7 The supersymmetry algebra

Def (~~Graded~~ Lie algebra) is a vector space L over ($\mathbb{R}$, $\mathbb{C}$ or some other field) with a bracket $[\cdot,\cdot] : L \times L \longrightarrow L$ with properties

- $[a,b] = -[b,a]$
- $[a+b,c] = [a,c] + [b,c]$
- $[a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0$ Jacobi identity.

Def: graded Lie algebra of grade n is a vector space

$$L = \bigoplus_{i=0}^{n} L_i$$

such that

- $[L_i, L_j\} \in L_{i+j} \mod n+1$
- $[L_i, L_j\} = (-)^{ij+1} [L_j, L_i\}$
- $[L_i,[L_j,L_k\}\}(-)^{ik} + [L_j,[L_k,L_i\}\}(-)^{ij} + [L_k,[L_i,L_j\}\}(-)^{jk} = 0$

A supersymmetry algebra is a graded Lie algebra of grade 1

$$L = L_0 \oplus L_1$$

L_0 contains the Lorentz group

$$[P_\mu, P_\nu] = 0$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i\eta_{\mu\rho}M_{\nu\sigma} - i\eta_{\nu\sigma}M_{\mu\rho} + i\eta_{\mu\sigma}M_{\nu\rho} + i\eta_{\nu\rho}M_{\mu\sigma}$$

$$[M_{\mu\nu}, P_\rho] = -i\eta_{\rho\mu}P_\nu + i\eta_{\rho\nu}P_\mu$$

L_1 contains Q^I_α, $\bar{Q}^I_{\dot\alpha}$ with $I = 1,\dots,\mathcal{N}$

More commutators

i) $[P_\mu, Q^I_\alpha] = 0$, $[P_\mu, \bar{Q}^I_{\dot\alpha}] = 0$

ii) $[M_{\mu\nu}, Q^I_\alpha] = i(\sigma_{\mu\nu})_\alpha{}^\beta Q^I_\beta$, $[M_{\mu\nu}, \bar{Q}^{I\dot\alpha}] = i(\bar\sigma_{\mu\nu})^{\dot\alpha}{}_{\dot\beta}\bar{Q}^{I\dot\beta}$

iii) $\{Q^I_\alpha, \bar{Q}^J_{\dot\alpha}\} = 2\sigma^\mu_{\alpha\dot\alpha}P_\mu\delta^{IJ}$

iv) $\{Q^I_\alpha, Q_\beta{}^J\} = \varepsilon_{\alpha\beta}Z^{IJ}$, $Z^{IJ} = -Z^{JI}$

v) $\{\bar{Q}^I_{\dot\alpha}, \bar{Q}^J_{\dot\beta}\} = \varepsilon_{\dot\alpha\dot\beta}(Z^{IJ})^*$

- ii) follow from the fact that Q, $\bar{Q}$ are Lorentz spinors.
- iii) the tensor product of a left Weyl and a right Weyl spinor is a vector. The obvious vector generator is P_μ

* We can consider theories with global supersymmetry, where the supersymmetry transformations do not depend on coordinates, but also theories with <u>local</u> supersymmetry, which are invariant under supersymmetry transformations which are local, i.e. they depend on coordinates. These transformations are really gauge transformations so we need to quotient by these transf

(3) In these theories, the anticommutator of two local supersymmetry transformations yields a translation by a coordinate-dependent quantity. These are diffeomorphisms. Theories invariant under diffeomorphisms are gravity theories, so we are dealing with supergravity theories.

The relations

$$[P_\mu, Q^I_\alpha] = 0, \qquad [P_\mu, \bar{Q}^I_{\dot\alpha}] = 0$$

are not so obvious.

We could consider a more general commutation relation

$$[P_\mu, Q^I_\alpha] = C^I{}_J\, \sigma_{\mu,\alpha\dot\beta}\, \bar{Q}^{J\dot\beta}$$

$$[P_\mu, \bar{Q}^I_{\dot\alpha}] = (C^I{}_J)^*\, \bar\sigma_{\mu,\dot\alpha\beta}\, Q^{J\beta}$$

The (P, P, Q) Jacobi identity implies $CC^* = 0$

The general Q, Q anticommutator is

$$\{Q^I_\alpha, Q^J_\beta\} = \varepsilon_{\alpha\beta} Z^{IJ} + (\sigma^{\mu\nu}\varepsilon)_{\alpha\beta} M_{\mu\nu} Y^{IJ}$$

$$Z^{IJ} = -Z^{JI} \qquad\qquad Y^{IJ} = Y^{JI}$$

Doing the (Q, Q, P) Jacobi identity we find $C_{IJ} = +C_{JI}$, which together with $CC^* = 0$ implies that $CC^\dagger = 0$ so $C = 0$

To show that Y vanishes, compute the Jacobi identity (Q, Q, P) which now reduces to

$$[\{Q_{I\alpha}, Q_{J\beta}\}, P_\mu] = 0$$

which implies that $Y = 0$

- Commutation relations of Q's with internal symmetries

$$[Q^I_\alpha, B_\ell] = (b_\ell)^I{}_J\, Q^J_\alpha \qquad\qquad b_\ell \text{ are hermitian}$$

$$[\bar{Q}_{I\dot\alpha}, B_\ell] = -\bar{Q}_{J\dot\alpha}(b_\ell)^J{}_I$$

Therefore, the largest symmetry group acting on Q's is $U(\mathcal{N})$.
(In some cases only a subgroup acts non-trivially).

CPT transformations*

Before studying the action of parity and time reversal on physical states, it is worthwhile to study the interplay of these transformations with the super-Poincaré algebra.

The action on coordinates is

$$(x^0, \vec{x}) \xrightarrow{T} (-x^0, \vec{x})$$
$$(x^0, \vec{x}) \xrightarrow{P} (x^0, -\vec{x})$$

Since the translations and Lorentz transformations are represented on coordinates by

$$P_\mu = i\partial_\mu$$
$$M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$$

we are tempted to propose the transformations

$$P_\mu \longrightarrow P P_\mu P^{-1} = \begin{cases} P_0 & \text{if } \mu = 0 \\ -P_i & \text{if } \mu = i \end{cases}$$

$$M_{\mu\nu} \longrightarrow P M_{\mu\nu} P^{-1} = \begin{cases} -M_{0i}, & \text{if } \mu = 0, \nu = i \\ M_{ij}, & \text{if } \mu = i, \nu = j \end{cases}$$

$$P_\mu \longrightarrow T P_\mu T^{-1} \stackrel{?}{=} \begin{cases} -P_0, & \text{if } \mu = 0 \\ P_i, & \text{if } \mu = i \end{cases}$$

$$M_{\mu\nu} \longrightarrow T M_{\mu\nu} T^{-1} \stackrel{?}{=} \begin{cases} -M_{0i} & \text{if } \mu = 0, \nu = i \\ M_{ij} & \text{if } \mu = i, \nu = j \end{cases}$$

There is a problem with these transformations. Since P_0 measures the energy, and the T transformations change the sign of the energy, it would force us to accept negative energy solutions. Instead, we impose the condition that T is anti-linear, which means that when acting on the representations $P_\mu = i\partial_\mu$ and $M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$ we pick up an extra sign. Therefore, we have

$$P_\mu \longrightarrow T P_\mu T^{-1} = \begin{cases} P_0 & \text{if } \mu = 0 \\ -P_i & \text{if } \mu = i \end{cases}$$

$$M_{\mu\nu} \longrightarrow T M_{\mu\nu} T^{-1} = \begin{cases} M_{0i} & \text{if } \mu = 0, \nu = i \\ -M_{ij} & \text{if } \mu = i, \nu = j \end{cases}$$

Observations

- Even if T is anti-linear, $T P_\mu T^{-1}$ and $T M_{\mu\nu} T^{-1}$ are linear.
- If T is anti-unitary $T^+ = T^{-1}$ where T^+ is defined by

$$(T^+\psi, \varphi) = (\psi, T\varphi)^*,$$

then $T P_\mu T^{-1}$ is hermitian.
- The new generators $P'_\mu = P P_\mu P^{-1}$, $M'_{\mu\nu} = P M_{\mu\nu} P^{-1}$ and similarly for T

satisfy the same commutation relations as before. In the case of T transformations we pick up extra signs from the factors of i in the RHS.

- The transformations under PT transformations are simpler. We have

$$P_\mu \longrightarrow (PT)\, P_\mu\, (PT)^{-1} = P_\mu$$
$$M_{\mu\nu} \longrightarrow (PT)\, M_{\mu\nu}\, (PT)^{-1} = -M_{\mu\nu}$$

What is the extension of this action to the supercharges $Q, \bar{Q}$?

We will use the (anti-) commutation relations

$$[M_{\mu\nu}, Q_\alpha] = i(\sigma_{\mu\nu})_\alpha{}^\beta Q_\beta \ , \qquad [M_{\mu\nu}, \bar{Q}^{\dot\alpha}] = i(\bar\sigma_{\mu\nu})^{\dot\alpha}{}_{\dot\beta}\, \bar{Q}^{\dot\beta}$$

$$\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = 2\sigma^\mu_{\alpha\dot\alpha} P_\mu$$

Acting with the anti-unitary operator PT, we find

$$[PT\, M_{\mu\nu} (PT)^{-1} , PT\, Q_\alpha (PT)^{-1}] = -i\, (\sigma^*_{\mu\nu})_{\dot\alpha}{}^{\dot\beta}\, PT\, Q_\beta (PT)^{-1}$$

Using the fact that $\quad PT\, M_{\mu\nu}\, (PT)^{-1} = -M_{\mu\nu}$

and $\quad (\sigma^*_{\mu\nu})_{\dot\alpha}{}^{\dot\beta} = \varepsilon_{\dot\alpha\dot\gamma} (\bar\sigma_{\mu\nu})^{\dot\gamma}{}_{\dot\delta}\, \varepsilon^{\dot\delta\dot\beta}$

we find that we can set

$$PT\, Q_\alpha (PT)^{-1} = \eta\, \bar{Q}_{\dot\alpha} ,$$

where η is a constant.

From the anti-commutator $\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta} P_\mu$ we find

$$\{PT\, Q_\alpha (PT)^{-1}, PT\, \bar{Q}_{\dot\beta} (PT)^{-1}\} = 2\, (\sigma^{\mu *})_{\dot\alpha\beta}\, PT\, P_\mu (PT)^{-1}$$
$$= 2\sigma^\mu_{\beta\dot\alpha} P_\mu$$
$$= \{\eta\, \bar{Q}_{\dot\alpha} , \bar\eta\, Q_\beta\} = |\eta|^2\, 2\sigma^\mu_{\beta\dot\alpha} P_\mu$$

This implies that η is a phase.

<u>Observations</u>:

- PT transformations preserve the canonical choices of momentum: rest mass frame for massive particles and $P_\mu = (E, 0, 0, E)$ for massless particles.
- PT transformations flip the spin or helicity label.
- PT transformations map Q to $\bar{Q}$ generators.

9 <u>Representations of the supersymmetry algebra</u>

Poincaré algebra representations

- Poincaré algebra has 2 Casimirs

$$p^2 = p_\mu p^\mu \quad , \quad W^2 = W_\mu W^\mu$$

where $W_\mu = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} M^{\nu\rho} p^\sigma$ is the Pauli – Lubanski vector.

1) <u>Massive particles</u>

Go to the rest mass frame, $p_\mu = (m, \vec{0})$

$\Rightarrow \quad p^2 = m^2$

$p \cdot W = 0 \quad$ so $\quad W_0 = 0$

$W = (0, \frac{1}{2} \varepsilon_{ijk0} M^{jk} m) = (0, -m\vec{J})$

$W^2 = -m^2 \vec{J}^2$ with eigenvalues $W^2 = -m^2 j(j+1)$

Therefore, massive particles are characterized by their mass and their spin.

2) <u>Massless particles</u>

Go to the rest frame $\quad p_\mu = (E, 0, 0, E) \quad , \quad p^\mu = (E, 0, 0, -E)$

$p^2 = 0, \quad W^2 = 0,$

$$W_\mu = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} M^{\nu\rho} p^\sigma$$

$$= \frac{E}{2} (\varepsilon_{\mu\nu\rho0} M^{\nu\rho} - \varepsilon_{\mu\nu\rho3} M^{\nu\rho})$$

$$W_0 = -\frac{E}{2} \varepsilon_{0\nu\rho3} M^{\nu\rho} = -E M^{12} \Rightarrow \boxed{W_0 = -E M_{12}}$$

$$W_3 = \frac{E}{2} \varepsilon_{3\nu\rho0} M^{\nu\rho} = -E M^{12} \Rightarrow \boxed{W_3 = -E M_{12}}$$

$$W_1 = \frac{E}{2} (\varepsilon_{1\nu\rho0} M^{\nu\rho} - \varepsilon_{1\nu\rho3} M^{\nu\rho})$$

$$= \frac{E}{2} (-2M^{23} + 2M^{02}) = E(-M_{23} - M_{02}) \Rightarrow \boxed{W_1 = -E(M_{02} + M_{23})}$$

$$W_2 = \frac{E}{2} (\varepsilon_{2\nu\rho0} M^{\nu\rho} - \varepsilon_{2\nu\rho3} M^{\nu\rho})$$

$$= \frac{E}{2} (2M^{13} - 2M^{01}) = E(M_{13} + M_{01}) \quad \boxed{W_2 = E(M_{13} + M_{01})}$$

If we do not set $W_1 = W_2 = 0$, then we obtain <u>continuous spin representations</u>. It is unknown how to make QFT's with such field content. We will set $W_1 = W_2 = 0$ in the following. Then,

$$W_\mu = - M_{12} P_\mu$$

The eigenstates of $M_{12} = \pm j$ are the helicity of the particle

For supersymmetry, W^2 is not a Casimir operator anymore. A supersymmetry irreducible representation splits into a sum of Poincaré irreducible representations. A supermultiplet contains particles of several spins, but with the same mass. This mass degeneracy is not realised in nature. It means that supersymmetry must be broken.

Properties of supermultiplets

1) The energy of any state is positive

$$\langle \phi | \{ Q^I_\alpha , \bar{Q}^I_{\dot\alpha} \} | \phi \rangle = 2 \sigma^\mu_{\alpha\dot\alpha} \langle \phi | P_\mu | \phi \rangle \delta^{II}$$

$$= \langle \phi | Q^I_\alpha (Q^I_\alpha)^\dagger + (Q^I_\alpha)^\dagger Q^I_\alpha | \phi \rangle$$

$$= \| (Q^I_\alpha)^\dagger | \phi \rangle \|^2 + \| Q^I_\alpha | \phi \rangle \|^2 \geq 0$$

Summing over $\alpha, \dot\alpha$ and recalling that $\mathrm{tr}(\sigma^\mu_{\alpha\dot\alpha}) = 2\delta^{\mu 0}$ we find $4 \langle \phi | P_0 | \phi \rangle \geq 0$

2) A supermultiplet contains an equal number of bosonic and fermionic degrees of freedom.

Define an operator $(-)^F = \begin{cases} -1 & \text{on a fermionic state} \\ 1 & \text{on a bosonic state.} \end{cases}$

$$(-)^F | B \rangle = | B \rangle$$
$$(-)^F | F \rangle = - | F \rangle$$

$$\{ Q^I_\alpha , (-)^F \} = 0$$

$$\cancel{0 = \mathrm{tr} \left(\{ Q^I_\alpha , (-)^F \} \bar{Q}^J_{\dot\beta} \right) = \mathrm{tr} \left(Q^I_\alpha (-)^F \bar{Q}^J_{\dot\beta} + (-)^F Q^I_\alpha \bar{Q}^J_{\dot\beta} \right)}$$
$$\cancel{= \mathrm{tr} \left(- (-)^F \bar{Q}^J_{\dot\beta} Q^I_\alpha \right.}$$

$$0 = \mathrm{tr} \left([(-)^F \bar{Q}^I_{\dot\alpha} , Q^J_\beta] \right) = \mathrm{tr} \left((-)^F \bar{Q}^I_{\dot\alpha} Q^J_\beta - Q^J_\beta (-)^F \bar{Q}^I_{\dot\alpha} \right)$$

$$= \mathrm{tr} \left((-)^F (\bar{Q}^I_{\dot\alpha} Q^J_\beta + Q^J_\beta \bar{Q}^I_{\dot\alpha}) \right)$$

$$= \mathrm{tr} \left((-)^F 2 \delta^{IJ} \sigma^\mu_{\beta\dot\alpha} P_\mu \right)$$

$$= \mathrm{tr} \left((-)^F \right) 2 \delta^{IJ} \sigma^\mu_{\beta\dot\alpha} P_\mu$$

$$\Rightarrow \mathrm{tr} \left((-)^F \right) = 0 \quad \Rightarrow \quad n_B = n_F$$