(5) In general we abbreviate

$$\psi\chi \equiv \psi^\alpha \chi_\alpha = (\varepsilon\psi)^\alpha \chi_\alpha = \varepsilon^{\alpha\beta}\psi_\beta \chi_\alpha = -\varepsilon^{\beta\alpha}\psi_\beta\chi_\alpha$$
$$= -\psi_\beta (\varepsilon\chi)^\beta = (\varepsilon\chi)^\beta \psi_\beta = \chi^\beta\psi_\beta = \chi\psi.$$

- Note that for a bilinear $\psi\chi$ the contraction is North-West – South-East.
- Above we assume that ψ, χ are Grassmann fields, i.e. $\psi_\alpha \chi_\beta = -\chi_\beta\psi_\alpha$

The convention for barred fields is

$$\bar\psi\bar\chi \equiv \bar\psi_{\dot\alpha}\bar\chi^{\dot\alpha} = \dots = \bar\chi\bar\psi$$

- In this case the contraction is South-West — North-East.
- This convention works well with hermitian conjugation

$$(\psi\chi)^+ = (\psi^\alpha\chi_\alpha)^+ = \chi_\alpha^+(\psi^\alpha)^+ = \bar\chi_{\dot\alpha}\bar\psi^{\dot\alpha} = \bar\chi\bar\psi$$

Let us also define

$$\bar X := \bar\sigma^\mu x_\mu$$

We have $\quad \bar\sigma^{\mu,\dot\alpha\alpha} = \varepsilon^{\dot\alpha\dot\beta}\varepsilon^{\alpha\beta}\sigma^\mu_{\beta\dot\beta} \quad , \quad \bar\sigma^\mu = -\varepsilon\,\sigma^{\mu,T}\varepsilon$

$$\Rightarrow \quad \bar X = -\varepsilon\,\sigma^{\mu T}\varepsilon\, x_\mu = -\varepsilon X^T \varepsilon$$

Then, $\quad \bar X' = -\varepsilon X'^T\varepsilon = -\varepsilon(AXA^+)^T\varepsilon = -\varepsilon A^* X^T A^T \varepsilon$

$$= -\varepsilon A^*\varepsilon(\varepsilon X^T\varepsilon)\varepsilon A^T\varepsilon$$
$$= \varepsilon A^*\varepsilon\,\bar X\,\varepsilon A^T\varepsilon$$

The transformation of $\bar\psi^T\bar X\chi$ is

$$\bar\psi^T\bar X\chi \longrightarrow \bar\psi'^T\bar X'\chi' = (A^*\bar\psi)^T\varepsilon A^*\varepsilon\bar X\varepsilon A^T\varepsilon(A\chi)$$
$$= \bar\psi^T(A^+\varepsilon A^*)\varepsilon\bar X\varepsilon(A^T\varepsilon A)\chi$$
$$= \bar\psi^T\bar X\chi,$$

where we used $A^T\varepsilon A = \varepsilon$ and $\varepsilon^2 = -1$ (this is $\varepsilon_{..}\varepsilon^{..}$ or $\varepsilon^{..}\varepsilon_{..}$ so not really ε^2, which is an abuse of notation)

$\Rightarrow \quad \bar\psi^T\bar X\chi$ is therefore a Lorentz invariant. In index notation,

$$\bar\psi^T\bar X\chi = \bar\psi_{\dot\alpha}\bar X^{\dot\alpha\alpha}\chi_\alpha$$

The same transformation holds if $\bar X$ is a differential operator $\bar X \to \bar\sigma^\mu\partial_\mu \equiv \bar\partial$

Then we obtain a term which looks like the Lagrangian of a massless spinor

$$\mathcal{L}_{\text{spinor}} = \bar\psi\,\bar\partial\,\psi$$

⑥ <u>Connection to the 4-component Dirac spinors</u>

In the Weyl representation the Dirac matrices read

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix} \qquad\qquad \gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\mathbb{1}_4 \iff \begin{cases} \sigma^\mu\bar\sigma^\nu + \sigma^\nu\bar\sigma^\mu = 2\eta^{\mu\nu}\mathbb{1}_2 \\ \bar\sigma^\mu\sigma^\nu + \bar\sigma^\nu\sigma^\mu = 2\eta^{\mu\nu}\mathbb{1}_2 \end{cases}$$

A Dirac spinor is $\quad \Psi = \begin{pmatrix} \psi_\alpha \\ \bar\chi^{\dot\alpha} \end{pmatrix}$

ψ_α, $\bar\chi^{\dot\alpha}$ <u>Weyl spinor with chirality</u> $+1$, -1

$$\bar\Psi = \Psi^+\gamma^0 = (\psi^+ \;\; \bar\chi^+)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = (\bar\chi^+ \;\; \psi^+)$$

$$\psi^c = \mathcal{C}\bar\Psi^T = \begin{pmatrix} +\varepsilon & 0 \\ 0 & +\varepsilon \end{pmatrix}\bar\Psi^T = \begin{pmatrix} +\varepsilon & 0 \\ 0 & +\varepsilon \end{pmatrix}\begin{pmatrix} \bar\chi^* \\ \psi^* \end{pmatrix} = \begin{pmatrix} +\varepsilon\bar\chi^* \\ +\varepsilon\psi^* \end{pmatrix}$$

A Majorana spinor is a spinor such that $\psi^c = \psi \Rightarrow \psi = +\varepsilon\bar\chi^*$

$$\Rightarrow \psi_\alpha = +\varepsilon_{\alpha\beta}(\bar\chi^*)^\beta = +\varepsilon_{\alpha\beta}(\bar\chi^{\dot\beta})^* = +\varepsilon_{\alpha\beta}(\varepsilon^{\dot\beta\dot\gamma}\bar\chi_{\dot\gamma})^*$$

$$= +\varepsilon_{\alpha\beta}\,\varepsilon^{\beta\gamma}(\bar\chi_{\dot\gamma})^* \Rightarrow \boxed{\psi^* = \bar\chi}$$

<u>Lorentz transformations</u>

The Lorentz transformations are expressed in terms of the generators

$$\Sigma^{\mu\nu} = \frac{i}{2}\gamma^{\mu\nu} = \frac{i}{2}\times\frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) = \frac{i}{4}\begin{pmatrix} \sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu & 0 \\ 0 & \bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu \end{pmatrix}$$

$$= i\begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \bar\sigma^{\mu\nu} \end{pmatrix}$$

Index structure: $\quad i(\sigma^{\mu\nu})_\alpha{}^\beta$ acts on ψ_β

$\qquad\qquad\qquad\quad i(\bar\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta}$ acts on $\bar\chi^{\dot\beta}$