King's College London

UNIVERSITY OF LONDON

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MSC AND MSCI EXAMINATION

7CCMMS40T SUPERSYMMETRY (MSC)
7CCMMS40U SUPERSYMMETRY (MSCI)

SUMMER 2017

TIME ALLOWED: TWO HOURS

ALL QUESTIONS CARRY EQUAL MARKS.

FULL MARKS WILL BE AWARDED FOR COMPLETE ANSWERS TO FOUR QUESTIONS.

IF MORE THAN FOUR QUESTIONS ARE ATTEMPTED, THEN ONLY THE BEST FOUR WILL COUNT.

NO CALCULATORS ARE PERMITTED.

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Introduction

The following questions are about supersymmetry in two-dimensional space-time. Some useful formulas follow. For every question, you may use without proof the results of the previous questions.

We will study the free action

$$S = \frac{1}{2\pi} \int d^2\sigma (\partial_\alpha X^\mu \partial^\alpha X_\mu + \bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu), \tag{1}$$

where $\alpha = 0, 1$, and $\mu = 0, 1, \ldots, D-1$, and σ^α are the coordinates of the two-dimensional space-time, X^μ are D real scalar fields, ψ^μ are D real fermionic fields. These fermions are two-component spinors ψ^μ_A, where $A = 1, 2$ but in the following we will mostly omit the spinor indices.

We can choose a representation of the two-dimensional Dirac matrices

$$\rho^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2}$$

and they satisfy the algebra

$$\{\rho^\alpha, \rho^\beta\} = 2\eta^{\alpha\beta} \mathbf{1}_2, \tag{3}$$

where $\eta = \mathrm{diag}(-1, 1)$.

The two components ψ_A with $A = 1, 2$ of the spinor ψ can be written in column form as

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{4}$$

We also define $\bar{\psi} = \psi^\dagger \rho^0$ which can be written as a row

$$\bar{\psi} = \begin{pmatrix} \psi_1^* & \psi_2^* \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \psi_2^* & -\psi_1^* \end{pmatrix}. \tag{5}$$

A real spinor is a spinor whose components are real, i.e. $\psi_A^* = \psi_A$.

Questions

1. (Properties of the Dirac matrices)

Show the following properties

1. $(\rho^\alpha)^\intercal = \rho^0\rho^\alpha\rho^0$, $(\rho^\alpha)^\dagger = \rho^0\rho^\alpha\rho^0$,
2. $\rho^\alpha\rho^\beta\rho_\alpha = 0$,
3. $\mathrm{tr}(\rho^\alpha) = 0$, $\mathrm{tr}(\rho^\alpha\rho^\beta) = 2\eta^{\alpha\beta}$,
4. $\{\rho^\alpha, \rho^0\rho^1\} = 0$.

2. (Properties of spinor contractions)

If ψ and χ are two *real* Grassmann spinors whose components anti-commute $\psi_A\chi_B = -\chi_B\psi_A$, show the following identities

1. $\bar{\psi}\chi = \bar{\chi}\psi$
2. $\bar{\psi}\rho^\alpha\chi = -\bar{\chi}\rho^\alpha\psi$
3. $\bar{\psi}\rho^\alpha\rho^\beta\chi = \bar{\chi}\rho^\beta\rho^\alpha\psi$

3. (Supersymmetry variations)

Consider the following variations

$$\delta X^\mu = \bar{\epsilon}\psi^\mu, \tag{6}$$
$$\delta\psi^\mu = c(\partial_\alpha X^\mu)\rho^\alpha\epsilon, \tag{7}$$

where ϵ is a spinor and c is a complex constant.

1. If ϵ is a real spinor, show that c should be real.
2. What are the dimensions of X^μ and ψ^μ? What are the dimensions of ϵ and c?
3. Compute the variation of $\bar{\psi}$.
4. Compute the variation of the action and find the value of c for which the action is invariant.

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4. (Supersymmetry algebra)

In this section we study the algebra of the transformations written above. We denote by δ_1 a transformation with parameter ϵ_1 and by δ_2 a transformation with parameter ϵ_2.

1. Write the equations of motion for X^μ and ψ^μ.

2. Without imposing the equations of motion (i.e. off-shell) how many degrees of freedom are there in the bosonic fields X^μ and in the fermionic fields ψ^μ?

3. Do you expect the supersymmetry transformations to close off-shell without adding auxiliary fields? If not, what type (bosonic or fermionic) of auxiliary fields do we need to add in order to close the supersymmetry off-shell?

4. Compute the commutator $[\delta_1, \delta_2]X^\mu$, when acting on X^μ and show that it corresponds to a translation by a real quantity.

5. If $\bar{\epsilon}_1$ and ϵ_2 are spinors and $A, B = 1, 2$ are spinor indices, show that

$$\bar{\epsilon}_1^A \epsilon_{2,B} = \frac{1}{2}(\bar{\epsilon}_1\epsilon_2)\delta_B^A + \frac{1}{2}(\bar{\epsilon}_1\rho^\alpha\epsilon_2)(\rho_\alpha)_B{}^A + \frac{1}{2}(\bar{\epsilon}_1\rho^0\rho^1\epsilon_2)(\rho^0\rho^1)_B{}^A. \quad (8)$$

6. Compute the commutator $[\delta_1, \delta_2]\psi^\mu$, when acting on ψ^μ and show that it contains a translation by the same quantity as for X^μ (you may assume without proof the result from the previous question). What happens to the extra term when the equations of motion for ψ is satisfied?

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5. (An interacting theory)

In this question we study a two-dimensional interacting supersymmetric field theory of a real boson ϕ and a real fermion ψ. The action is

$$S = \frac{1}{2\pi} \int d^2\sigma \left(\partial_\alpha \phi \partial^\alpha \phi + \mu^2 e^{2b\phi} + \bar{\psi} \rho^\alpha \partial_\alpha \psi - b\mu \bar{\psi} \psi e^{b\phi} \right) . \tag{9}$$

This theory is called the super-Liouville theory and is the supersymmetrization of the Liouville theory (which can be obtained by setting ψ to zero).

1. What is the dimension of the fields ϕ, ψ and of the parameters μ, b?

2. Given the transformations

$$\delta\phi = \bar{\epsilon}\psi, \tag{10}$$
$$\delta\psi = \rho^\alpha \epsilon \partial_\alpha \phi + \mu e^{b\phi} \epsilon, \tag{11}$$

   compute the variation of $\bar{\psi}$.

3. Show that the interacting action above is invariant under these supersymmetry transformations.