$$W_\alpha = -\frac{1}{4}\bar{D}^2\left(e^{-2V} D_\alpha e^{2V}\right)$$

Under a gauge transformation

$$e^{2V'} = e^{i\bar{\Lambda}} e^{2V} e^{-i\Lambda}, \quad e^{-2V'} = \left(e^{2V'}\right)^{-1} = e^{i\Lambda} e^{-2V} e^{-i\bar{\Lambda}}$$

$$\begin{aligned}
W'_\alpha &= -\frac{1}{4}\bar{D}^2\left[\left(e^{i\Lambda} e^{-2V} e^{-i\bar{\Lambda}}\right) D_\alpha \left(e^{i\bar{\Lambda}} e^{2V} e^{-i\Lambda}\right)\right] \qquad (\text{since } \bar{\Lambda} \text{ is antichiral so } D_\alpha\bar{\Lambda} = 0)\\
&= -\frac{1}{4}\bar{D}^2\left[e^{i\Lambda} e^{-2V} D_\alpha \left(e^{2V} e^{-i\Lambda}\right)\right]\\
&= -\frac{1}{4}\bar{D}^2\left[e^{i\Lambda} e^{-2V}\left(\left(D_\alpha e^{2V}\right) e^{-i\Lambda} + e^{2V}\left(D_\alpha e^{-i\Lambda}\right)\right)\right]\\
&= -\frac{1}{4}\bar{D}^2\left[e^{i\Lambda}\left(e^{-2V} D_\alpha e^{2V}\right) e^{-i\Lambda} + \underbrace{e^{i\Lambda} D_\alpha e^{-i\Lambda}}_{\text{chiral}}\right]\\
&= -\frac{1}{4} e^{i\Lambda}\bar{D}^2\left(e^{-2V} D_\alpha e^{2V}\right) e^{-i\Lambda}\\
&= e^{i\Lambda} W_\alpha e^{-i\Lambda}
\end{aligned}$$

The field W_α transforms covariantly under gauge transformations.

Let us compute the expansion of W_α

$$W_\alpha = -\frac{1}{4}\bar{D}^2\left(e^{-2V} D_\alpha e^{2V}\right)$$

in the Wess-Zumino gauge $V^3 = 0$ so

$$e^{2V} = 1 + 2V + 2V^2$$

$$\begin{aligned}
\Rightarrow \quad W_\alpha &= -\frac{1}{4}\bar{D}^2\left(\left(1 - 2V + 2V^2\right) D_\alpha \left(\cancel{1} + 2V + 2V^2\right)\right)\\
&= -\frac{1}{4}\bar{D}^2\left(\left(1 - 2V + 2V^2\right)\left(2D_\alpha V + 2(D_\alpha V)V + 2V(D_\alpha V)\right)\right)\\
&= -\frac{1}{4}\bar{D}^2\Big(2(D_\alpha V) - 4V(D_\alpha V) + 4V^2(D_\alpha V)\\
&\qquad + 2(D_\alpha V)V - 4V(D_\alpha V)V + 4V^2(D_\alpha V)V\\
&\qquad + 2V(D_\alpha V) - 4V^2(D_\alpha V)V + 4\cancel{V^3}(D_\alpha V)\Big)
\end{aligned}$$

- $V^2(D_\alpha V)V$ vanishes because each of the V's which appear without a derivative contains a $\theta\sigma\bar{\theta}$ term.
- $V(D_\alpha V)V$ and $V^2(D_\alpha V)$ vanish because $D_\alpha V$ contains at least a $\bar{\theta}$ and so do the other 2 V's.

$$\Rightarrow \quad W_\alpha = -\frac{1}{4}\bar{D}^2\left(2D_\alpha V + 2\left[D_\alpha V, V\right]\right)$$

The Abelian part was computed before

$$-\frac{1}{4}\bar{D}^2(D_\alpha V) = -i\lambda_\alpha + \theta_\alpha D + i(\sigma^{\mu\nu}\theta)_\alpha(\partial_\mu v_\nu - \partial_\nu v_\mu) - \theta^2(\sigma^\mu\partial_\mu\bar{\lambda})_\alpha$$

Let us now compute $[D_\alpha V, V]$

$D_\alpha V$ contains at least a $\bar\theta$ so we should drop all $\bar\theta^2$ from the expansion of V

$$V_{WZ} = (\theta\sigma^\mu\bar\theta)v_\mu + i\theta^2\bar\theta\bar\lambda - i\bar\theta^2\theta\lambda + \frac{1}{2}\theta^2\bar\theta^2 D$$

$$D_\alpha V = (\partial_\alpha + \bar\theta^{\dot\alpha}\sigma^\mu_{\alpha\dot\alpha}\partial_\mu)V$$

Notice that $\bar\theta\sigma^\mu_{\alpha\dot\alpha}\partial_\mu V$ contains a $\bar\theta^2$. Since V also contains at least a $\bar\theta$, then this term does not contribute

$$[D_\alpha V, V] = [\partial_\alpha V, V] = [(\sigma^\mu\bar\theta)_\alpha v_\mu + 2i\theta_\alpha\bar\theta\bar\lambda - i\bar\theta^2\lambda_\alpha + \theta_\alpha\bar\theta^2 D, (\theta\sigma^\nu\bar\theta)v_\nu + i\theta^2\bar\theta\bar\lambda]$$

$$= (\sigma^\mu\bar\theta)_\alpha(\theta\sigma^\nu\bar\theta)[v_\mu, v_\nu] + 2i\,\theta_\alpha\bar\theta_{\dot\beta}(\theta\sigma^\nu\bar\theta)[\bar\lambda^{\dot\beta}, v_\nu]$$
$$+ i(\sigma^\mu\bar\theta)_\alpha\theta^2\bar\theta_{\dot\beta}[v_\mu, \bar\lambda^{\dot\beta}]$$

Next, analyze each term in turn

$$(\sigma^\mu\bar\theta)_\alpha(\theta\sigma^\nu\bar\theta) = \sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\theta^\beta\sigma^\nu_{\beta\dot\beta}\bar\theta^{\dot\beta} = -\sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\beta}\left(-\frac{1}{2}\varepsilon^{\dot\alpha\dot\beta}\bar\theta^2\right)\theta^\beta$$

$$= -\frac{1}{2}\sigma^\mu_{\alpha\dot\alpha}\underbrace{\sigma^\nu_{\beta\dot\beta}\varepsilon^{\dot\alpha\dot\beta}\varepsilon^{\beta\gamma}}_{-\bar\sigma^{\nu,\dot\alpha\gamma}}\theta_\gamma\bar\theta^2$$

$$= \frac{1}{2}(\sigma^\mu\bar\sigma^\nu\theta)_\alpha\bar\theta^2$$

$$(\sigma^\mu\bar\theta)_\alpha(\theta\sigma^\nu\bar\theta)[v_\mu, v_\nu] = \frac{1}{2}(\sigma^\mu\bar\sigma^\nu\theta)_\alpha\bar\theta^2[v_\mu, v_\nu]$$

$$= \frac{1}{4}((\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu)\theta)_\alpha\bar\theta^2[v_\mu, v_\nu]$$

$$= (\sigma^{\mu\nu}\theta)_\alpha\bar\theta^2[v_\mu, v_\nu]$$

The last two terms are

$$2i\,\theta_\alpha\bar\theta_{\dot\beta}(\theta\sigma^\nu\bar\theta)[\bar\lambda^{\dot\beta}, v_\nu] + i(\sigma^\mu\bar\theta)_\alpha\theta^2\bar\theta_{\dot\beta}[v_\mu, \bar\lambda^{\dot\beta}]$$

$$= [v_\mu, \bar\lambda^{\dot\beta}]\left(-2i\,\theta_\alpha\bar\theta_{\dot\beta}(\theta\sigma^\mu\bar\theta) + i(\sigma^\mu\bar\theta)_\alpha\theta^2\bar\theta_{\dot\beta}\right)$$

Now,

$$-2i\,\theta_\alpha\bar\theta_{\dot\beta}(\theta\sigma^\mu\bar\theta) = -2i\,\theta_\alpha\bar\theta_{\dot\beta}\theta^\beta\sigma^\mu_{\beta\dot\alpha}\bar\theta^{\dot\alpha}$$

$$= 2i(\theta_\alpha\theta^\beta)(\bar\theta_{\dot\beta}\bar\theta^{\dot\alpha})\sigma^\mu_{\beta\dot\alpha} = 2i\left(-\frac{1}{2}\delta^\beta_\alpha\theta^2\right)\left(-\frac{1}{2}\delta^{\dot\alpha}_{\dot\beta}\bar\theta^2\right)\sigma^\mu_{\beta\dot\alpha}$$

$$= \frac{i}{2}\theta^2\bar\theta^2\sigma^\mu_{\alpha\dot\beta}$$

Also,

$$i(\sigma^\mu\bar\theta)_\alpha\theta^2\bar\theta_{\dot\beta} = i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\theta^2\bar\theta_{\dot\beta} = i\sigma^\mu_{\alpha\dot\alpha}\theta^2(\bar\theta^{\dot\alpha}\bar\theta_{\dot\beta})$$

$$= i\sigma^\mu_{\alpha\dot\alpha}\theta^2\left(\frac{1}{2}\delta^{\dot\alpha}_{\dot\beta}\bar\theta^2\right) = \frac{i}{2}\sigma^\mu_{\alpha\dot\beta}\theta^2\bar\theta^2$$

Adding up, we obtain

$$[D_\alpha V, V] = (\sigma^{\mu\nu}\theta)_\alpha\bar\theta^2[v_\mu, v_\nu] + i\sigma^\mu_{\alpha\dot\beta}\theta^2\bar\theta^2[v_\mu, \bar\lambda^{\dot\beta}]$$

Finally, let us compute $\bar{D}^2[D_\alpha V, V]$

$$\bar{D}_{\dot\alpha} = \partial_{\dot\alpha} + i\,\theta^\alpha \sigma^\mu_{\alpha\dot\alpha}\,\partial_\mu$$

$$\bar{D}_{\dot\alpha}\,[D_\alpha V, V] = \left(\partial_{\dot\alpha} + i\,\theta^\beta \sigma^\mu_{\beta\dot\alpha}\,\partial_\mu\right)\left((\sigma^{\nu\rho}\theta)_\alpha\,\bar\theta^2\,[v_\nu, v_\rho] + i\,\sigma^\nu_{\alpha\dot\rho}\,\theta^2\bar\theta^2\,[v_\nu, \bar\lambda^{\dot\rho}]\right)$$

$$= -(\sigma^{\nu\rho}\theta)_\alpha\,2\bar\theta_{\dot\alpha}\,[v_\nu, v_\rho] + i\,\sigma^\nu_{\alpha\dot\rho}\,\theta^2\,2\bar\theta_{\dot\alpha}\,[v_\nu, \bar\lambda^{\dot\rho}]$$
$$+ i(\theta\sigma^\mu)_{\dot\alpha}(\sigma^{\nu\rho}\theta)_\alpha\,\bar\theta^2\,\partial_\mu([v_\nu, v_\rho])$$

The last term can be simplified

$$(\theta\sigma^\mu)_{\dot\alpha}(\sigma^{\nu\rho}\theta)_\alpha = \theta^\beta\sigma^\mu_{\beta\dot\alpha}\,(\sigma^{\nu\rho})_\alpha{}^\gamma\,\theta_\gamma = \sigma^\mu_{\beta\dot\alpha}(\sigma^{\nu\rho})_\alpha{}^\gamma\left(\tfrac{1}{2}\delta^\beta_\gamma\,\theta^2\right)$$
$$= \tfrac{1}{2}(\sigma^{\nu\rho}\sigma^\mu)_{\alpha\dot\alpha}\,\theta^2$$

$$\Rightarrow \quad \bar{D}_{\dot\alpha}\,[D_\alpha V, V] = -(\sigma^{\nu\rho}\theta)_\alpha\,2\bar\theta_{\dot\alpha}\,[v_\nu, v_\rho] + i\,\sigma^\nu_{\alpha\dot\rho}\,\theta^2\,2\bar\theta_{\dot\alpha}\,[v_\nu, \bar\lambda^{\dot\rho}]$$
$$+ \tfrac{i}{2}(\sigma^{\nu\rho}\sigma^\mu)_{\alpha\dot\alpha}\,\theta^2\bar\theta^2\,\partial_\mu([v_\nu, v_\rho])$$

Next, compute

$$\bar{D}_{\dot\rho}\,\bar{D}_{\dot\alpha}\,[D_\alpha V, V] = \left(\partial_{\dot\rho} + i\,\theta^\gamma\sigma^{\mu'}_{\gamma\dot\rho}\,\partial_{\mu'}\right)\left(\bar{D}_{\dot\alpha}[D_\alpha V, V]\right)$$

$$= 2(\sigma^{\nu\rho}\theta)_\alpha\,\varepsilon_{\dot\alpha\dot\rho}\,[v_\nu, v_\rho] + i\,\sigma^\nu_{\alpha\dot\rho}\,\theta^2\,2\varepsilon_{\dot\alpha\dot\rho}\,[v_\nu, \bar\lambda^{\dot\rho}]$$
$$+ i(\sigma^{\nu\rho}\sigma^\mu)_{\alpha\dot\alpha}\,\theta^2\,\bar\theta_{\dot\rho}\,\partial_\mu([v_\nu, v_\rho])$$
$$+ i\,\theta^\gamma\sigma^{\mu'}_{\gamma\dot\rho}\,\partial_{\mu'}\left(-(\sigma^{\nu\rho}\theta)_\alpha\,2\bar\theta_{\dot\alpha}\,[v_\nu, v_\rho]\right)$$

The last term can be rewritten using

$$\theta^\gamma\sigma^{\mu'}_{\gamma\dot\rho}\,(\sigma^{\nu\rho}\theta)_\alpha = \theta^\gamma\sigma^{\mu'}_{\gamma\dot\rho}\,(\sigma^{\nu\rho})_\alpha{}^\beta\,\theta_\beta = \sigma^{\mu'}_{\gamma\dot\rho}(\sigma^{\nu\rho})_\alpha{}^\beta\,\tfrac{1}{2}\delta^\gamma_\beta\,\theta^2$$
$$= \tfrac{1}{2}(\sigma^{\nu\rho}\sigma^{\mu'})_{\alpha\dot\rho}\,\theta^2$$

Therefore, the last two terms in $\bar{D}_{\dot\rho}\,\bar{D}_{\dot\alpha}\,[D_\alpha V, V]$ can be combined

$$i(\sigma^{\nu\rho}\sigma^\mu)_{\alpha\dot\alpha}\,\theta^2\bar\theta_{\dot\rho}\,\partial_\mu([v_\nu, v_\rho]) - (\dot\alpha \leftrightarrow \dot\rho)$$

$$\bar{D}^2 = \bar{D}_{\dot\alpha}\bar{D}^{\dot\alpha} = -\varepsilon^{\dot\alpha\dot\rho}\,\bar{D}_{\dot\alpha}\bar{D}_{\dot\rho} = \varepsilon^{\dot\alpha\dot\rho}\,\bar{D}_{\dot\rho}\bar{D}_{\dot\alpha}$$

$$\Rightarrow \quad \bar{D}^2\,[D_\alpha V, V] = -4(\sigma^{\nu\rho}\theta)_\alpha\,[v_\nu, v_\rho] - 4i\,\theta^2\,[v_\nu, (\sigma^\nu\bar\lambda)_\alpha]$$

$$+ 2i(\sigma^{\nu\rho}\sigma^\mu\bar\theta)_\alpha\,\theta^2\,\partial_\mu[v_\nu, v_\rho]$$

Notice that so far all the fields are evaluated at x not at $y = x + i\theta\sigma\bar\theta$ as would be appropriate for a chiral superfield. For most of the terms we can perform the shif for free since they are multiplied by θ^2. For the first term we need to do this explicitly.

$$-4(\sigma^{\nu\rho}\theta)_\alpha\,[v_\nu(x), v_\rho(x)] = ?$$

$$v_\nu(x) = v_\nu(y - i\theta\sigma\bar\theta) = v_\nu(y) - i(\theta\sigma^\mu\bar\theta)\partial_\mu v_\nu + \dots$$

$$\Rightarrow \quad -4(\sigma^{\nu\rho}\theta)_\alpha\,[v_\nu(x), v_\rho(x)] = -4(\sigma^{\nu\rho}\theta)_\alpha\,[v_\nu(y), v_\rho(y)]$$

$$-4(\sigma^{\nu\rho}\theta)_\alpha\,(-i\theta\sigma^\mu\bar\theta)\,\partial_\mu\left([v_\nu(y), v_\rho(y)]\right)$$

The second term can be rewritten

$$+4i(\sigma^{\nu\rho})_\alpha{}^\beta\theta_\beta\,\theta^\gamma(\sigma^\mu\bar\theta)_\gamma = 4i(\sigma^{\nu\rho})_\alpha{}^\beta\left(-\tfrac{1}{2}\delta_\beta^\gamma\theta^2\right)(\sigma^\mu\bar\theta)_\gamma$$

$$= -2i\,(\sigma^{\nu\rho}\sigma^\mu\bar\theta)_\alpha\,\theta^2$$

We conclude that

$$-4(\sigma^{\nu\rho}\theta)_\alpha\,[v_\nu(x), v_\rho(x)] + 2i(\sigma^{\nu\rho}\sigma^\mu\bar\theta)_\alpha\,\theta^2\,\partial_\mu[v_\nu(x), v_\rho(x)]$$

$$= -4(\sigma^{\nu\rho}\theta)_\alpha\,[v_\nu(y), v_\rho(y)]$$

Let us add the Abelian and non-abelian pieces

$$W_\alpha = -\frac{1}{2}\bar D^2(D_\alpha V) - \frac{1}{2}\bar D^2\,[D_\alpha V, V]$$

$$= -2i\lambda_\alpha + 2\theta_\alpha D + 2i(\sigma^{\mu\nu}\theta)_\alpha(\partial_\mu v_\nu - \partial_\nu v_\mu) - \theta^2(\sigma^\mu\partial_\mu\bar\lambda)_\alpha$$

$$+ 2(\sigma^{\nu\rho}\theta)_\alpha[v_\nu, v_\rho] + 2i\,\theta^2\,[v_\nu, (\sigma^\nu\bar\lambda)_\alpha]$$

$$\boxed{\begin{aligned} W_\alpha = {} & -2i\lambda_\alpha + 2\theta_\alpha D + 2i(\sigma^{\mu\nu}\theta)_\alpha\left(\partial_\mu v_\nu - \partial_\nu v_\mu - i[v_\mu, v_\nu]\right) \\ & - \theta^2\,[\partial_\mu - iv_\mu, (\sigma^\mu\bar\lambda)_\alpha] \end{aligned}}$$

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$$W_\alpha = \cdots 2\theta_\alpha D + \cdots$$

$$W^\alpha W_\alpha \ni 4 D^2 \theta^2$$

$$\frac{1}{32\pi} \mathrm{Im}\left(\frac{4\pi i}{g^2} \int d^4x\, 4\, \mathrm{tr}(D^2) + \cdots \right) = \frac{1}{2g^2}\int d^4x\, \mathrm{tr}(D^2) + \cdots$$

The contribution from the Kähler potential is

$$\int d^4x\, d^2\theta\, d\bar\theta^2 \left(\bar\varphi\, \theta^2 \bar\theta^2 D \varphi \right) = \int d^4x \left(\bar\varphi D \varphi \right)$$

Decompose D over a basis of the algebra $D = D^a T^a$ where $\mathrm{tr}(T^a T^b) = \delta^{ab}$

$\Rightarrow$ The action involving D's is

$$\int d^4x \left(\frac{1}{2g^2} D^a D^a + (\bar\varphi T^a \varphi) D^a \right)$$

The equations of motion for the D's are

$$\frac{1}{g^2} D^a + \bar\varphi T^a \varphi = 0 \quad \Rightarrow \quad D^a = -g^2 (\bar\varphi T^a \varphi)$$

Plugging back in the action we obtain

$$\boxed{V(\varphi, \bar\varphi) = \frac{1}{2} g^2 (\bar\varphi T^a \varphi)(\bar\varphi T^a \varphi)}$$