Massless supermultiplets

Go to the rest frame $P_\mu = (E, 0, 0, E)$

$$\sigma^\mu P_\mu = \begin{pmatrix} 0 & 0 \\ 0 & 2E \end{pmatrix}$$

$$\Rightarrow \quad \{ Q_\alpha^I , \bar{Q}_{\dot\beta}^J \} = 2 \sigma^\mu_{\alpha\dot\beta} P_\mu \delta^{IJ} = 2 \delta^{IJ} \begin{pmatrix} 0 & 0 \\ 0 & 2E \end{pmatrix}_{\alpha\dot\beta}$$

$$\Rightarrow \quad \{ Q_1^I , \bar{Q}_{\dot 1}^J \} = 0$$

These generators are trivially realised on the Hilbert space of states

$$0 = \langle \phi | \{ Q_1^I , \bar{Q}_{\dot 1}^I \} | \phi \rangle = \| Q_1^I | \phi \rangle \|^2 + \| \bar{Q}_{\dot 1}^I | \phi \rangle \|^2$$

$$\Rightarrow \quad \| Q_1^I | \phi \rangle \|^2 = 0 , \quad \| \bar{Q}_{\dot 1}^I | \phi \rangle \|^2 = 0$$

$$\Rightarrow \quad Q_1^I | \phi \rangle = 0 , \quad \bar{Q}_{\dot 1}^I | \phi \rangle = 0 \qquad \forall \ I, \text{ and } | \phi \rangle = \mathcal{H}$$

$$\Rightarrow \quad Q_1^I = 0 , \quad \bar{Q}_{\dot 1}^I = 0$$

Define $a_I = \frac{1}{\sqrt{4E}} Q_2^I$, $a_I^+ = \frac{1}{\sqrt{4E}} \bar{Q}_{\dot 2}^I$

They satisfy the algebra

$$\{ a_I , a_J^+ \} = \delta_{IJ} , \quad \{ a_I , a_J \} = 0 , \quad \{ a_I^+ , a_J^+ \} = 0$$

These operators lower and raise helicity by $1/2$ (Compute the commutator with M_{12}.)

To build a representation choose a highest weight state (also called Clifford vacuum in some references) $| \lambda \rangle$ such that $a_I | \lambda \rangle = 0 \ \forall \ I$.

Then, the states in the representation are

$$| \lambda \rangle , \quad a_I^+ | \lambda \rangle =: | \lambda + \tfrac{1}{2} , I \rangle$$

$$a_I^+ a_J^+ | \lambda \rangle =: | \lambda + 1 , IJ \rangle \quad \leftarrow \text{antisymmetric in } IJ$$

$$\dots$$

$$a_1^+ \cdots a_N^+ | \lambda \rangle =: | \lambda + \tfrac{N}{2} \rangle$$

At level k we have $\binom{N}{k}$ states of helicity $\lambda + \frac{k}{2}$

The total number of states is

$$\sum_{k=0}^{N} \binom{N}{k} = 2^N = \underbrace{2^{N-1}}_{\text{bosonic}} + \underbrace{2^{N-1}}_{\text{fermionic}}.$$

These representations exist as abstract representations, but in order to be symmetries of a QFT more constraints need to be satisfied. A basic requirement is that the spectrum be CPT invariant, which means that helicities should be symmetrically distributed around zero. Some multiplets are CPT invariant by themselves. For others, we need to add a CPT conjugate multiplet in order to achieve CPT invariance.

Examples of representations

1) $\mathcal{N}=1$ supersymmetry.

- Matter multiplet. In this case we choose $\lambda = 0$. Acting with Q^+ we obtain a state with helicity $1/2$. To this we need to add the CPT conjugate multiplet $(0, -\frac{1}{2})$. These degrees of freedom can be described by a complex scalar and a Weyl fermion.

- Gauge multiplet. Here we take $\lambda = 1/2$. Acting with a^+ we obtain a state with helicity 1. To this we must add the CPT conjugate multiplet of helicities $(-1, -\frac{1}{2})$. These degrees of freedom can be described by a gauge field and Weyl fermion. The gauge fields and the Weyl fermions must transform under the same representation of the gauge group, otherwise supersymmetry can not exist.

- Gravitino multiplet. Start with $\lambda = 1$. We obtain helicities

$$(1, 3/2) \oplus (-1, -3/2)$$

These are the degrees of freedom of a gauge field and of a gravitino. This multiplet mostly appears in extended supergravity theories where we make only part of the supersymmetry manifest.

- Gravity multiplet. Start with $\lambda = 3/2$

$$\rightarrow (3/2, 2) \oplus (-3/2, -2)$$

The degrees of freedom are those of a graviton and a gravitino.

2. $\mathcal{N}=2$ supersymmetry.

- Hypermultiplet $\quad \lambda = -1/2$

$$\begin{tikzcd}
 & |0;1\rangle \arrow[dr, "a_2^+"] & \\
|-1/2\rangle \arrow[ur, "a_1^+"] \arrow[dr, "a_2^+"'] & & |1/2\rangle \\
 & |0;2\rangle \arrow[ur, "a_1^+"'] &
\end{tikzcd}$$

Recall that the CPT transformation flips the helicity but also conjugates all the other representations under which particles transform. The groups which will be relevant here are the R-symmetry group and the gauge group. The two scalars transform in the 2 representation of $SU(2)$ R-symmetry, which is a pseudo-real representation. If the gauge representation is also pseudo-real, then the tensor product is real and we can impose a reality condition.

For the fermions a similar story holds. Above we have states corresponding to two Weyl fermions. We can impose a Majorana condition which is a pseudo-reality condition. When tensored with the pseudo-real gauge representation we obtain a real representation and we can impose a reality condition again. The hypermultiplet with this reality condition is called a half-hypermult. In general, we need to add a conjugate to make CPT hold

$$\begin{tikzcd}[row sep=small] & 0 \arrow[dr] & \\ -\frac{1}{2} \arrow[ur] \arrow[dr] & & \frac{1}{2} \\ & 0 \arrow[ur] & \end{tikzcd} \quad \oplus \quad \begin{tikzcd}[row sep=small] & 0 \arrow[dr] & \\ -\frac{1}{2} \arrow[ur] \arrow[dr] & & \frac{1}{2} \\ & 0 \arrow[ur] & \end{tikzcd}$$

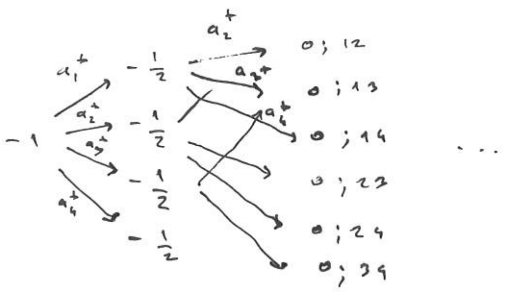
This can be done for any representations of the gauge group and the result is called a hypermultiplet. A hypermultiplet has the same states as a chiral plus antichiral multiplet.

- Gauge multiplet $\quad \lambda = 0$

$$\begin{tikzcd}[row sep=small] & \frac{1}{2} \arrow[dr] & \\ 0 \arrow[ur] \arrow[dr] & & 1 \\ & \frac{1}{2} \arrow[ur] & \end{tikzcd} \quad \oplus \quad \begin{tikzcd}[row sep=small] & -\frac{1}{2} \arrow[dl] & \\ -1 & & 0 \arrow[ul] \arrow[dl] \\ & -\frac{1}{2} \arrow[ul] & \end{tikzcd}$$

(Contains an $\mathcal{N}=1$ vector multiplet and a matter multiplet (in the same rep. of the gauge group.)

<u>$\mathcal{N}=4$ supersymmetry</u> $\quad \lambda = -1$



States

$$\begin{pmatrix} -1 & -\frac{1}{2} & 0 & \frac{1}{2} & 1 \\ & -\frac{1}{2} & 0 & \frac{1}{2} & \\ & -\frac{1}{2} & 0 & \frac{1}{2} & \\ & -\frac{1}{2} & 0 & \frac{1}{2} & \\ & & 0 & & \\ & & 0 & & \end{pmatrix}$$

1 – vector
4 – Weyl fermions
3 – complex scalars.

The multiplet is self-conjugate since
- vectors are singlets of $SU(4)$
- left spinors (helicity $-\frac{1}{2}$) transform under 4 rep. of $SU(4)$
  and right spinors (helicity $\frac{1}{2}$) ———— $\bar{4}$ ————
  and these get mapped to each other.
- Scalars transform as 6 of $SU(4)$ $\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}$ which is a real representation and therefore scalars map to scalars.

Notice that in this case an R-symmetry $U(4)$ is incompatible with CPT conjugation.

(14) Representations of Supersymmetry on fields

We start with a scalar field ϕ and create new fields by action of supercharges Q_α, $\bar{Q}_{\dot\alpha}$ (we restrict to $\mathcal{N}=1$).

Assume
$$[Q_\alpha, \phi(x)] = 0$$

If $\phi^\dagger = \phi$, then from $\bar{Q}_{\dot\alpha} = Q_\alpha^\dagger$ we obtain $[\bar{Q}_{\dot\alpha}, \phi(x)] = 0$ which implies $[P_\mu, \phi] = 0$ which then shows that $\phi(x)$ is a constant. Does not describe dynamical degrees of freedom.

Instead, we take $\phi(x)$ be complex and we set
$$\boxed{[\bar{Q}_{\dot\alpha}, \phi] = 0}, \quad \boxed{[Q_\alpha, \phi] = \psi_\alpha}$$

We need to take more ~~the~~ commutators with supercharges. In general
$$\boxed{\{Q_\alpha, \psi_\beta\} = F_{\alpha\beta}}, \quad \boxed{\{\bar{Q}_{\dot\alpha}, \psi_\beta\} = X_{\dot\alpha\beta}}$$

$$X_{\dot\alpha\beta} = \{\bar{Q}_{\dot\alpha}, \psi_\beta\} = \{\bar{Q}_{\dot\alpha}, [Q_\beta, \phi]\} = [\{\bar{Q}_{\dot\alpha}, Q_\beta\}, \phi]$$
$$- \{Q_\beta, \cancel{[\bar{Q}_{\dot\alpha}, \phi]}\} = 2\sigma^\mu_{\beta\dot\alpha}\partial_\mu\phi$$
$$\Rightarrow \quad \boxed{X_{\dot\alpha\beta} = 2\sigma^\mu_{\beta\dot\alpha}\partial_\mu\phi}$$

$$F_{\alpha\beta} = \{Q_\alpha, \psi_\beta\} = \{Q_\alpha, [Q_\beta, \phi]\} = [\cancel{\{Q_\alpha, Q_\beta\}}, \phi] - \{Q_\beta, [Q_\alpha, \phi]\}$$
$$= -\{Q_\beta, \psi_\alpha\} = -F_{\beta\alpha}$$

Since $F_{\alpha\beta} = -F_{\beta\alpha}$ we find $\boxed{F_{\alpha\beta} = \varepsilon_{\alpha\beta}F}$

Next, using the Jacobi identity we find
$$\boxed{[Q_\alpha, F] = 0}, \quad \boxed{[\bar{Q}_{\dot\alpha}, F] = 2\varepsilon^{\beta\gamma}\partial_{\gamma\dot\alpha}\psi_\beta}$$

Notice that the representation closes, i.e. we don't need any new fields. This field theory contains two complex scalar fields ϕ and F and one Weyl fermion ψ_α. The two scalar fields have 4 real degrees of freedom while ψ_α has two complex component, which are also 4 real degrees of freedom. The numbers of degrees of freedom between bosons and fermions match! Since we haven't imposed the equations of motion, we say that these degrees of freedom match off-shell.

On-shell, the equation of motion for the fermion reduces the number of degrees of freedom to 2, while the field F is completely auxiliary, i.e. its value on-shell is determined by ϕ and ψ.