.

Non-zero coefficients for $3 - n \geq 0 \Rightarrow n = 1, 2, 3$.

So the most general, renormalisable superpotential is :-

$$W(\Phi) = c_1 \Phi + c_2 \Phi^2 + c_3 \Phi^3$$

$$\Phi = \varphi(x) + \theta(\theta, \bar{\theta})$$

Taylor series

$$W(\Phi) = W(\varphi) + (\Phi - \varphi) W'(\varphi) + \frac{1}{2} (\Phi - \varphi)^2 W''(\varphi)$$

$$\ldots \uparrow$$

Expand around $\Phi = \varphi$

no more terms as $(\Phi - \varphi)^3 = 0$

11.48

$$W(\Phi) = W(\varphi) + \sqrt{2}(\theta\psi) W' - \theta^2 \left(W' F + \tfrac{1}{2} W'' \psi\psi\right)$$

$$(\Phi - \varphi)^2 = \left(\sqrt{2}\,\theta\psi - \cancel{\theta^2 F}\right)^2 = 2(\theta\psi)^2 = -\theta^2\psi^2$$

~~missing 2 ??~~ No !

$$\int d^4x \, d^2\theta \, W(\Phi) = -\int d^4x \left(W' F + \frac{1}{2} W'' \psi\psi\right)$$

$$-\theta^\alpha \theta_\beta^{\ \beta} \psi_\alpha \psi_\beta = -\tfrac{1}{2} \varepsilon^{\alpha\beta} \ldots\ldots$$

$$S_{int} = -\left[\int d^4x \left(W' F + \tfrac{1}{2} W'' \psi\psi\right) + c.c.\right]$$

Last step:

Solve EofM for F and plug the sol^n back into the Action.

$$S_{kin} \longrightarrow \bar{F} F$$

$$S_{int} \longrightarrow -W' F - \bar{W}' \bar{F}$$

Variation w.r.t. F

$$\Rightarrow \quad \bar{F} - W' = 0$$

Variation w.r.t. $\bar{F}$

$$\Rightarrow \quad F - \bar{W}' = 0$$

$\leftarrow$ scalar potential.

Plug back $\cancel{|W'|^2 - 2|W'(\varphi)|^2}$

$$|W'|^2 - 2|W'|^2 = -|W'(\varphi)|^2$$