①

1. $\rho^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\rho^{0,T} = -\rho^0 \overset{?}{=} \rho^0\rho^0\rho^0 = -\rho^0 \quad ✓$$

$$\rho^{1,T} = \rho^1 \overset{?}{=} \rho^0\rho^1\rho^0 = -\rho^1\rho^0\rho^0 = \rho^1 \quad ✓$$

$\rho^{\alpha,+} = \rho^{\alpha,T}$ since the matrices are real

$$\Rightarrow \quad \rho^{\alpha,+} = \rho^0\rho^\alpha\rho^0$$

2. $$\rho^\alpha\rho^\beta\rho_\alpha = (2\eta^{\alpha\beta} - \rho^\beta\rho^\alpha)\rho_\alpha = 2\rho^\beta - 2\rho^\beta = 0$$

3. $$\mathrm{tr}(\rho^0) = 0, \qquad \mathrm{tr}(\rho^1) = 0$$

$$\mathrm{tr}(\rho^\alpha\rho^\beta) = \frac{1}{2}\mathrm{tr}(\rho^\alpha\rho^\beta + \rho^\beta\rho^\alpha) = \frac{1}{\cancel{2}}\mathrm{tr}(\cancel{2}\eta^{\alpha\beta}\mathbb{1}_2) = \eta^{\alpha\beta}\cdot 2$$

4. $$\{\rho^\alpha, \rho^0\rho^1\} = 0$$

$$\{\rho^1, \rho^0\rho^1\} = \rho^1\rho^0\rho^1 + \rho^0\rho^1\rho^1 = (-\rho^0\rho^1)\rho^1 + \rho^0\rho^1\rho^1 = 0$$

$$\{\rho^0, \rho^0\rho^1\} = \rho^0\rho^0\rho^1 + \rho^0\rho^1\rho^0 = \rho^0\rho^0\rho^1 + \rho^0(-\rho^0\rho^1) = 0$$

②

1. $$\bar{\Psi}_1\Psi_2 = \Psi_1^+\rho^0\Psi_2 = \Psi_1^T\rho^0\Psi_2 = \Psi_{1,A}(\rho^0)_{AB}\Psi_{2,B}$$

$$= -\Psi_{2,B}\Psi_{1,A}\rho^0_{AB} = \Psi_{2,B}\rho^0_{BA}\Psi_{1,A} = \bar{\Psi}_2\Psi_1$$

2. $$\bar{\Psi}_1\rho^\alpha\Psi_2 = \Psi_1^T\rho^0\rho^\alpha\Psi_2 = (\Psi_1^T\rho^0\rho^\alpha\Psi_2)^T = -\Psi_2^T\rho^{\alpha,T}\rho^{0,T}\Psi_1$$

(from the interchange of Ψ_1 and Ψ_2)

$$= -\Psi_2^T\rho^0\rho^\alpha\rho^0(-\rho^0)\Psi_1 = -\Psi_2^T\rho^0\rho^\alpha\Psi_1 = -\bar{\Psi}_2\rho^\alpha\Psi_1$$

3. $$\bar{\Psi}_1\rho^\alpha\rho^\beta\Psi_2 = \Psi_1^T\rho^0\rho^\alpha\rho^\beta\Psi_2 = (\Psi_1^T\rho^0\rho^\alpha\rho^\beta\Psi_2)^T$$

$$= -\Psi_2^T\rho^{\beta,T}\rho^{\alpha,T}\rho^{0,T}\Psi_1 = -\Psi_2^T(\rho^0\rho^\beta\rho^0)(\rho^0\rho^\alpha\rho^0)(-\rho^0)\Psi_1$$

$$= \Psi_2^T\rho^0\rho^\beta\rho^\alpha\Psi_1$$

$$= \bar{\Psi}_2\rho^\beta\rho^\alpha\Psi_1$$

(3)

$$\delta X^\mu = \bar{\varepsilon}\psi^\mu$$

$$\delta \psi^\mu = c(\partial_\alpha X^\mu)\rho^\alpha \varepsilon$$

1.

$$\delta\psi^{\mu,*} = c^*(\partial_\alpha X^\mu)\rho^{\alpha,*}\varepsilon^* = c^*(\partial_\alpha X^\mu)\rho^\alpha\varepsilon$$

$$= \delta\psi^\mu = c(\partial_\alpha X^\mu)\rho^\alpha\varepsilon$$

$$\Rightarrow \quad c = c^* \quad \Rightarrow \quad c \in \mathbb{R}.$$

2. In d dimensions the scalar field has dimension $\frac{d-2}{2}$ while the fermion field has dimension $\frac{d-1}{2}$. For $d=2$ we find $[X^\mu]=0$, $[\psi]=\frac{1}{2}$. In order for the transformation $\delta X^\mu = \bar{\varepsilon}\psi^\mu$ to be consistent, we need to take $[\varepsilon] = -\frac{1}{2}$. Also, we have $[\partial] = 1$ so

$$[\delta\psi^\mu] = [\psi^\mu] = [c] + [\partial_\alpha] + [X^\mu] + [\varepsilon]$$

$$\Rightarrow \quad \frac{1}{2} = [c] + 1 + 0 + \left(-\frac{1}{2}\right)$$

$$\Rightarrow \quad [c] = 0 \quad (c \text{ is a dimensionless real number})$$

3.

$$\bar{\psi} = \psi^+\rho^0$$

$$\delta\bar{\psi} = (\delta\psi)^+\rho^0 = \left(c\ \varepsilon^+(\rho^\alpha)^+(\partial_\alpha X^\mu)\right)\rho^0$$

$$= c\ \varepsilon^+(\rho^0\rho^\alpha\rho^0)(\partial_\alpha X^\mu)\rho^0$$

$$= -c(\bar{\varepsilon}\rho^\alpha)(\partial_\alpha X^\mu)$$

4.

$$\delta\left((\partial_\alpha X^\mu)(\partial^\alpha X_\mu)\right) = 2(\partial_\alpha X^\mu)(\partial^\alpha \delta X_\mu)$$

$$= 2(\partial_\alpha X^\mu)\partial^\alpha(\bar{\varepsilon}\psi^\mu) = 2(\partial_\alpha X^\mu)(\bar{\varepsilon}\partial^\alpha\psi^\mu)$$

$$= 2\partial^\alpha\left((\partial_\alpha X^\mu)(\bar{\varepsilon}\psi^\mu)\right) - 2(\Box X^\mu)(\bar{\varepsilon}\psi_\mu)$$

$$\delta(\bar{\psi}^\mu\rho^\alpha\partial_\alpha\psi_\mu) = \delta\bar{\psi}^\mu\rho^\alpha\partial_\alpha\psi_\mu + \bar{\psi}^\mu\rho^\alpha\partial_\alpha\delta\psi_\mu$$

$$= -c(\bar{\varepsilon}\rho^\alpha\rho^\beta\partial_\beta\psi_\mu)(\partial_\alpha X^\mu) + \bar{\psi}^\mu\rho^\alpha\partial_\alpha\left(c(\partial_\beta X_\mu)\rho^\beta\varepsilon\right)$$

$$= -c(\bar{\varepsilon}\rho^\alpha\rho^\beta\partial_\beta\psi_\mu)(\partial_\alpha X^\mu) + c(\bar{\psi}^\mu\rho^\alpha\rho^\beta\varepsilon)(\partial_\alpha\partial_\beta X_\mu)$$

$$= \partial_\beta\left(-c(\bar{\varepsilon}\rho^\alpha\rho^\beta\psi_\mu)\partial_\alpha X^\mu\right) + c(\bar{\varepsilon}\rho^\alpha\rho^\beta\psi_\mu)\partial_\beta\partial_\alpha X^\mu$$

$$+ c(\bar{\psi}^\mu\rho^\alpha\rho^\beta\varepsilon)\partial_\alpha\partial_\beta X_\mu$$

$$= \partial_\beta \left(-c (\bar\varepsilon \rho^\alpha \rho^\beta \psi_\mu)(\partial_\alpha X^\mu) \right)$$
$$+ c \left[(\bar\Psi^\mu \rho^\beta \rho^\alpha \varepsilon) + (\bar\Psi^\mu \rho^\alpha \rho^\beta \varepsilon) \right] \partial_\alpha \partial_\beta X_\mu$$

$$= \partial_\beta (\dots) + 2c\, (\bar\Psi^\mu \varepsilon)\, \Box X_\mu$$
$$= \partial_\beta (\dots) + 2c\, (\bar\epsilon \psi^\mu)\, \Box X_\mu$$

Adding the two contributions, the $\Box X$ part cancels when $\boxed{c = 1}$

The total derivative also cancels if we impose the right boundary conditions.

(4)

1. Taking the variations wrt. X^μ and ψ^μ yields

$$\Box X^\mu = 0 \qquad , \qquad \rho^\alpha \partial_\alpha \psi^\mu = 0$$

2. Since X^μ is real, it contains D degrees of freedom

Similarly, since ψ^μ is real and is a 2-component spinor, it contains $2D$ degrees of freedom.

3. No. Linear realization of supersymmetry requires an equal number of bosonic and fermionic degrees of freedom. Therefore, we need to add D bosonic degrees of freedom in order to close the supersymmetry algebra off-shell.

4. Compute $[\delta_1, \delta_2] X^\mu$

$$\delta_2 X^\mu = \bar\epsilon_2 \psi^\mu \quad , \quad \delta_1 \delta_2 X^\mu = \bar\epsilon_2 (\partial_\alpha X^\mu \rho^\alpha \epsilon_1) = (\bar\epsilon_2 \rho^\alpha \epsilon_1)\, \partial_\alpha X^\mu$$

$$\Rightarrow [\delta_1, \delta_2] X^\mu = (\bar\epsilon_2 \rho^\alpha \epsilon_1 - \bar\epsilon_1 \rho^\alpha \epsilon_2)\, \partial_\alpha X^\mu$$
$$= 2 (\bar\epsilon_2 \rho^\alpha \epsilon_1)\, \partial_\alpha X^\mu$$

This is a translation by $2(\bar\epsilon_2 \rho^\alpha \epsilon_1)$. Since $\epsilon_2^* = \epsilon_2$, $\epsilon_1^* = \epsilon_1$, $(\rho^\alpha)^* = \rho^\alpha$

$\Rightarrow$ $\bar\epsilon_2 \rho^\alpha \epsilon_1$ is real.

5. Notice that $1, \rho^0, \rho^1, \rho^0\rho^1$ are a basis for the space of 2×2 matrices. We will compute the coefficients of $\bar{\epsilon}_1{}^A \epsilon_{2,B}$ in this basis

$$\bar{\epsilon}_1{}^A \epsilon_{2,B} = a\,\delta_B^A + b_\alpha (\rho^\alpha)_B{}^A + c\,(\rho^0\rho^1)_B{}^A$$

Take the trace over A, B

$$\Rightarrow \quad \bar{\epsilon}_1 \epsilon_2 = 2a \quad \Rightarrow \quad a = \tfrac{1}{2}\bar{\epsilon}_1\epsilon_2$$

Multiply by $(\rho^\beta)_A{}^B$ and contract

$$\Rightarrow \quad \bar{\epsilon}_1 \rho^\beta \epsilon_2 = b_\alpha \,\mathrm{tr}(\rho^\alpha\rho^\beta) = 2b^\beta \quad \Rightarrow \quad b^\beta = \tfrac{1}{2}\bar{\epsilon}_1\rho^\beta\epsilon_2$$

Multiply by $(\rho^0\rho^1)_A{}^B$ and contract

$$\bar{\epsilon}_1 \rho^0\rho^1 \epsilon_2 = c\,\mathrm{tr}(\rho^0\rho^1\rho^0\rho^1) = -c\,\mathrm{tr}(\rho^0\rho^0\rho^1\rho^1) = c\,\mathrm{tr}\,1 = 2c$$

$$c = \tfrac{1}{2}\bar{\epsilon}_1\rho^0\rho^1\epsilon_2$$

In conclusion

$$\bar{\epsilon}_1{}^A \epsilon_{2,B} = \tfrac{1}{2}(\bar{\epsilon}_1\epsilon_2)\,\delta_B^A + \tfrac{1}{2}(\bar{\epsilon}_1\rho_\alpha\epsilon_2)(\rho^\alpha)_B{}^A + \tfrac{1}{2}(\bar{\epsilon}_1\rho^0\rho^1\epsilon_2)(\rho^0\rho^1)_B{}^A$$

6. $[\delta_1, \delta_2]\Psi^\mu = ?$

$$\delta_2 \Psi^\mu = (\partial_\alpha X^\mu)\rho^\alpha\epsilon_2$$

$$\delta_1\delta_2\Psi^\mu = \partial_\alpha(\bar{\epsilon}_1\Psi^\mu)\rho^\alpha\epsilon_2 = (\bar{\epsilon}_1\partial_\alpha\Psi^\mu)\rho^\alpha\epsilon_2$$

$$[\delta_1,\delta_2]\Psi^\mu = (\bar{\epsilon}_1\partial_\alpha\Psi^\mu)\rho^\alpha\epsilon_2 - (\bar{\epsilon}_2\partial_\alpha\Psi^\mu)\rho^\alpha\epsilon_1$$

If we contract the identity from the previous question by $\partial_\alpha\Psi^\mu_A$ we find

$$-(\bar{\epsilon}_1\partial_\alpha\Psi^\mu)\epsilon_2 = \tfrac{1}{2}(\bar{\epsilon}_1\epsilon_2)\partial_\alpha\Psi^\mu + \tfrac{1}{2}(\bar{\epsilon}_1\rho^\beta\epsilon_2)\rho_\beta\partial_\alpha\Psi^\mu$$
$$+\tfrac{1}{2}(\bar{\epsilon}_1\rho^0\rho^1\epsilon_2)\rho^0\rho^1\partial_\alpha\Psi^\mu$$

$$(\bar{\epsilon}_1\partial_\alpha\Psi^\mu)\epsilon_2 - (\bar{\epsilon}_2\partial_\alpha\Psi^\mu)\epsilon_1 = (\bar{\epsilon}_1\rho^\beta\epsilon_2)\rho_\beta\partial_\alpha\Psi^\mu$$

$$\Rightarrow \quad [\delta_1,\delta_2]\Psi^\mu = -(\bar{\epsilon}_1\rho^\beta\epsilon_2)\rho^\alpha\rho_\beta\partial_\alpha\Psi^\mu$$
$$= -(\bar{\epsilon}_1\rho^\beta\epsilon_2)(2\delta_\beta^\alpha - \rho_\beta\rho^\alpha)\partial_\alpha\Psi^\mu$$
$$= -2\underbrace{(\bar{\epsilon}_1\rho^\alpha\epsilon_2)}_{-(\bar{\epsilon}_2\rho^\alpha\epsilon_1)}\partial_\alpha\Psi^\mu + (\bar{\epsilon}_1\rho^\beta\epsilon_2)\rho_\beta(\rho^\alpha\partial_\alpha\Psi^\mu)$$

(5) 1. $[\phi] = 0, \qquad [\psi] = 1/2$

$[b] = 0 \quad , \qquad [\mu] = 1$

2.

$$\delta\varphi = \bar\epsilon\psi$$

$$\delta\psi = \rho^\alpha \epsilon \partial_\alpha \varphi + \mu e^{b\varphi}\epsilon$$

$$\delta\bar\psi = (\delta\psi)^+ \rho^0 = \partial_\alpha\varphi\ \epsilon^+ \rho^{\alpha,+}\rho^0 + \mu e^{b\varphi}\epsilon^+\rho^0$$

$$= -(\partial_\alpha\varphi)\ \bar\epsilon\rho^\alpha + \mu e^{b\varphi}\bar\epsilon$$

(μ and b have to be real to preserve the reality of $\delta\psi$)

3.

$$\delta(\partial_\alpha\varphi\,\partial^\alpha\varphi) = 2\,\partial_\alpha\varphi\,\partial^\alpha\delta\varphi = 2\partial_\alpha\varphi\,\partial^\alpha(\bar\epsilon\psi)$$

$$\delta\left(\mu^2 e^{2b\varphi}\right) = 2b\mu^2 e^{2b\varphi}(\bar\epsilon\psi)$$

$$\delta(\bar\psi\rho^\alpha\partial_\alpha\psi) = \left(-\partial_\beta\varphi\,\bar\epsilon\rho^\beta + \mu e^{b\varphi}\bar\epsilon\right)\rho^\alpha\partial_\alpha\psi$$

$$+ \bar\psi\rho^\alpha\partial_\alpha\left(\rho^\beta\epsilon\,\partial_\beta\varphi + \mu e^{b\varphi}\epsilon\right)$$

$$= -\partial_\beta\varphi\,(\bar\epsilon\rho^\beta\rho^\alpha\partial_\alpha\psi) + \mu e^{b\varphi}(\bar\epsilon\rho^\alpha\partial_\alpha\psi)$$
$$+ (\bar\psi\rho^\alpha\rho^\beta\epsilon)\,\partial_\alpha\partial_\beta\varphi + (\bar\psi\rho^\alpha\epsilon)\,\mu b e^{b\varphi}(\partial_\alpha\varphi)$$

$$= -(\partial_\beta\varphi)\,(\bar\epsilon\rho^\beta\rho^\alpha\partial_\alpha\psi) + \mu e^{b\varphi}(\bar\epsilon\rho^\alpha\partial_\alpha\psi)$$

$$+ (\bar\psi\epsilon)\,\Box\varphi + (\bar\psi\rho^\alpha\epsilon)\,\mu b e^{b\varphi}(\partial_\alpha\varphi)$$

$$\delta\left(-b\mu\bar\psi\psi e^{b\varphi}\right) = -b\mu\left(-(\partial_\alpha\varphi)\bar\epsilon\rho^\alpha + \mu e^{b\varphi}\bar\epsilon\right)\psi\ e^{b\varphi}$$

$$- b\mu\bar\psi\left(\rho^\alpha\epsilon\,\partial_\alpha\varphi + \mu e^{b\varphi}\epsilon\right)e^{b\varphi}$$

$$- b\mu\bar\psi\psi\ e^{b\varphi}\ b\,\bar\epsilon\psi$$

$$= b\mu\,\partial_\alpha\varphi\ e^{b\varphi}\ \bar\epsilon\rho^\alpha\psi - b\mu^2 e^{2b\varphi}\ \bar\epsilon\psi$$

$$- b\mu\,\partial_\alpha\varphi\,e^{b\varphi}\ \bar\psi\rho^\alpha\epsilon - b\mu^2 e^{2b\varphi}\ \bar\psi\epsilon$$

$$- b^2\mu(\bar\psi\psi)(\bar\epsilon\psi)\,e^{b\varphi}$$

$$= 2b\mu(\partial_\alpha\varphi)\,e^{b\varphi}\ \bar\epsilon\rho^\alpha\psi - 2b\mu^2 e^{2b\varphi}\ \bar\epsilon\psi - b^2\mu\ (\bar\psi\psi)(\bar\epsilon\psi)\,e^{b\varphi}$$

Adding all the terms together, we obtain

$$2(\partial_\alpha\varphi)\,\partial^\alpha(\bar\epsilon\psi) + 2b\mu^2\,\cancel{e^{2b\varphi}}\,(\bar\epsilon\psi)$$

$$-(\partial_\beta\varphi)(\bar\epsilon\rho^\beta\rho^\alpha\partial_\alpha\psi) + \mu e^{b\varphi}(\bar\epsilon\rho^\alpha\partial_\alpha\psi)$$

$$+(\bar\psi\epsilon)\,\Box\varphi + \underline{(\bar\psi\rho^\alpha\epsilon)\,\mu b\, e^{b\varphi}(\partial_\alpha\varphi)}$$

$$+\underline{2b\mu(\partial_\alpha\varphi)\,e^{b\varphi}(\bar\epsilon\rho^\alpha\psi)} - 2b\mu^2\,\cancel{e^{2b\varphi}}\,(\bar\epsilon\psi)$$

$$\boxed{-b^2\mu\,(\bar\psi\psi)(\bar\epsilon\psi)\,e^{b\varphi}}$$

The circled terms produce

$$2(\partial_\alpha\varphi)\,\partial^\alpha(\bar\epsilon\psi) - (\partial_\beta\varphi)\,\bar\epsilon\,(2\eta^{\alpha\beta} - \rho^\alpha\rho^\beta)\,\partial_\alpha\psi + (\bar\psi\epsilon)\,\Box\varphi$$

$$= \partial_\alpha\left((\partial_\beta\varphi)\,\bar\epsilon\rho^\alpha\rho^\beta\psi\right)$$

The boxed term cubic in the fermionic fields vanishes since the spinors ψ are anti-commuting and they only have two components.

The underlined terms can be combined together. In the end, we are left with.

$$\mu e^{b\varphi}(\bar\epsilon\rho^\alpha\partial_\alpha\psi) + b\mu e^{b\varphi}(\partial_\alpha\varphi)(\bar\epsilon\rho^\alpha\psi) + \partial_\alpha\left((\partial_\beta\varphi)(\bar\epsilon\rho^\alpha\rho^\beta\psi)\right)$$

$$= \partial_\alpha\left(\mu e^{b\varphi}(\bar\epsilon\rho^\alpha\psi) + (\partial_\beta\varphi)(\bar\epsilon\rho^\alpha\rho^\beta\psi)\right)$$

Hence, the Lagrangian density is invariant up to a total derivative. If we impose appropriate boundary conditions, the action is invariant.