(18) Supersymmetric actions

Consider the integral

$$\int d^4x \, d^2\theta \, d^2\bar\theta \; Y(x, \theta, \bar\theta)$$

We will show that this is invariant under supersymmetry transformations.

Consider a supersymmetry transformation

$$\delta x^\mu = i\theta\sigma^\mu\bar\varepsilon - i\varepsilon\sigma^\mu\bar\theta$$

$$\delta\theta = \varepsilon, \qquad \delta\bar\theta = \bar\varepsilon$$

$$x' = x + \delta x, \qquad \theta' = \theta + \delta\theta, \qquad \bar\theta' = \bar\theta + \delta\bar\theta$$

There is a notion of Jacobian of change of coordinates, also in the supersymmetric case. There, it is

$$\left(\begin{array}{c|cc} \frac{\partial x'^\mu}{\partial x^\nu} & \frac{\partial x'^\mu}{\partial \theta^\beta} & \frac{\partial x'^\mu}{\partial \bar\theta^{\dot\beta}} \\ \hline \frac{\partial \theta'^\alpha}{\partial x^\nu} & \frac{\partial \theta'^\alpha}{\partial \theta^\beta} & \frac{\partial \theta'^\alpha}{\partial \bar\theta^{\dot\beta}} \\ \frac{\partial \bar\theta'^{\dot\alpha}}{\partial x^\nu} & \frac{\partial \bar\theta'^{\dot\alpha}}{\partial \theta^\beta} & \frac{\partial \bar\theta'^{\dot\alpha}}{\partial \bar\theta^{\dot\beta}} \end{array}\right) = \left(\begin{array}{c|cc} \delta^\mu_\nu & i(\sigma^\mu\bar\varepsilon)_\beta & i(\varepsilon\sigma^\mu)_{\dot\beta} \\ \hline 0 & \delta^\alpha_\beta & 0 \\ 0 & 0 & \delta^{\dot\alpha}_{\dot\beta} \end{array}\right)$$

I will not prove this, but in this case the supersymmetric version of the determinant of this supermatrix is identity. This should be plausible since it's upper triangular.

In conclusion, we have $\quad d^4x' \, d^2\theta' \, d^2\bar\theta' = d^4x \, d^2\theta \, d^2\bar\theta.$

Using this in the formula for the action we have

$$\int d^4x' \, d^2\theta' \, d^2\bar\theta' \; Y(x', \theta', \bar\theta') = \int d^4x \, d^2\theta \, d^2\bar\theta \Big(Y(x, \theta, \bar\theta) +$$
$$+ \delta x \cdot \partial Y(x, \theta, \bar\theta) + \delta\theta \cdot \partial Y(x, \theta, \bar\theta) + \delta\bar\theta \cdot \bar\partial Y(x, \theta, \bar\theta) \Big)$$

→ The terms of type $\quad \delta\theta \, \partial Y \quad$ vanish because they lead to triple derivatives in the odd direction.

→ The term $\quad \delta x \, \partial Y \quad$ can be written as a total derivative $\partial_\mu(\delta x^\mu Y)$ since δx does not depend on x.

We conclude that

$$\int d^4x \, d^2\theta \, d^2\bar\theta \; Y(x, \theta, \bar\theta)$$

is invariant under supersymmetry, up to boundary terms.

In general, Y will be a product of superfields.

(19) <u>Chiral superfields</u>

we can construct supersymmetry covariant derivatives

$$\begin{cases} D_\alpha = \partial_\alpha + i\,\sigma^\mu_{\alpha\dot\alpha}\,\bar\theta^{\dot\beta}\,\partial_\mu \\ \bar D_{\dot\alpha} = \bar\partial_{\dot\alpha} + i\,\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\,\partial_\mu \end{cases}$$

which anti-commute with the representation of supersymmetry on superfields

$$\{D_\alpha, r(Q_\beta)\} = 0 \qquad \{D_\alpha, r(\bar Q_{\dot\beta})\} = 0$$

$$\{\bar D_{\dot\alpha}, r(Q_\beta)\} = 0 \qquad \{\bar D_{\dot\alpha}, r(\bar Q_{\dot\beta})\} = 0$$

We also have

$$\{D_\alpha, D_\beta\} = 0 \quad , \quad \{\bar D_{\dot\alpha}, \bar D_{\dot\beta}\} = 0$$

$$\{D_\alpha, \bar D_{\dot\beta}\} = -2\sigma^\mu_{\alpha\dot\beta}\, r(P_\mu)$$

These relations imply that

$$\delta_{\varepsilon,\bar\varepsilon}(D_\alpha Y) = D_\alpha(\delta_{\varepsilon,\bar\varepsilon} Y),$$

where Y is a superfield and $\delta_{\varepsilon,\bar\varepsilon}$ is a supersymmetry transformation of parameters ε, $\bar\varepsilon$. A similar condition holds for $\bar D_{\dot\alpha}$.

Therefore, we can impose a supersymmetry-invariant constraint

$$\bar D_{\dot\alpha}\phi = 0$$

A superfield ϕ which satisfies this constraint is called a <u>chiral superfield</u>.

An anti-chiral superfield is such that $D_\alpha\phi = 0$.

Note: If ϕ is real $\phi = \phi^+$, then $D_\alpha\phi = 0$ implies $\bar D_{\dot\alpha}\phi = 0$ which then implies $\{D_\alpha, \bar D_{\dot\alpha}\}\phi = 0$ which yields $\partial_\mu\phi = 0$. In order to obtain a non-trivial result, we need to take ϕ complex.

In order to write the Grassmann expansion of a chiral superfield in the simplest way, define

$$y^\mu = x^\mu + i\theta\sigma^\mu\bar\theta \quad , \quad \bar y^\mu = x^\mu - i\theta\sigma^\mu\bar\theta$$

Since $\bar D_{\dot\alpha} y^\mu = 0$, $\bar D_{\dot\alpha}\theta^\beta = 0$ a function of y^μ, θ is automatically chiral, i.e. $\phi(y,\theta)$ is a chiral superfield. The expansion in θ reads

$$\phi(y,\theta) = \varphi(y) + \sqrt{2}\,\theta\psi(y) - \theta^2 F(y)$$

Let us now work out the supersymmetry transformations

$$\delta_{\varepsilon,\bar\varepsilon}\phi(y,\theta) = i(\varepsilon r(Q) + \bar\varepsilon r(\bar Q))\,\phi(y,\theta)$$

write the differential operators $r(Q)$, $r(\bar Q)$ in $y, \theta, \bar\theta$ coordinates

$$\Rightarrow \begin{cases} r(Q_\alpha) = -i\,\partial_\alpha \\ r(\bar Q_{\dot\alpha}) = i\,\bar\partial_{\dot\alpha} + 2\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\,\partial/\partial y^\mu \end{cases}$$

$$\Rightarrow \quad \delta_{\varepsilon,\bar\varepsilon}\,\phi(y,\theta) = i\left(\varepsilon\, r(Q) + \bar\varepsilon\, r(\bar Q)\right)\phi(y,\theta)$$

$$= i\left(\varepsilon^\alpha(-i\partial_\alpha) + \left(\cancel{i\bar\partial_{\dot\alpha}} + 2\theta^\alpha \sigma^\mu_{\alpha\dot\alpha}\frac{\partial}{\partial y^\mu}\right)\bar\varepsilon^{\dot\alpha}\right)\phi(y,\theta)$$

$$= \left(\varepsilon^\alpha\partial_\alpha + 2i(\theta\sigma^\mu\bar\varepsilon)\frac{\partial}{\partial y^\mu}\right)\phi(y,\theta)$$

$$= \sqrt2\,\varepsilon\psi - 2\varepsilon\theta F + 2i(\theta\sigma^\mu\bar\varepsilon)\frac{\partial\phi}{\partial y} + 2i(\theta\sigma^\mu\bar\varepsilon)\sqrt2\,\theta\frac{\partial\psi}{\partial y^\mu}$$

Now we use some rearrangement identity

$$(\theta\psi)(\theta\chi) = (\theta^\alpha\psi_\alpha)(\theta^\beta\chi_\beta)$$

$$\theta^2 = \theta^\alpha\theta_\alpha = \theta^\alpha\varepsilon_{\alpha\beta}\theta^\beta = -\theta^1\theta^2 + \theta^2\theta^1$$

$$= -2\,\theta^1\theta^2$$

$$\boxed{\varepsilon_{\alpha\beta} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}}$$

$$\Rightarrow \quad \boxed{\theta^\alpha\theta^\beta = -\frac12\,\varepsilon^{\alpha\beta}\,\theta^2}$$

$$\Rightarrow (\theta\psi)(\theta\chi) = -\theta^\alpha\theta^\beta\psi_\alpha\chi_\beta = \frac12\,\varepsilon^{\alpha\beta}\theta^2\psi_\alpha\chi_\beta$$

$$= -\frac12\,\theta^2(\psi^\beta\chi_\beta) \Rightarrow \boxed{(\theta\psi)(\theta\chi) = -\frac12\,\theta^2(\psi\chi)}$$

$$\Rightarrow \quad 2i(\theta\sigma^\mu\bar\varepsilon)\sqrt2\,\theta\frac{\partial\psi}{\partial y^\mu} = -i\sqrt2\,\theta^2\left[-(\sigma^\mu\bar\varepsilon)_\alpha\frac{\partial\psi^\alpha}{\partial y^\mu}\right]$$

$$= +i\sqrt2\,\theta^2\,\bar\varepsilon^{\dot\alpha}\sigma^\mu_{\alpha\dot\alpha}\frac{\partial\psi^\alpha}{\partial y^\mu}$$

$$= -i\sqrt2\,\theta^2\,\bar\varepsilon^{\dot\alpha}\epsilon_{\dot\alpha\dot\beta}\bar\sigma^{\mu,\dot\beta\beta}\epsilon_{\beta\alpha}\frac{\partial\psi^\alpha}{\partial y^\mu}$$

$$\sigma^\mu_{\alpha\dot\alpha} = -\varepsilon_{\dot\alpha\dot\beta}\bar\sigma^{\mu,\dot\beta\beta}\varepsilon_{\beta\alpha}$$

$$= -i\sqrt2\,\theta^2\left(-\bar\varepsilon_{\dot\beta}\right)\bar\sigma^{\mu,\dot\beta\beta}\frac{\partial\psi_\beta}{\partial y^\mu}$$

$$= +i\sqrt2\,\theta^2\left(\bar\varepsilon\bar\sigma^\mu\frac{\partial}{\partial y^\mu}\right)\psi$$

Finally,

$$\delta_{\varepsilon,\bar\varepsilon}\,\phi(y,\theta) = \sqrt2\,\varepsilon\psi + \sqrt2\,\theta\left(\sqrt2\,\varepsilon F + i\sqrt2(\sigma^\mu\bar\varepsilon)\frac{\partial\phi}{\partial y}\right) + \theta^2\left(+i\sqrt2\left(\bar\varepsilon\bar\sigma^\mu\frac{\partial\psi}{\partial y^\mu}\right)\right)$$

$$\Rightarrow \begin{cases} \delta_{\varepsilon,\bar\varepsilon}\,\phi = \sqrt2\,\varepsilon\psi \\ \delta_{\varepsilon,\bar\varepsilon}\,\psi = \sqrt2\,\varepsilon F + i\sqrt2(\sigma^\mu\bar\varepsilon)\frac{\partial\phi}{\partial y} \\ \delta_{\varepsilon,\bar\varepsilon}\,F = -i\sqrt2\,\bar\varepsilon\bar\sigma^\mu\frac{\partial\psi}{\partial y^\mu}\end{cases}$$