Lecture 7

as a reminder we discussed chiral multiplets

$$\bar{D}_{\dot\alpha} \Phi = 0$$

Anti-chiral

$$D_\alpha \bar{\Phi} = 0$$

Chiral ~~superfield~~ mult $\Phi \longrightarrow \bar{\Phi}$ anti-chiral superfield

$\bar{D}_{\dot\alpha} \Phi = 0$ $\quad (D_\alpha \bar{\Phi} = 0)$ are supersymmetric

$D, \bar{D}$ anti-commute with $Q, \bar{Q}$

its a supersymetric invariant notion.

So we have these chiral superfields

$$\Phi(x, \theta, \bar{\theta}) = \Phi(y, \theta)$$

$$y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$$

best written with shifted coord

$\Phi(y, \theta)$ automatically satisfies the constraints $\bar{D}_{\dot\alpha}\Phi = 0$.

- Grassman expansion

$$\Phi(y, \theta) = \varphi(y) + \sqrt{2}\,\theta\psi(y) - \theta^2 F(y)$$

Complex-Scalar field

Spinor field

Auxiliary field 'F'

Dimensions of fields.

So if we want to write the action for a free complex scalar

$$S = \int d^d x \, (\partial_\mu \bar{\phi})(\partial^\mu \phi)$$ for complex scalar

for units $\hbar = c = 1$ the action is dimensionless

$[S]=0$ $\qquad$ (mass dim) $= L^{-1}$

$[d^4x] = -4$

$[\partial_\mu] = 1$

$[S] = -4 + 2 + 2[\phi] = \boxed{[\phi] = 1}$

Scalar - side

The action for a fermion

$$S = \int d^4x \, \bar{\psi} \not{\partial} \psi$$

fermion

$$[S] = 0 = \underbrace{-4}_{d^4x} + \underbrace{1}_{\not{\partial}} + 2[\psi] \quad \therefore \quad \boxed{[\psi] = \tfrac{3}{2}}$$

$\therefore$ dim of θ

$$[x] = -1 = 2[\theta] \quad \therefore \quad \boxed{[\theta] = -\tfrac{1}{2}}.$$

using $y = x + i\theta\sigma^\mu\bar{\theta}$

Looking at the Grassmann expansion (superview)

$$\Phi(y,\theta) = \underbrace{\varphi(y)}_{[1]} + \sqrt{2}\,\underbrace{\theta}_{-1/2}\,\underbrace{\psi(y)}_{3/2} - \underbrace{\theta^2}_{[-1]}\,\underbrace{F(y)}_{[2]}$$

So we have

$$[\phi] = [\Phi] = 1, \quad [\psi] = 3/2, \quad [\theta] = -1/2, \quad [F] = 2$$

$$[D_\alpha] = [\bar{D}_{\dot{\alpha}}] = 1/2.$$

To build up the action we will have to integrate over superspace.

$$S = \int d^4x\, d^2\theta\, d^2\bar{\theta}\; ?\,(\Phi, \bar{\Phi})$$

- the action is real (Φ and $\bar{\Phi}$ have to appear symmetrically)
  $\uparrow$ complex $\bar{\Phi}$

- The action is dimension 0

$$0 = \underbrace{-4}_{\int d^4x} \underbrace{+1}_{\int d^2\theta = \frac{1}{4}D^2} \underbrace{+1}_{\int d^2\bar{\theta} = \frac{1}{4}\bar{D}^2} + ?$$

$\therefore$ $[?] = 2$ as our integrand has dimension [2].

i.e. $\theta\bar{\theta}$.

So the action is

$$S = \int d^4x \, d^2\theta \, d^2\bar{\theta} \; \bar{\Phi}\Phi$$

↓ Space-time

$$= \frac{1}{16}\int d^4x \; D^2\bar{D}^2(\bar{\Phi}\Phi)$$

Tutorial

The space-time derivatives arise from integrating the odd variables θ and $\bar{\theta}$

Spacetime Action

$$= \int d^4x \Big[\underbrace{\partial_\mu\bar{\phi}\,\partial^\mu\phi}_{\text{free complex scalar.}} + \frac{i}{2}\underbrace{\left(\partial_\mu\psi\,\sigma^\mu\bar{\psi} - \psi\,\sigma^\mu\partial_\mu\bar{\psi}\right)}_{\text{free fermion}}$$

$$+ \underbrace{\bar{F}F}_{\text{Auxiliary}} + \text{Total derivative} \Big]$$

E.O.M for F is $\boxed{F = 0}$

if you want to close the supersymmetry off-shell you need to transform F as well - however it isn't needed for on-shell

E.O.M for ϕ, $\Box\phi = 0$, $\Box\bar{\phi} = 0$

complex conjugate

ψ, $\partial^\mu\partial_\mu\bar{\psi} = 0$, $(\bar{\sigma}^\mu\partial_\mu\psi = 0)$

E.O.M in Superspace.

$$S = \int d^4x \, d^2\theta \underbrace{d^2\bar{\theta}}_{\frac{1}{2}\bar{D}^2} \bar{\Phi}\Phi$$

note

$\int d^2\bar{\theta} = \frac{1}{4}\bar{\partial}^2$

$\bar{D}_{\dot{\alpha}} = \bar{\partial}_{\dot{\alpha}} + i(\overset{\theta}{\bar{\sigma}^{\mu}})_{\dot{\alpha}}\,\partial_{\mu}$, $\bar{\partial}_{\dot{\alpha}} = \bar{D}_{\dot{\alpha}} - i(\theta\sigma^{\mu})_{\dot{\alpha}}\,\partial_{\mu}$

$\bar{D}_{\dot{\alpha}} - \partial_{\mu}(\,i(\theta\sigma^{\mu})_{\dot{\alpha}})$

$\therefore\quad \bar{\partial}_{\dot{\alpha}} = \bar{D}_{\dot{\alpha}} +$ Total derivative.

$\therefore S = \int d^4x\, d^2\theta\, d^2\bar{\theta}\, \bar{\Phi}\Phi = \int d^4x\, d^2\theta\, \frac{1}{4}\bar{\partial}^2(\bar{\Phi}\Phi)$

$= \frac{1}{4}\int d^4x\, d^2\theta\, \bar{D}^2(\bar{\Phi}\Phi)$

$S = \frac{1}{4}\int d^4x\, d^2\theta\, \bar{D}^2(\bar{\Phi}\Phi)$

note $\bar{D}_{\dot{\alpha}}\bar{\Phi} = 0$, $\bar{D}^2 = \bar{D}_{\dot{\alpha}}\bar{D}^{\dot{\alpha}} = \bar{D}_{\dot{\alpha}}(-\varepsilon^{\dot{\alpha}\dot{\beta}}\bar{D}_{\dot{\beta}})$

$\hookrightarrow \bar{D}^{\dot{\alpha}}\bar{\Phi} = 0.\quad -\varepsilon^{\dot{\alpha}\dot{\beta}}\underbrace{(\bar{D}_{\dot{\beta}}\bar{\Phi})}_{=0.} = 0$

we want to just compute.

$\bar{D}^2(\bar{\Phi}\Phi)$

so $\bar{D}^2(\bar{\Phi}\Phi) = \bar{D}_{\dot{\alpha}}\bar{D}^{\dot{\alpha}}(\bar{\Phi}\Phi)$

$\bar{\Phi}$ is even

$= \bar{D}_{\dot{\alpha}}\left[(\bar{D}^{\dot{\alpha}}\bar{\Phi})\,\Phi + \bar{\Phi}\,\cancel{(\bar{D}^{\dot{\alpha}}\Phi)}\right]$

0

so $\bar{D}^2(\bar{\Phi}\bar{\Phi}) = \bar{D}_{\dot\alpha}\left((\bar{D}^{\dot\alpha}\bar{\Phi})\bar{\Phi}\right)$

$$= (\bar{D}_{\dot\alpha}\bar{D}^{\dot\alpha}\bar{\Phi})\bar{\Phi} - (\bar{D}^{\dot\alpha}\bar{\Phi})\underbrace{\cancel{(\bar{D}_{\dot\alpha}\bar{\Phi})}}_{0}.$$

(−) since $(\bar{D}^{\dot\alpha}\bar{\Phi})$ is odd.

in the end we are left with

$$\bar{D}^2(\bar{\Phi}\bar{\Phi}) = (\bar{D}_{\dot\alpha}\bar{D}^{\dot\alpha}\bar{\Phi})\bar{\Phi}$$

∴ in conclusion the action

$$S = \frac{1}{4}\int d^4x\, d^2\theta\, (\bar{D}^2\bar{\Phi})\bar{\Phi}$$

we had

$S = \int d^4x\, d^2\theta\, d^2\bar\theta\, \bar{\Phi}\Phi$

A naive variation w.r.t Φ

→ E.O.M, $\bar{\Phi} = 0$

This is wrong though

when you vary w.r.t Φ

we get E.O.M $\boxed{\bar{D}^2\bar{\Phi} = 0}$ ← This is the correct E.O.M.

To prove this

That is the chiral condition

The 2 terms

* Φ · chiral $(\bar{D}_{\dot\alpha}\Phi = 0)$

* $\bar{D}^2\bar{\Phi}$ chiral since $\bar{D}_{\dot\alpha}\bar{D}^2 = 0$

Theorem

$$\left.\delta S = \int d^4x\, d^2\theta\, \delta Y \cdot G = 0.\right\} \Rightarrow G = 0.$$

Y, G are chiral

in our case

$$\left.\delta_\Phi S = \int d^4x\, d^2\theta\, (\bar{D}^2\bar{\Phi})\delta\Phi = 0\right\} \rightarrow \boxed{\bar{D}^2\bar{\Phi} = 0}$$

Φ, $\bar{D}^2\bar{\Phi}$; chiral

note! if you now change $d^2\theta \rightarrow D^2$ and work it out you get the space-time action.

Side note:

$$D^2\bar{\Phi} = 0 \qquad \bar{\Phi}(Y,\theta)$$

- write D in Y,θ coords
- plug in $\Phi(Y,\theta)$ Expansion $\Phi(Y,\theta) = \varphi(y) + \sqrt{2}\,\theta\psi(y) - \theta^2 F($
- ~~Step~~: identify the components.

$$S = \int d^4x\, d^2\theta\, d^2\bar{\theta}\, (\bar{\Phi}\Phi)$$

is it the most General?

$$S = \int d^4x\, d^2\theta\, d^2\bar{\theta}\, K(\bar{\Phi},\Phi)$$

lets try sine something like this.

we need to constrain K

- $K(\bar{\Phi},\Phi)$ should be real
- $[K] = 2$.
- K does not depend on $D_\alpha\Phi$ or $\bar{D}_{\dot\alpha}\bar{\Phi}$

(if it did depend on this, we would get high order derivatives in the action.)

- Redefining

$$K(\bar{\Phi},\Phi) \to K(\bar{\Phi},\Phi) + \Lambda\Phi + \bar{\Lambda}(\bar{\Phi})$$

holomorphic function, $\bar{\Lambda}$ is complex conjugate

This does not change the action.

i.e.

$$\int d^4x\, d^2\theta \underbrace{d^2\bar\theta}_{\frac{1}{4}\bar D^2} \Lambda(\bar\Phi) = 0$$

$$\bar D^2 \Lambda(\bar\Phi) = \bar D_{\dot\alpha}\left(\underbrace{\cancel{\bar D^{\dot\alpha}\Phi}}_{0}\,\Lambda'(\bar\Phi)\right) = 0.$$

Take K as a power series.

$$K(\bar\Phi, \Phi) = \sum_{m,n=1}^{\infty} c_{mn}\, \bar\Phi^m \Phi^n \qquad c_{mn}^* = c_{nm}$$

from reality.

$[K] = 2. = [c_{m,n}] + m + n$

$\Rightarrow [c_{m,n}] = 2 - m - n$

in a naively Renormalisable Theory the couplings cannot have negative mass dimensions

$\Rightarrow 2 - m - n \geq 0$

If $2 - m - n < 0$ ← non-Renormalisable, then $c_{m,n} = 0$

← These are just set to zero.

Possible terms for a Renormalisable Theory

$\boxed{m=0,\ n=0}$ → real const.

$m=0,\ n=1$
$m=0,\ n=2$ → Anti-holomorphic

$m=1,\ n=0$
$m=2,\ n=0$ → Anti-holomorphic

$\boxed{m=1,\ n=1}$ → The only terms that have an effect.

$$K(\bar{\Phi}, \Phi) = c_{11} \bar{\Phi}\Phi$$

We get back what we started with.

However its slightly worrying as we only have 3 fields, does this mean we can only have a supersymmetric theory with 3 fields, How can we get interactions?

- There are Renormalisable supersymmetric field theories - How to get them?

- Chiral Integrals.

If Σ is a Chiral Superfield, Then

$$\int d^4x \, d^2\theta \, \Sigma$$

is susy invariant, The proof of this is similar to what we did for the full supersymmetry - i.e do a change of coords Solve the Sup-Jacobian etc.

- Chiral Integrals are more general than non-chiral ~~that~~ ~~full superspace~~ full superspace integrals. Since Full superspace integrals

$$\int d^4x \, d^2\theta \, d^2\bar{\theta} \, \Lambda = \int d^4x \, d^2\theta \, \underbrace{\frac{1}{4} \bar{D}^2 \Lambda}_{\text{chiral}}$$

You can't always transform the other way $\rightarrow$ to a anti-chi

i.e. not always possible to write a Chiral (anti-chiral) Superspace integral as a non-chiral - full superspace integral

Interactions for chiral (anti-chiral) superfields and are chiral (anti-chiral) ~~integra~~ superspace integrals.

$$S_{INT} = \int d^4x \, d^2\theta \, \underbrace{W(\Phi)}_{\text{Holomorphic function}} + \int d^4x \, d^2\bar{\theta} \, \overline{W}(\bar{\Phi})$$

(called superpotential)

Constraints on W

- W is holomorphic (Tiny ~~[illegible]~~ minority Holomorphic functions)

- $[W] = 3$.

- As a power series

$$W(\Phi) = \sum_{n=1}^{\infty} c_n \Phi^n \qquad c_n \in \mathbb{C}$$

we start at $n=1$ as $n=0$ is just a constant.

$$[W] = 3 = [c_n] + n$$

$$[c_n] = 3 - n$$, Renormalisability implies if $3-n < 0$ then $c_n = 0$ (set bad terms to 0)

non-zero coeff only for $3 - n \geq 0$, $3 \geq n \geq 1$

The most General Renormalisable Super potential is

$$W(\Phi) = c_1 \Phi + c_2 \Phi^2 + c_3 \Phi^3$$

if we interact

$$\Phi = \varphi + \mathcal{O}(\theta\xi, \bar\theta)$$

so $W(\Phi) = W(\varphi) + (\Phi - \varphi) W'(\varphi) + \frac{1}{2}(\Phi - \varphi)^2 W''(\varphi)$

Expand around $\Phi = \varphi$ Stops here since $(\Phi - \varphi)^3 = 0$.

$$W(\Phi) = W(\varphi) + \sqrt{2}(\theta\psi) W' - \theta^2 (W'F + \tfrac{1}{2} W'' \psi\psi)$$

$$(\Phi - \varphi)^2 = (\sqrt{2}\theta\psi - \theta^2 F)^2 = 2(\theta\psi)^2 = -\theta^2\psi^2$$

$$\int d^4x \, d^2\theta \, W(\Phi) = -\int d^4x \left(W'F + \frac{1}{2} W'' \psi\psi \right)$$

$\therefore$

$$S_{int} = -\int d^4x \left(W'F + \tfrac{1}{2} W'' \psi\psi) + \text{complex. conjugate} \right)$$

The last step solve E.O.M for F and plug ~~in~~ the solution back into the action.

$S_{kin} \to \bar{F}F$

$S_{int} \to -W'F - \bar{W}'\bar{F}$

variation w.r.t F implies $\bar{F} - W' = 0$

" " $\bar{F}$ " $F - \bar{W}' = 0$

Scalar potential.

plug back: $|W'|^2 - 2|W'|^2 = -|W'_\varphi|^2$