King's College London

University Of London

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Candidate No: .... **Desk No:** ....

MSc and MSci Examination

7CCMMS40T Supersymmetry (MSc)
7CCMMS40U Supersymmetry (MSci)

August 2017

Time Allowed: Two Hours

All questions carry equal marks.
Full marks will be awarded for complete answers to FOUR questions.
If more than four questions are attempted, then only the best FOUR will count.

NO calculators are permitted.

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Note

The questions below are largely independent, but you are advised to attempt them in the order in which they are presented. When solving a given question you are encouraged to try to use results from previous questions, even when you have not been able to show them.

Questions

1. *Algebra of supersymmetry covariant derivatives*

The supersymmetry covariant derivatives satisfy the algebra

$$\{D_\alpha, D_\beta\} = 0, \qquad \{\bar{D}_{\dot\alpha}, \bar{D}_{\dot\beta}\} = 0, \qquad \{D_\alpha, \bar{D}_{\dot\alpha}\} = -2i\partial_{\alpha\dot\alpha}. \tag{1}$$

We use the notations $D^2 = D^\alpha D_\alpha$, $\bar{D}^2 = \bar{D}_{\dot\alpha}\bar{D}^{\dot\alpha}$. Our conventions for raising indices for the supersymmetry covariant derivatives are $D^\alpha = -\epsilon^{\alpha\beta}D_\beta$, $\bar{D}^{\dot\alpha} = -\epsilon^{\dot\alpha\dot\beta}\bar{D}_{\dot\beta}$. Furthermore, we have $\epsilon_{\alpha\beta}\epsilon^{\beta\gamma} = \delta_\alpha^\gamma$ and similarly for the dotted index case. Raising the indices of spinor fields is done as for supersymmetry covariant derivatives, but without the extra minus sign, $\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta$.

Show that

1. $D_\alpha D_\beta D_\gamma = 0$,
2. $D_\alpha D_\beta = -\frac{1}{2}\epsilon_{\alpha\beta}D^2, \qquad \bar{D}_{\dot\alpha}\bar{D}_{\dot\beta} = \frac{1}{2}\epsilon_{\dot\alpha\dot\beta}\bar{D}^2$,
3. $D^\alpha\bar{D}^2D_\alpha = \bar{D}_{\dot\alpha}D^2\bar{D}^{\dot\alpha}$,
4. $[D_\alpha, \bar{D}^2] = -4i\partial_{\alpha\dot\alpha}\bar{D}^{\dot\alpha}, \qquad [\bar{D}_{\dot\alpha}, D^2] = 4i\partial_{\alpha\dot\alpha}D^\alpha$,
5. $[D^2, \bar{D}^2] = -4i\partial_{\alpha\dot\alpha}[D^\alpha, \bar{D}^{\dot\alpha}]$.

2. *Reality conditions*

In this question we study reality conditions on superfields and their supersymmetry covariant derivatives. We take the supersymmetry covariant derivatives to satisfy $(D_\alpha)^\dagger = \bar{D}_{\dot{\alpha}}$. In order to study the action of $\dagger$ on derivatives of superfields, we take $D_\alpha \Psi = [D_\alpha, \Psi\}$, where $[\cdot,\cdot\}$ stands for commutator if Ψ is even and anti-commutator if Ψ is odd. Our conventions are such that $(AB)^\dagger = B^\dagger A^\dagger$ irrespective of parities of A and B. We also take $(A^\dagger)^\dagger = A$. For the anti-symmetric tensor $\epsilon_{\alpha\beta}$ we have $(\epsilon_{\alpha\beta})^\dagger = \epsilon_{\dot{\alpha}\dot{\beta}}$.

1. Show that $[A, B]^\dagger = -[A^\dagger, B^\dagger]$ and $\{A, B\}^\dagger = \{A^\dagger, B^\dagger\}$.

2. If η_α is a parity odd superfield such that $(\eta_\alpha)^\dagger = \bar{\eta}_{\dot{\alpha}}$, show that $(D^\alpha \eta_\alpha)^\dagger = \bar{D}_{\dot{\alpha}} \bar{\eta}^{\dot{\alpha}}$.

3. If $G = G^\dagger$ is an even real superfield, show that $(D_\alpha G)^\dagger = -\bar{D}_{\dot{\alpha}} G$.

4. Using the fact that $D^2 \Psi = D^\alpha (D_\alpha \Psi)$, show that $(D^2 \Psi)^\dagger = \bar{D}^2 \bar{\Psi}$, where $\bar{\Psi} = \Psi^\dagger$. Show this in both cases: Ψ even and Ψ odd.

5. Show that $([D_\alpha, \bar{D}_{\dot{\alpha}}] G)^\dagger = [D_\alpha, \bar{D}_{\dot{\alpha}}] G$.

3. *Real linear multiplet*

A real linear multiplet is described by a superfield G satisfying the reality condition $G^\dagger = G$ and the constraints $D^2 G = \bar{D}^2 G = 0$. In the following questions we will study the properties of this superfield.

1. Show that the constraints on G can be solved by $G = D^\alpha \eta_\alpha + \bar{D}_{\dot\alpha} \bar{\eta}^{\dot\alpha}$, where η_α is a chiral superfield i.e. $\bar{D}_{\dot\alpha} \eta_\alpha = 0$.

2. Consider a variation of η given by $\delta\eta_\alpha = i\bar{D}^2 D_\alpha V$ with V real $V = V^\dagger$. Show that G is invariant under such a transformation.

3. Define

$$\phi = G\big| \,, \qquad \psi_\alpha = i\, D_\alpha G\big| \,, \qquad \bar{\psi}_{\dot\alpha} = i\, \bar{D}_{\dot\alpha} G\big| \,, \qquad v_{\alpha\dot\alpha} = [D_\alpha, \bar{D}_{\dot\alpha}] G\big| \,, \tag{2}$$

   where the vertical bar stands for projection on the lowest component (or setting all the odd coordinates to zero). Using the constraints on G show that the other components in the expansion of G can be expressed as derivatives of the fields listed above.

4. Assuming that G contains a scalar field with the usual canonical dimension, what are the dimensions of the other component fields of G?

5. Assuming that all the components of $v_{\alpha\dot\alpha}$ are independent, count the number of degrees of freedom in the bosonic and fermionic component fields of G. Do you find a match?

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4. *Action and equations of motion*

We take the superspace action of the linear multiplet to be

$$S = \int d^4x d^2\theta d^2\bar{\theta} \frac{1}{2} G^2. \tag{3}$$

1. In order to find the equations of motion for G we can not naively take the variation with respect to G since it is constrained. However, we have shown in the previous problem that for a linear multiplet G we always have $G = D^\alpha \eta_\alpha + \bar{D}_{\dot{\alpha}} \bar{\eta}^{\dot{\alpha}}$, with η_α chiral and $(\eta_\alpha)^\dagger = \bar{\eta}_{\dot{\alpha}}$. Taking the variation with respect to η and $\bar{\eta}$ show that

$$\bar{D}^2 D_\alpha G = 0, \qquad D^2 \bar{D}_{\dot{\alpha}} G = 0. \tag{4}$$

   You may assume the form of the variational principle in chiral superspace. That is,

$$\int d^4x d^2\theta d^2\bar{\theta} W(\eta, \bar{\eta}) \delta\eta = 0 \implies \int d^4x d^2\theta (\bar{D}^2 W(\eta, \bar{\eta})) \delta\eta = 0. \tag{5}$$

   Then, if the last equality holds for all chiral $\delta\eta$, this implies $\bar{D}^2 W(\eta, \bar{\eta}) = 0$.

2. Show that these equations of motion imply the Dirac equations for the component fields $\psi_\alpha = i\, D_\alpha G|$ and $\bar{\psi}_{\dot{\alpha}} = i\, \bar{D}_{\dot{\alpha}} G\big|$.

3. Acting with the commutator $[D^2, \bar{D}^2]$ on G, show that $\partial_{\alpha\dot{\alpha}} v^{\dot{\alpha}\alpha} = 0$, where $v^{\dot{\alpha}\alpha} = v^\mu \bar{\sigma}_\mu^{\dot{\alpha}\alpha}$, $v_{\alpha\dot{\alpha}} = \sigma^\mu_{\alpha\dot{\alpha}} v_\mu$, $\bar{\sigma}_\mu^{\dot{\alpha}\alpha} = \epsilon^{\dot{\alpha}\dot{\beta}} \epsilon^{\alpha\beta} \sigma_{\mu,\beta\dot{\beta}}$. This shows that not all the components of $v_{\alpha\dot{\alpha}}$ are independent. How many are independent?

5. *Duality transformation*

In this question we will show that the action of a real linear multiplet can be transformed to the action of a chiral multiplet. So in a sense both multiplets capture the same physics. For this purpose, we start with an action

$$S = \int d^4x d^2\theta d^2\bar{\theta} \left(\frac{1}{2}\Gamma^2 - \Gamma(\Phi + \bar{\Phi}) \right), \tag{6}$$

where Γ is a real superfield and Φ is chiral with $\bar{\Phi}$ its anti-chiral conjugate.

1. Show that the variation with respect to Φ and $\bar{\Phi}$ implies $D^2\Gamma = \bar{D}^2\Gamma = 0$. Note: this can be interpreted as saying that Φ and $\bar{\Phi}$ are Lagrange multipliers for the linear multiplet constraints on Γ.

2. Find the equations of motion obtained by taking the variation with respect to Γ.

3. Plug the solution for Γ back in the action and show that, after a small transformation, one obtains the action for a free chiral superfield (up to a global sign).