1) Algebra of supersymmetry covariant derivatives

1. $D_\alpha D_\beta D_\gamma = 0$

since $\{D_\alpha, D_\beta\} = 0 \Rightarrow D_\alpha D_\beta = -D_\beta D_\alpha$

So $D_\alpha D_\beta D_\gamma$ is antisymmetric in all 3 indices $\alpha\beta\gamma$.
However, $\alpha, \beta, \gamma$ only take 2 values each so there are always at least two which are identical. Then by antisymmetry $\Rightarrow 0$.

2. $D_\alpha D_\beta = -\frac{1}{2}\varepsilon_{\alpha\beta} D^2$

We know that $D_\alpha D_\beta$ is antisymmetric so it has to be proportional to $\varepsilon_{\alpha\beta}$

Hence, $D_\alpha D_\beta = c\, \varepsilon_{\alpha\beta}$. Now contract with $-\varepsilon^{\beta\alpha} \Rightarrow$

$$D^2 = D^\beta D_\beta = -c\, \varepsilon^{\beta\alpha}\varepsilon_{\alpha\beta} = -2c \Rightarrow c = -\frac{1}{2} D^2$$

$$\Rightarrow \underline{D_\alpha D_\beta = -\frac{1}{2}\varepsilon_{\alpha\beta} D^2}$$

Similarly, $\bar{D}_{\dot\alpha}\bar{D}_{\dot\beta} = c\, \varepsilon_{\dot\alpha\dot\beta} \Rightarrow \bar{D}_{\dot\alpha}\left(-\varepsilon^{\dot\alpha\dot\beta}\bar{D}_{\dot\beta}\right) = \bar{D}^2$
$= -c\, \varepsilon^{\dot\alpha\dot\beta}\varepsilon_{\dot\alpha\dot\beta} = 2c$

$$\Rightarrow \underline{\bar{D}_{\dot\alpha}\bar{D}_{\dot\beta} = \frac{1}{2}\varepsilon_{\dot\alpha\dot\beta}\bar{D}^2}$$

3. $D^\alpha \bar{D}^2 D_\alpha = \bar{D}_{\dot\alpha} D^2 \bar{D}^{\dot\alpha}$

$$\begin{aligned}
\text{LHS} = D^\alpha \bar{D}^2 D_\alpha &= D^\alpha \bar{D}_{\dot\alpha}\bar{D}^{\dot\alpha} D_\alpha = -D^\alpha \bar{D}^{\dot\alpha}\bar{D}_{\dot\alpha} D_\alpha \\
&= -\left(-2i\partial^{\dot\alpha\alpha} - \bar{D}^{\dot\alpha}D^\alpha\right)\left(-2i\partial_{\alpha\dot\alpha} - D_\alpha \bar{D}_{\dot\alpha}\right) \\
&= 4\partial^{\dot\alpha\alpha}\partial_{\alpha\dot\alpha} - 2i\,\partial^{\dot\alpha\alpha} D_\alpha \bar{D}_{\dot\alpha} - 2i\,\bar{D}^{\dot\alpha}D^\alpha \partial_{\alpha\dot\alpha} - \bar{D}^{\dot\alpha}D^\alpha D_\alpha \bar{D}_{\dot\alpha} \\
&= 4\partial^{\dot\alpha\alpha}\partial_{\alpha\dot\alpha} - 2i\,\partial_{\alpha\dot\alpha}\left(D^\alpha\bar{D}^{\dot\alpha} + \bar{D}^{\dot\alpha}D^\alpha\right) + \bar{D}_{\dot\alpha}D^\alpha D_\alpha \bar{D}^{\dot\alpha} \\
&= 4\partial^{\dot\alpha\alpha}\partial_{\alpha\dot\alpha} - 4\partial_{\alpha\dot\alpha}\partial^{\dot\alpha\alpha} + \bar{D}_{\dot\alpha}D^2\bar{D}^{\dot\alpha} = \text{RHS}
\end{aligned}$$

4.
$$\begin{aligned}
[D_\alpha, \bar{D}^2] &= [D_\alpha, \bar{D}_{\dot\alpha}\bar{D}^{\dot\alpha}] = -\varepsilon^{\dot\alpha\dot\beta}[D_\alpha, \bar{D}_{\dot\alpha}\bar{D}_{\dot\beta}] \\
&= -\varepsilon^{\dot\alpha\dot\beta}\left(\{D_\alpha, \bar{D}_{\dot\alpha}\}\bar{D}_{\dot\beta} - \bar{D}_{\dot\alpha}\{D_\alpha, \bar{D}_{\dot\beta}\}\right) \\
&= -\varepsilon^{\dot\alpha\dot\beta}\left(-2i\,\partial_{\alpha\dot\alpha}\bar{D}_{\dot\beta} - (-2i\,\partial_{\alpha\dot\beta})\bar{D}_{\dot\alpha}\right) \\
&= -2i\,\partial_{\alpha\dot\alpha}\bar{D}^{\dot\alpha} - 2i\,\partial_{\alpha\dot\beta}\bar{D}^{\dot\beta} \\
&= \underline{-4i\,\partial_{\alpha\dot\alpha}\bar{D}^{\dot\alpha}}
\end{aligned}$$

$$[\bar{D}_{\dot{\alpha}}, D^2] = [\bar{D}_{\dot{\alpha}}, D^\alpha D_\alpha] = -\varepsilon^{\alpha\beta}[\bar{D}_{\dot{\alpha}}, D_\beta D_\alpha]$$

$$= -\varepsilon^{\alpha\beta}\left(\{\bar{D}_{\dot{\alpha}}, D_\beta\} D_\alpha - D_\beta \{\bar{D}_{\dot{\alpha}}, D_\alpha\}\right)$$

$$= -\varepsilon^{\alpha\beta}\left(-2i\,\partial_{\beta\dot{\alpha}} D_\alpha - (-2i\,\partial_{\alpha\dot{\alpha}}) D_\beta\right)$$

$$= 2i\,\partial_{\beta\dot{\alpha}} D^\beta + 2i\,\partial_{\alpha\dot{\alpha}} D^\alpha$$

$$= \underline{4i\,\partial_{\alpha\dot{\alpha}} D^\alpha}$$

5. $$[D^2, \bar{D}^2] = [D^2, \bar{D}_{\dot{\alpha}}\bar{D}^{\dot{\alpha}}] = -\varepsilon^{\dot{\alpha}\dot{\beta}}[D^2, \bar{D}_{\dot{\alpha}}\bar{D}_{\dot{\beta}}]$$

$$= -\varepsilon^{\dot{\alpha}\dot{\beta}}\left([D^2, \bar{D}_{\dot{\alpha}}]\bar{D}_{\dot{\beta}} + \bar{D}_{\dot{\alpha}}[D^2, \bar{D}_{\dot{\beta}}]\right)$$

$$= -\varepsilon^{\dot{\alpha}\dot{\beta}}\left(-4i\,\partial_{\alpha\dot{\alpha}} D^\alpha \bar{D}_{\dot{\beta}} + \bar{D}_{\dot{\alpha}}(-4i\,\partial_{\alpha\dot{\beta}} D^\alpha)\right)$$

$$= -4i\,\partial_{\alpha\dot{\alpha}} D^\alpha \bar{D}^{\dot{\alpha}} + 4i\,\partial_{\alpha\dot{\beta}} \bar{D}^{\dot{\beta}} D^\alpha$$

$$= \underline{-4i\,\partial_{\alpha\dot{\alpha}}[D^\alpha, \bar{D}^{\dot{\alpha}}]}$$

## 2.) Reality conditions

1. $$[A,B]^+ = (AB - BA)^+ = B^+A^+ - A^+B^+ = -[A^+, B^+]$$

$$\{A,B\}^+ = (AB + BA)^+ = B^+A^+ + A^+B^+ = \{A^+, B^+\}$$

2. $$(D^\alpha \eta_\alpha)^+ = \{D^\alpha, \eta_\alpha\}^+ = +\{(D^\alpha)^+, \eta_\alpha^+\} = +\{\bar{D}^{\dot\alpha}, \bar\eta_{\dot\alpha}\}$$

$$= +\bar{D}^{\dot\alpha}\bar\eta_{\dot\alpha} = +(-\varepsilon^{\dot\alpha\dot\beta}\bar{D}_{\dot\beta})\bar\eta_{\dot\alpha} = \bar{D}_{\dot\beta}(\varepsilon^{\dot\beta\dot\alpha}\bar\eta_{\dot\alpha})$$

$$= \bar{D}_{\dot\beta}\bar\eta^{\dot\beta}$$

3. $$(D_\alpha G)^+ = ([D_\alpha, G])^+ = -[D_\alpha^+, G^+] = -[\bar{D}_{\dot\alpha}, G]$$

$$= -(\bar{D}_{\dot\alpha} G)$$

4. Consider $\Psi$ fermionic (parity odd).

Then

$$(D^2\Psi)^+ = (D^\alpha(D_\alpha\Psi))^+ = ([D^\alpha, \{D_\alpha, \Psi\}])^+$$

$$= -[D^{\alpha+}, \{D_\alpha^+, \Psi^+\}]$$

$$= -[\bar{D}^{\dot\alpha}, \{\bar{D}_{\dot\alpha}, \bar\Psi\}] = -\bar{D}^{\dot\alpha}\bar{D}_{\dot\alpha}\bar\Psi$$

$$= -(-\varepsilon^{\dot\alpha\dot\beta}\bar{D}_{\dot\beta})\bar{D}_{\dot\alpha}\bar\Psi = \bar{D}_{\dot\beta}(-\varepsilon^{\dot\beta\dot\alpha}\bar{D}_{\dot\alpha})\bar\Psi$$

$$= \bar{D}_{\dot\beta}\bar{D}^{\dot\beta}\bar\Psi = \bar{D}^2\bar\Psi$$

If $\Psi$ is bosonic (parity even) we have instead

$$(D^2\Psi)^+ = (D^\alpha(D_\alpha\Psi))^+ = (\{D^\alpha, [D_\alpha, \Psi]\})^+$$

$$= \{D^{\alpha+}, -[D_\alpha^+, \Psi^+]\} = -\bar{D}^{\dot\alpha}\bar{D}_{\dot\alpha}\bar\Psi \underbrace{= \ldots =}_{\text{same as above}} \bar{D}^2\bar\Psi$$

5. $$([D_\alpha, \bar{D}_{\dot\alpha}]G)^+ = (\{D_\alpha, [\bar{D}_{\dot\alpha}, G]\} - \{\bar{D}_{\dot\alpha}, [D_\alpha, G]\})^+$$

$$= \{D_\alpha^+, -[\bar{D}_{\dot\alpha}^+, G^+]\} - \{\bar{D}_{\dot\alpha}^+, -[D_\alpha^+, G^+]\}$$

$$= -\{\bar{D}_{\dot\alpha}, [D_\alpha, G]\} + \{D_\alpha, [\bar{D}_{\dot\alpha}, G]\}$$

$$= [D_\alpha, \bar{D}_{\dot\alpha}]G$$

3) Real linear multiplet

1.
$$D^2 G = D^2 (D^\alpha \eta_\alpha + \bar{D}_{\dot\alpha} \bar\eta^{\dot\alpha})$$
$$= \underbrace{D^2 D^\alpha \eta_\alpha}_{0 \text{ by } D^3 = 0} + \left( [D^2, \bar D_{\dot\alpha}] + \underbrace{\bar D_{\dot\alpha} D^2}_{} \right) \bar\eta^{\dot\alpha}$$

0 by anti-chirality of $\bar\eta$

$$= -4i \partial_{\alpha\dot\alpha} D^\alpha \bar\eta^{\dot\alpha} = 0 \quad \text{by antichirality of } \bar\eta$$

$\bar D^2 G = 0$ is entirely analogous.

2.
$$\delta \bar\eta_{\dot\alpha} = i D^2 \bar D_{\dot\alpha} V \quad \Rightarrow \quad \delta \bar\eta^{\dot\alpha} = \varepsilon^{\dot\alpha\dot\beta} \delta \bar\eta_{\dot\beta} = -i D^2 \bar D^{\dot\alpha} V$$

Then,
$$\delta G = D^\alpha \delta \eta_\alpha + \bar D_{\dot\alpha} \delta \bar\eta^{\dot\alpha}$$
$$= D^\alpha (i \bar D^2 D_\alpha V) + \bar D_{\dot\alpha} (-i D^2 \bar D^{\dot\alpha} V)$$
$$= i \underbrace{(D^\alpha \bar D^2 D_\alpha - \bar D_{\dot\alpha} D^2 \bar D^{\dot\alpha})}_{=0 \text{ by the identity shown above}} V = 0$$

3. The other components contain either a $D^2$ or a $\bar D^2$ or both. The order of the various covariant derivatives is not important since changing it produces either signs or derivatives of the lower components. So if we have a $D^2$ or a $\bar D^2$ we can move it to the right while producing only derivatives of lower components. But $D^2$ or $\bar D^2$ on the right, acting on $G$ yields 0.

4. We set $[G] = [\phi] = 1$. Then, since $[D_\alpha] = [\bar D_{\dot\alpha}] = \frac{1}{2}$ we find $[\psi_\alpha] = [\bar\psi_{\dot\alpha}] = 3/2$ and $[v_{\alpha\dot\alpha}] = 2$

5.
$\phi$ — 1 dof (real)

$v_{\alpha\dot\alpha}$ — 4 dof ($v_{\alpha\dot\alpha}$ is hermitian $\Rightarrow$ 4 real dof)

$\psi_\alpha$ with complex-conjugate $\bar\psi_{\dot\alpha}$ — 4 dof.

$\Rightarrow$ 5 bosonic $>$ 4 fermionic dof.

(this is not inconsistent since the components of $v_{\alpha\dot\alpha}$ are not independent, as we show below)

## 4) Action and equations of motion

1.
$$\delta S = \int d^4x\, d^2\theta\, d^2\bar\theta\; G\,\delta G$$
$$= \int d^4x\, d^2\theta\, d^2\bar\theta\; G\left(D^\alpha \delta\eta_\alpha + \bar D_{\dot\alpha}\delta\bar\eta^{\dot\alpha}\right)$$
$$= \int d^4x\, d^2\theta\, d^2\bar\theta\left(-(D^\alpha G)\delta\eta_\alpha - (\bar D_{\dot\alpha} G)\delta\bar\eta^{\dot\alpha}\right)$$
$$= \int d^4x\, d^2\theta\, \tfrac{1}{4}\bar D^2\left(-(D^\alpha G)\,\delta\eta_\alpha\right)$$
$$+ \int d^4x\, d^2\bar\theta\, \tfrac{1}{4} D^2\left(-(\bar D_{\dot\alpha} G)\,\delta\bar\eta^{\dot\alpha}\right)$$

since $\delta\eta_\alpha$ chiral, $\delta\bar\eta_{\dot\alpha}$ anti-chiral

$$= -\tfrac{1}{4}\int d^4x\, d^2\theta\left(\bar D^2 D^\alpha G\right)\delta\eta_\alpha - \tfrac{1}{4}\int d^4x\, d^2\bar\theta\left(D^2\bar D_{\dot\alpha}G\right)\delta\bar\eta^{\dot\alpha}$$

$$\Rightarrow \quad \bar D^2 D^\alpha G = 0\,, \qquad D^2\bar D_{\dot\alpha} G = 0$$

2.
$$\bar D^2 D_\alpha G = 0 \quad\Rightarrow\quad \Big([\bar D^2, D_\alpha] + \underbrace{\cancel{D_\alpha \bar D^2}}_{\text{by linearity constraint}}\Big)G = 0$$

$$\Rightarrow \quad 4i\partial_{\alpha\dot\alpha}\bar D^{\dot\alpha} G = 0$$

projecting on the lowest component implies
$$\partial_{\alpha\dot\alpha}\bar\Psi^{\dot\alpha} = 0,$$
which is the Dirac equation for $\bar\Psi^{\dot\alpha}$. The Dirac equation for $\Psi$ is entirely analogous.

3.
$$[D^2, \bar D^2]\,G = 0 \qquad \text{by the linearity constraints.}$$

However, we also have $\quad [D^2, \bar D^2] = -4i\partial_{\alpha\dot\alpha}[D^\alpha, \bar D^{\dot\alpha}]$

$$\Rightarrow \quad 4i\partial_{\alpha\dot\alpha}[D^\alpha, \bar D^{\dot\alpha}]\,G = 0$$
$$\Rightarrow \quad \partial_{\alpha\dot\alpha} v^{\dot\alpha\alpha} = 0$$

In vector language this means that $\partial_\mu v^\mu = 0$ so the vector field $v_\mu$ has divergence zero.

This implies that not all components of $v$ are independent. Going to momentum space we find $p_\mu v^\mu = 0$. Therefore one of the components of $v_\mu$ can always be solved in terms of the others. In the end, $v_{\alpha\dot\alpha}$ has three independent components.

5.) Duality transformation

1. $$S = \int d^4x\, d^2\theta\, d^2\bar\theta \left( \frac{1}{2}\Gamma^2 - \Gamma(\phi + \bar\phi)\right)$$

$$\delta_{\phi,\bar\phi} S = \int d^4x\, d^2\theta\, d^2\bar\theta \left(-\Gamma\,\delta\phi - \Gamma\,\delta\bar\phi\right)$$

$$= \int d^4x\, d^2\theta \left(-\frac{1}{4}\left(\bar D^2 \Gamma\right)\delta\phi\right)$$

since $\phi$ is chiral
$\bar\phi$ anti-chiral

$$+ \int d^4x\, d^2\bar\theta \left(-\frac{1}{4}\left(D^2\Gamma\right)\delta\bar\phi\right)$$

$\Rightarrow \quad \bar D^2 \Gamma = 0, \quad D^2\Gamma = 0$ linear superfield constraints.

So, integrating out $\phi, \bar\phi$ has the effect of imposing linearity constraints on $\Gamma$. and reproduces the action in the previous question

2. We can take the variation w.r.t. $\Gamma$ straightforwardly since it's unconstrained (apart from the reality condition)

$$\Rightarrow \quad \delta_\Gamma S = \int d^4x\, d^2\theta\, d^2\bar\theta \left(\Gamma\,\delta\Gamma - \delta\Gamma(\phi + \bar\phi)\right) = 0$$

$$\Rightarrow \quad \Gamma = \phi + \bar\phi$$

3. Plug back the solution

$$\Rightarrow \quad S' = \int d^4x\, d^2\theta\, d^2\bar\theta \left(\frac{1}{2}(\phi + \bar\phi)^2 - (\phi+\bar\phi)^2\right)$$

$$= -\frac{1}{2}\int d^4x\, d^2\theta\, d^2\bar\theta\, (\phi + \bar\phi)^2$$

$$= -\frac{1}{2}\int d^4x\, d^2\theta\, d^2\bar\theta \left(\phi^2 + \bar\phi^2 + 2\bar\phi\phi\right)$$

However,
$$\int d^4x\, d^2\theta\, d^2\bar\theta\, \phi^2 = \int d^4x\, d^2\theta \left(\frac{1}{4}\bar D^2 \phi^2\right) = 0$$
chirality

and similarly $$\int d^4x\, d^2\theta\, d^2\bar\theta\, \bar\phi^2 = 0$$

$$\Rightarrow \quad S' = -\int d^4x\, d^2\theta\, d^2\bar\theta\, \bar\phi\phi$$

which is the action for a chiral superfield (up to the global sign).