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Superspace

In general it's hard to construct supersymmetric Lagrangians. One can start with a representation of supersymmetry on fields, write the most general ansatz for the Lagrangian and fix the coefficients such that the Lagrangian is invariant up to a total derivative.

A more systematic way is to use superspace. Just like to the generator P_μ corresponds a coordinate x^μ, we associate to $Q_\alpha, \bar{Q}_{\dot\alpha}$ coordinates $\theta^\alpha, \bar\theta^{\dot\alpha}$. Since $Q_\alpha, \bar{Q}_{\dot\alpha}$ have odd grading, we take $\theta^\alpha, \bar\theta^{\dot\alpha}$ to have odd grading too. In particular,

$$\{\theta^\alpha, \theta^\beta\} = 0 \,, \qquad \{\bar\theta_{\dot\alpha}, \bar\theta_{\dot\beta}\} = 0 \,, \qquad \{\theta^\alpha, \bar\theta_{\dot\alpha}\} = 0$$

Define $\quad \theta Q = \theta^\alpha Q_\alpha \,, \quad \bar\theta\bar Q = \bar\theta_{\dot\alpha}\bar Q^{\dot\alpha}$

For a field $\phi(x)$ we have

$$\phi(x) = e^{-iPx}\,\phi(0)\,e^{+iPx}$$

when we go to the supersymmetric case we replace

$$e^{iPx} \longrightarrow e^{i(Px + \theta Q + \bar\theta\bar Q)}$$

A superfield is then

$$\phi(x,\theta,\bar\theta) = e^{-i(Px+\theta Q+\bar\theta\bar Q)}\,\phi(0)\,e^{+i(Px+\theta Q+\bar\theta\bar Q)}$$

This superfield can be expanded in powers of $\theta, \bar\theta$

$$\phi(x,\theta,\bar\theta) = \phi(x) + \theta\psi + \bar\theta\bar\psi + \theta\sigma^\mu\bar\theta\, a_\mu + \theta^2 m(x) + \bar\theta^2 n(x)$$
$$+ \theta^2\bar\theta\bar\lambda + \bar\theta^2\theta\rho + \theta^2\bar\theta^2 d$$

Note that $\theta^3 = 0$, $\bar\theta^3 = 0$

Differentiations and integration

we introduce fermionic derivatives

$$\partial_\alpha\theta^\beta = \delta_\alpha^\beta \,, \quad \bar\partial_{\dot\alpha}\bar\theta^{\dot\beta} = \delta_{\dot\alpha}^{\dot\beta} \,, \quad \partial_\alpha\bar\theta^{\dot\beta} = 0 \,, \quad \bar\partial_{\dot\alpha}\theta^\beta = 0$$

where $\quad \partial_\alpha = \dfrac{\partial}{\partial\theta^\alpha} \,, \quad \bar\partial_{\dot\alpha} = \dfrac{\partial}{\partial\bar\theta^{\dot\alpha}}$

Note that this implies that $\partial^\alpha = -\varepsilon^{\alpha\beta}\partial_\beta$ in order to have $\partial^\alpha\theta_\beta = \delta^\alpha_\beta$, where, in our conventions, $\theta_\beta = \varepsilon_{\beta\gamma}\theta^\gamma$.

A function of a Grassmann variable θ can be expanded

$$f(\theta) = f_0 + \theta f_1$$

(16) We define integration to be translation invariant; if ξ is also a Grassmann variable, then

$$\int d\theta\, f(\theta) = \int d(\theta+\xi)\, f(\theta+\xi)$$

$$f(\theta+\xi) = f_0 + (\theta+\xi) f_1 = (f_0 + \xi f_1) + \theta f_1$$

→ The coefficient f_0 of the expansion after translation is not invariant.

→ The coefficient f_1 is invariant.

We set

$$\int d\theta\, f(\theta) = \int d\theta\, (f_0 + \theta f_1) = f_1$$

Integration is the same as derivation!

$$\int d\theta\, f(\theta) = \frac{\partial}{\partial\theta} f(\theta) = f_1$$

Representation of supersymmetry generators as differential operators on superfields

Recall the situation for translations

$$\varphi(x+a) = e^{-iaP}\varphi(x)\, e^{iaP} = \varphi(x) - ia\,[P, \varphi(x)] + \dots$$

If we equate with the Taylor expansion of the LHS we find

$$-i\,[P_\mu, \varphi(x)] = \partial_\mu \varphi(x) \quad , \quad \boxed{[P_\mu, \varphi(x)] = i\partial_\mu \varphi(x)}$$

This is the representation of the Poincaré generator P_μ on the field $\varphi(x)$.

We can do the same for superfields

$$Y(x+\delta x, \theta+\delta\theta, \bar\theta+\delta\bar\theta) = \exp(-i(\varepsilon Q + \bar\varepsilon\bar Q))\, Y(x,\theta,\bar\theta)\, \exp(i(\varepsilon Q + \bar\varepsilon \bar Q))$$

$$= \exp(-i(\varepsilon Q + \bar\varepsilon\bar Q))\, \exp(-i(x\cdot P + \theta Q + \bar\theta\bar Q))\, Y(0,0,0)\,\cdot$$

$$\exp(i(x\cdot P + \theta Q + \bar\theta\bar Q))\, \exp(i(\varepsilon Q + \bar\varepsilon\bar Q))$$

Next, we compute the product $e^A e^B = e^C$ by using the Baker-Campbell Hausdorff formula.

$$e^A e^B = \exp\left(A + B + \frac{1}{2}[A,B] + \frac{1}{12}\left([A,[A,B]] - [B,[B,A]]\right)\dots\right)$$

We are in the situation where $[A,B]$ commutes with A and B so all terms after $[A,B]$ actually vanish.

$$A = i(x\cdot P + \theta Q + \bar\theta\bar Q)$$

$$B = i(\varepsilon Q + \bar\varepsilon\bar Q)$$

$$[A,B] = [i\theta Q, i\bar\varepsilon\bar Q] + [i\bar\theta\bar Q, i\varepsilon Q]$$

$$= +2(\theta\sigma^\mu\bar\varepsilon)P_\mu + 2(\varepsilon\sigma^\mu\bar\theta)P_\mu$$

Hint: replace $\bar\varepsilon\bar Q \Rightarrow \bar Q\bar\varepsilon$

$$\Rightarrow \quad A + B + \frac{1}{2}[A,B] = i(x^\mu + i\theta\sigma^\mu\bar\varepsilon - i\varepsilon\sigma^\mu\bar\theta)P_\mu + i(\theta+\varepsilon)Q + i(\bar\theta+\bar\varepsilon)\bar Q.$$

$$\Rightarrow \begin{cases} \delta x^\mu = i\theta\sigma^\mu\bar\varepsilon - i\varepsilon\sigma^\mu\bar\theta \\ \delta\theta = \varepsilon \\ \delta\bar\theta = \bar\varepsilon \end{cases}$$

Proceeding just as for translation, we find

$$Y(x+\delta x, \theta+\delta\theta, \bar\theta+\delta\bar\theta) = \delta x^\mu \partial_\mu Y(x,\theta,\bar\theta) + \delta\theta\cdot\partial Y(x,\theta,\bar\theta) + \delta\bar\theta\cdot\bar\partial Y(x,\theta,\bar\theta)$$

$$= -i[\varepsilon Q, Y(x,\theta,\bar\theta)] - i[\bar\varepsilon\bar Q, Y(x,\theta,\bar\theta)]$$

$$\Rightarrow \begin{cases} [Q_\alpha, Y(x,\theta,\bar\theta)] = -(-i\partial_\alpha - \sigma^\mu_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_\mu)Y(x,\theta,\bar\theta) \\ [\bar Q_{\dot\alpha}, Y(x,\theta,\bar\theta)] = -(i\bar\partial_{\dot\alpha} + \theta^\beta\sigma^\mu_{\beta\dot\alpha}\partial_\mu)Y(x,\theta,\bar\theta) \end{cases}$$

We denote by

$$r(Q_\alpha) = -i\partial_\alpha - \sigma^\mu_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_\mu$$

$$r(\bar Q_{\dot\alpha}) = i\bar\partial_{\dot\alpha} + \theta^\beta\sigma^\mu_{\beta\dot\alpha}\partial_\mu$$

$$r(P_\mu) = -i\partial_\mu$$

the representation of the operators Q_α, $\bar Q_{\dot\alpha}$, P_μ, respectively.

There are some extra signs which are not completely intuitive. To understand them, consider an algebra with generators G_α acting on some other generator Θ. Define the representation on Θ to be $r(G_\alpha)$ below.

$$[G_\alpha, \Theta] = -r(G_\alpha)\Theta$$

then, it's easy to convince ourself that compatibility with Jacobi identities and with the requirement $r([G_\alpha, G_\beta]) = [r(G_\alpha), r(G_\beta)]$ requires the sign in the right-hand side.

Check that the algebra

$$\{r(Q_\alpha), r(\bar Q_{\dot\alpha})\} = 2\, r(P_\mu)\, \sigma^\mu_{\alpha\dot\alpha}$$

$$\{r(Q_\alpha), r(Q_\beta)\} = 0$$

$$\{r(\bar Q_{\dot\alpha}), r(\bar Q_{\dot\beta})\} = 0$$

is satisfied.