1) SUPERSYMMETRY

What is SUSY?

- Space-time symmetry mapping bosons to fermions and fermions to bosons.

Denote the generators by Q

$$Q\,|\text{boson}\rangle = |\text{fermion}\rangle$$
$$Q\,|\text{fermion}\rangle = |\text{boson}\rangle$$

- Q transforms as a spin $1/2$ particle
- one particle states have superpartners.
- Commutation relations with other symmetry generators

$[Q, P_\mu] = 0$, $[Q, G] = 0$ where G is an internal symmetry

$[Q, M_{\mu\nu}] \neq 0$ does not commute with Lorentz transformations

The particles in a supersymmetry multiplet have the same mass and quantum numbers (like charges).

Def : A supersymmetric field theory is field theory (a set of fields and a Lagrangian) which is invariant under a supersymmetry.

- For supersymmetry we do not have a symmetry algebra, but a super-algebra. For odd generators $Q, \bar{Q}$ have a symmetric bracket

$$\{Q, \bar{Q}\} = \{\bar{Q}, Q\}$$

instead of an anti-symmetric bracket as for usual symmetries.

- Extended supersymmetry. We can have multiple supersymmetry charges Q^I $I = 1, \dots, \mathcal{N}$. However, if we want to have interacting theories, the number of supercharges is limited. Otherwise, we obtain higher spin particles which are difficult to make interacting.

For theories with maximum spin 1 $\mathcal{N} \leq 4$ while for theories with maximum spin 2 (supergravity theories) $\mathcal{N} \leq 8$.

Why study supersymmetry?

- Coleman & Mandula asked if it is possible to have a symmetry in a QFT which mixes space-time symmetries P_μ, $M_{\mu\nu}$ with internal symmetries G. Under some conditions, such as locality, unitarity, finiteness of number of particles, etc. They proved this was not possible. That is, $[P_\mu, G] = 0$, $[M_{\mu\nu}, G] = 0$. The symmetry group is Poincaré $\times$ Internal

Supersymmetry provides a way around this no-go theorem. Haag, Lopuszanski & Sohnius showed that superPoincaré $\times$ Internal is also possible. This can sometimes be enlarged to superconformal symmetry.

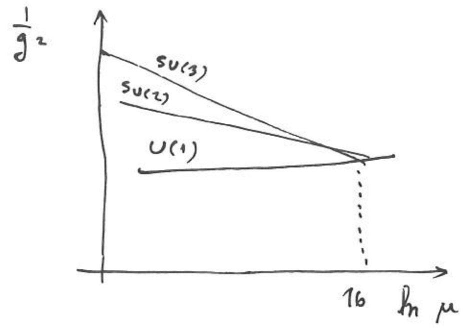
(2) - String theory predicts supersymmetry and higher dimensions

- Naturalness and the hierarchy problem

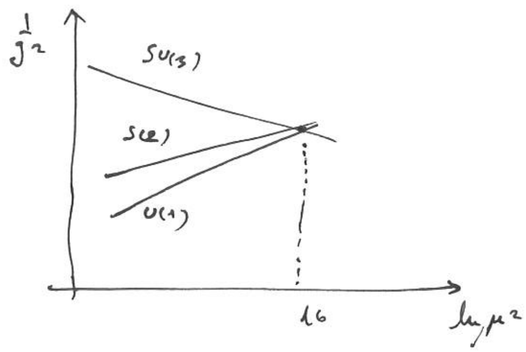
- Gauge coupling unification

There are 3 gauge couplings g_1, g_2, g_3 which run with the energy scale

$$\frac{4\pi}{g_i^2(\mu)} = \frac{b_i}{2\pi} \ln \frac{\mu}{\Lambda_i}$$



Standard Model



- Supersymmetry and cosmology

26 % of energy density in the universe is dark matter. Supersymmetry provides a candidate for dark matter, the neutralino. One can make the neutralino stable and it becomes a dark matter candidate.

- Supersymmetry is a laboratory for strongly coupled gauge dynamics.
  - The description of QCD is in terms of quarks and gluons, but the light degrees of freedom are mesons, baryons, glueballs, etc. It is very hard to understand these states in terms of quarks and gluons.
  - Many strong-coupling phenomena: confinement, charge screening, mass gap, etc.
  - vacuum condensates $\langle \Omega | \Psi \bar{\Psi} | \Omega \rangle \neq 0$ which breaks chiral symmetry.

Supersymmetric theories have nicer renormalization properties which allows better control over strong coupling regimes. As examples, we have non-renormalization theorems and properties such as holomorphy which allow some non-perturbative contributions to be computed.

THE SUPERSYMMETRY ALGEBRA

- Lorentz group $SO(1,3) = \{ \Lambda \in GL(4,\mathbb{R}) \mid \Lambda^T \eta \Lambda = \eta, \ \det \Lambda = 1 \}$

where $\eta = \mathrm{diag}(+,-,-,-)$

The algebra $so(1,3)$ has generators $M_{\mu\nu} = -M_{\nu\mu}$. They are the generators of rotations in the $(\mu\nu)$ plane. The $so(1,3)$ Lie algebra is

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i\eta_{\mu\rho} M_{\nu\sigma} - i\eta_{\nu\sigma} M_{\mu\rho} + i\eta_{\mu\sigma} M_{\nu\rho} + i\eta_{\nu\rho} M_{\mu\sigma}$$

The Lie algebra $so(1,3)$ exponentiates to the connected part of $SO(1,3)$ to identity. We denote it by $SO(1,3)_0$.

Lorentz transformations act on vectors as $x'^\mu = \Lambda^\mu{}_\nu x^\nu$

In order to define the action on spinors we need to consider the covering group of $SO(1,3)_0$, which is $Spin(1,3) \simeq SL(2,\mathbb{C})$. This is a double cover, which means that there is a homomorphism

$$SL(2,\mathbb{C}) \longrightarrow SO(1,3)_0$$
$$A \longmapsto \Lambda(A)$$

$$\Lambda(A)\Lambda(B) = \Lambda(AB)$$

In order to construct this map explicitly, define

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sigma^\mu = (\sigma^0, \sigma^i) = (\sigma_0, -\sigma_i)$$

The matrices σ^μ are a complete set for the 2×2 matrices. This means any 2×2 matrix can be decomposed on a basis of σ^μ. Start with a 4-dimensional vector x^μ and define $X = x^\mu \sigma_\mu$. The matrices σ_μ are hermitian and the components x^μ are real, therefore X is hermitian $X^+ = X$.

$$\det X = \det \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} = (x^0)^2 - (x^3)^2 - (x^1)^2 - (x^2)^2 = x^\mu x_\mu$$

Let us now act with a matrix A such that $\det A = 1$ on X

$$X \longrightarrow X' = A X A^+$$

This preserves the norm of X (since $\det A = 1$) and the hermiticity. Therefore, it is a Lorentz transformation. Let us find what it is as a 4×4 matrix.

Define

$$\bar\sigma^\mu = (\sigma^0, -\sigma^i) = (\sigma_0, \sigma_i)$$

We have

$$\begin{cases} \sigma^\mu \bar\sigma^\nu + \sigma^\nu \bar\sigma^\mu = 2\eta^{\mu\nu} \mathbb{1}_2 \\ \bar\sigma^\mu \sigma^\nu + \bar\sigma^\nu \sigma^\mu = 2\eta^{\mu\nu} \mathbb{1}_2 \end{cases}$$

$$\mathrm{tr}(\sigma^\mu \bar\sigma^\nu) = 2\eta^{\mu\nu}$$

Then,

$$x^\mu = \tfrac{1}{2}\mathrm{tr}(X\bar\sigma^\mu) \quad \Rightarrow \quad x'^\mu = \tfrac{1}{2}\mathrm{tr}(X'\bar\sigma^\mu) = \tfrac{1}{2}\mathrm{tr}(AXA^+\bar\sigma^\mu)$$
$$= \tfrac{1}{2}\mathrm{tr}(A\sigma_\nu A^+\bar\sigma^\mu)\, x^\nu$$

$$\Rightarrow \quad \boxed{\Lambda^\mu{}_\nu(A) = \frac{1}{2}\mathrm{tr}\left(\bar\sigma^\mu A \sigma_\nu A^+\right)}$$

The relation between $\Lambda(A)$ and A is not 1-to-1. For example,

$$\Lambda(-A) = \Lambda(A)$$

Therefore, given $\Lambda(A)$ we can not uniquely determine A, but given A we can uniquely determine $\Lambda(A)$.

Spinors

The spinors are the 2-dimensional (complex) representations of $SL(2,\mathbb{C})$. The group $SL(2,\mathbb{C})$ acts on $\mathbb{C}^2$ by matrix multiplication, so one of the spinor representations has the transformation law

$$\psi \to \psi' = A\psi \qquad \text{where} \quad A \in SL(2,\mathbb{C}), \quad \psi \in \mathbb{C}^2$$

The complex-conjugate representation is

$$\bar\psi \to \bar\psi' = A^* \bar\psi$$

and is not isomorphic to the first representation. If the complex conjugate representation $\bar R$ is not isomorphic to R, then we are dealing with complex representations.

It is conventional to use Greek indices for spinors. We have ψ_α, $\alpha = 1,2$ for the components of ψ. The components of $\bar\psi$ are written with dotted indices $\bar\psi_{\dot\alpha}$ $\dot\alpha = 1,2$. This emphasizes the fact that they transform differently under Lorentz transformations.

$$\psi_\alpha \to \psi'_\alpha = A_\alpha{}^\beta \psi_\beta \qquad \text{(called left)}$$

$$\bar\psi_{\dot\alpha} \to \bar\psi'_{\dot\alpha} = (A^*)_{\dot\alpha}{}^{\dot\beta}\bar\psi_{\dot\beta} \qquad \text{(called right)}$$

How to make Lorentz invariant quantities of spinors? For example, we want to write Lorentz-invariant Lagrangians...

We can use an $SL(2,\mathbb{C})$ invariant tensor

$$\varepsilon_{\alpha\beta} = \varepsilon_{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\varepsilon^{\alpha\beta} = \varepsilon^{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$A_\alpha{}^\beta A_\gamma{}^\delta \varepsilon_{\beta\delta} = \varepsilon_{\alpha\gamma} \det A = \varepsilon_{\alpha\gamma}$$

In index-free notation this reads $\quad A\varepsilon A^T = \varepsilon$

For the ε tensor with upper indices we have

$$\varepsilon^{\beta\delta} A_\beta{}^\alpha A_\delta{}^\gamma = \varepsilon^{\alpha\gamma}\det A = \varepsilon^{\alpha\gamma} \quad \Rightarrow \quad A^T\varepsilon A = \varepsilon$$

Now consider the transformation properties of $\psi^T\varepsilon\chi$ where ψ, χ are spinors

$$\psi^T\varepsilon\chi \to \psi'^T\varepsilon\chi' = (A\psi)^T\varepsilon(A\chi) = \psi^T A^T\varepsilon A\chi = \psi^T\varepsilon\chi$$

The quantity $\psi^T\varepsilon\chi = \psi_\alpha\varepsilon^{\alpha\beta}\chi_\beta$ is a Lorentz scalar.