$$V = C(x) + i\theta\chi - i\bar\theta\bar\chi + (\theta\sigma^\mu\bar\theta)v_\mu + \frac{i}{2}\theta^2(M+iN) - \frac{i}{2}\bar\theta^2(M-iN)$$
$$+ i\theta^2\bar\theta\left(\bar\lambda + \frac{i}{2}\bar\sigma^\mu\partial_\mu\chi\right) - i\bar\theta^2\theta\left(\lambda + \frac{i}{2}\sigma^\mu\partial_\mu\bar\chi\right) + \frac{1}{2}\theta^2\bar\theta^2\left(D - \frac{1}{2}\Box C\right)$$

1) $C^+ = C$

$$(i\theta\chi)^+ = -i\bar\chi\bar\theta = -i\bar\theta\bar\chi$$
$$((\theta\sigma^\mu\bar\theta)v_\mu)^+ = (\bar\theta^+\sigma^{\mu,+}\theta^+)v_\mu^+ = (\theta\sigma^\mu\bar\theta)v_\mu$$
$$\left(\frac{i}{2}\theta^2(M+iN)\right)^+ = -\frac{i}{2}\bar\theta^2(M^+ - iN^+) = -\frac{i}{2}\bar\theta^2(M-iN)$$
$$\left[i\theta^2\bar\theta\left(\bar\lambda + \frac{i}{2}\bar\sigma^\mu\partial_\mu\chi\right)\right]^+ = -i(\bar\lambda^+\theta)\bar\theta^2 - \frac{1}{2}(\bar\theta\bar\sigma^\mu\partial_\mu\chi)^+\bar\theta^2$$
$$= -i\bar\theta^2(\theta\lambda) - \frac{1}{2}\bar\theta^2\underbrace{(\partial_\mu\bar\chi\,\bar\sigma^\mu\theta)}_{-\theta\sigma^\mu\partial_\mu\bar\chi}$$
$$= -i\bar\theta^2\theta\left(\lambda + i/_2\,\sigma^\mu\partial_\mu\bar\chi\right)$$

$$\left(\frac{1}{2}\theta^2\bar\theta^2\left(D - \frac{1}{2}\Box C\right)\right)^+ = \frac{1}{2}\left(D^+ - \frac{1}{2}\Box C^+\right)\theta^2\bar\theta^2 = \frac{1}{2}\theta^2\bar\theta^2\left(D - \frac{1}{2}\Box C\right)$$

2) Bosonic $\underbrace{C}_{1}$ $\underbrace{v}_{4}$ $\underbrace{M}_{1}$ $\underbrace{N}_{1}$ $\underbrace{D}_{1}$ $\Rightarrow 8$

Fermionic $\underbrace{\chi + \bar\chi}_{4}$ $\underbrace{\lambda + \bar\lambda}_{4}$ $\Rightarrow 8$

3) $[C] = 0$, $[\chi] = \frac{1}{2}$, $[v] = 1$, $[M] = [N] = 1$

$[\lambda] = 3/_2$, $[D] = 2$

The only fields with physical dimensions are v, M, N, λ. It turns out that M, N can be gauged away (see below).

4)
$$\Lambda = \frac{1}{2}C + i\theta\chi + \frac{i}{2}\theta^2(M+iN)$$
$$\Lambda(x,\theta,\bar\theta) = \frac{1}{2}C(x + i\theta\sigma\bar\theta) + i\theta\chi(x + i\theta\sigma\bar\theta) + \frac{i}{2}\theta^2(M+iN)(x + i\theta\sigma\bar\theta)$$
$$C(x + i\theta\sigma\bar\theta) = C(x) + i(\theta\sigma^\mu\bar\theta)\partial_\mu C - \frac{1}{2}(\theta\sigma^\mu\bar\theta)(\theta\sigma^\nu\bar\theta)\partial_\mu\partial_\nu C$$
$$(\theta\sigma^\mu\bar\theta)(\theta\sigma^\nu\bar\theta) = \theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\theta^\beta\sigma^\nu_{\beta\dot\beta}\bar\theta^{\dot\beta}$$
$$= \left(-\frac{1}{2}\varepsilon^{\alpha\beta}\theta^2\right)\left(-\frac{1}{2}\varepsilon^{\dot\alpha\dot\beta}\bar\theta^2\right)\sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\beta}$$
$$= \frac{1}{4}\theta^2\bar\theta^2\sigma^\mu_{\alpha\dot\alpha}\bar\sigma^{\nu,\dot\alpha\alpha} = \frac{1}{2}\theta^2\bar\theta^2\eta^{\mu\nu}$$

$$\Rightarrow \quad C(x+i\theta\sigma\bar\theta) = C(x) + i(\theta\sigma^\mu\bar\theta)\partial_\mu C - \frac{1}{4}\theta^2\bar\theta^2\Box C$$

$$i\theta\chi(x+i\theta\sigma\bar\theta) = i\theta\chi(x) + i\theta\partial_\mu\chi(x)\, i(\theta\sigma^\mu\bar\theta)$$

$$= i\theta\chi(x) - \theta^\alpha\partial_\mu\chi_\alpha\,\theta^\beta(\sigma^\mu\bar\theta)_\beta$$

$$= i\theta\chi(x) - \frac{1}{2}\varepsilon^{\alpha\beta}\theta^2\,\partial_\mu\chi_\alpha(\sigma^\mu\bar\theta)_\beta$$

$$= i\theta\chi(x) + \frac{1}{2}\theta^2\,\partial_\mu\chi\,\sigma^\mu\bar\theta$$

$$= i\theta\chi(x) - \frac{1}{2}\theta^2\left(\bar\theta\bar\sigma^\mu\partial_\mu\chi\right)$$

$$\Rightarrow \quad \Lambda(x,\theta,\bar\theta) = \frac{1}{2}C(x) + \frac{i}{2}(\theta\sigma^\mu\bar\theta)\partial_\mu C(x) - \frac{1}{8}\theta^2\bar\theta^2\Box C$$

$$+ i\theta\chi(x) - \frac{1}{2}\theta^2(\bar\theta\bar\sigma^\mu\partial_\mu\chi)(x) + \frac{i}{2}\theta^2(M+iN)$$

Adding $\bar\Lambda$ we obtain

$$V(x,\theta,\bar\theta) = \underbrace{\Lambda(x,\theta,\bar\theta)}_{\Lambda(y,\theta)} + \underbrace{\bar\Lambda(x,\theta,\bar\theta)}_{\bar\Lambda(\bar y,\bar\theta)} + (\theta\sigma^\mu\bar\theta)v_\mu + i\theta^2\bar\theta\bar\lambda(x) - i\bar\theta^2\theta\lambda(x)$$

$$+ \frac{1}{2}\theta^2\bar\theta^2 D$$

5)

$$\delta V(x,\theta,\bar\theta) = i\phi(y) - i\phi(\bar y) = i\phi(x+i\theta\sigma\bar\theta) - i\phi(x-i\theta\sigma\bar\theta)$$

$$= i\left(\cancel{\varphi(x)} + i(\theta\sigma^\mu\bar\theta)\partial_\mu\varphi(x) - \frac{1}{2}(\theta\sigma^\mu\bar\theta)(\theta\sigma^\nu\bar\theta)\partial_\mu\partial_\nu\varphi(x)\right)$$

$$- i\left(\cancel{\varphi(x)} - i(\theta\sigma^\mu\bar\theta)\partial_\mu\varphi(x) - \frac{1}{2}(\theta\sigma^\mu\bar\theta)(\theta\sigma^\nu\bar\theta)\partial_\mu\partial_\nu\varphi(x)\right)$$

$$= -2(\theta\sigma^\mu\bar\theta)\partial_\mu\varphi$$

In terms of the components in the WZ gauge, only v_μ is transformed

$$\delta v_\mu = -2\partial_\mu\varphi$$

while the other components are invariant.
This gauge transformation therefore preserves the WZ gauge.

6) A vector field has 4 components. One of them can be set to zero by gauge transformations $\delta v_\mu = \partial_\mu\varphi$. Hence, there are 3 degrees of freedom off-shell.

D is a real field so it has one degree of freedom

λ has four real degrees of freedom (2 complex).

The number of bosonic and fermionic degrees of freedom matches

7) $W_\alpha = -\frac{1}{4}\bar{D}^2 D_\alpha V$, $\bar{W}_{\dot\alpha} = -\frac{1}{4}D^2\bar{D}_{\dot\alpha}V$

$$\delta W_\alpha = -\frac{1}{4}\bar{D}^2 D_\alpha \delta V = -\frac{1}{4}\bar{D}^2 D_\alpha(\phi + \cancel{\bar\phi})$$

$$= -\frac{1}{4}\bar{D}_{\dot\alpha}\bar{D}^{\dot\alpha}D_\alpha\phi = \frac{1}{4}\bar{D}^{\dot\alpha}\bar{D}_{\dot\alpha}D_\alpha\phi = \frac{1}{4}\bar{D}^{\dot\alpha}(2i\partial_{\alpha\dot\alpha} - D_\alpha\cancel{\bar{D}_{\dot\alpha}})\phi$$

$$= \frac{i}{2}\partial_{\alpha\dot\alpha}\bar{D}^{\dot\alpha}\phi = 0$$

Similarly for $\delta \bar{W}_{\dot\alpha}$.

$$V_{WZ} = (\theta\sigma^\mu\bar\theta)v_\mu + i\theta^2\bar\theta\bar\lambda - i\bar\theta^2\theta\lambda + \frac{1}{2}\theta^2\bar\theta^2 D$$

$$D_\alpha V_{WZ} = (\partial_\alpha + i\bar\theta^{\dot\alpha}\partial_{\alpha\dot\alpha})V_{WZ}$$

$$D^2\Gamma| = \partial^\alpha\partial_\alpha\Gamma| \quad , \quad \bar{D}^2\Gamma| = \partial_{\dot\alpha}\partial^{\dot\alpha}\Gamma|$$

The only term which contributes to $D^2 D_\alpha V_{WZ}|$ is $-i\bar\theta^2\theta\lambda$

$$\bar{D}^2 D_\alpha V_{WZ}| = \partial_{\dot\beta}\partial^{\dot\beta}(\partial_\alpha + \cancel{\bar\theta^{\dot\alpha}\partial_{\alpha\dot\alpha}})(-i\bar\theta^2\theta\lambda)\Big|$$

$$= -i(\partial_{\dot\beta}\partial^{\dot\beta}\bar\theta^2)\lambda_\alpha$$

$$\partial_{\dot\beta}\partial^{\dot\beta}\bar\theta^2 = -\varepsilon^{\dot\beta\dot\gamma}\partial_{\dot\beta}\partial_{\dot\gamma}\bar\theta^2 = -\varepsilon^{\dot\beta\dot\gamma}\partial_{\dot\beta}(2\bar\theta_{\dot\gamma}) = -\varepsilon^{\dot\beta\dot\gamma}2\varepsilon_{\dot\gamma\dot\beta} = -4$$

$$\Rightarrow \quad \bar{D}^2 D_\alpha V_{WZ}| = 4i\lambda_\alpha$$

$$\Rightarrow \quad \boxed{W_\alpha| = -i\lambda_\alpha}$$

8) $\bar{D}_{\dot\alpha}W_\alpha = \bar{D}_{\dot\alpha}\left(-\frac{1}{4}\bar{D}^2 D_\alpha V\right) = 0$ since $\bar{D}^3 = 0$ by antisymmetry

similarly, $D_\alpha\bar{W}_{\dot\alpha} = 0$

$D^\alpha W_\alpha = -\frac{1}{4}D^\alpha\bar{D}^2 D_\alpha V$ we have shown in a previous tutorial that

$D^\alpha\bar{D}^2 D_\alpha = \bar{D}_{\dot\alpha}D^2\bar{D}^{\dot\alpha}$ so

$$D^\alpha W_\alpha = -\frac{1}{4}D^\alpha\bar{D}^2 D_\alpha V = -\frac{1}{4}\bar{D}_{\dot\alpha}D^2\bar{D}^{\dot\alpha}V = \bar{D}_{\dot\alpha}\bar{W}^{\dot\alpha}$$

9)

$$W_\alpha(y,\theta) = -i\lambda_\alpha + \theta_\alpha D + i(\sigma^{\mu\nu}\theta)_\alpha f_{\mu\nu} + \theta^2(\sigma^\mu\partial_\mu\bar\lambda)_\alpha$$

$$D_\alpha = \partial_\alpha + 2i\bar\theta^{\dot\alpha}\sigma^\mu_{\alpha\dot\alpha}\frac{\partial}{\partial y^\mu}$$

$$W^\alpha = -i\lambda^\alpha + \theta^\alpha D + i(\varepsilon\sigma^{\mu\nu}\theta)^\alpha f_{\mu\nu} + \theta^2(\varepsilon\sigma^\mu\partial_\mu\bar\lambda)^\alpha$$

$D_\alpha W^\alpha = 0$

$\Rightarrow \quad \theta^0, \bar\theta^0 \quad \Rightarrow \quad \boxed{D = 0} \leftarrow$ since $tr(\sigma^{\mu\nu}) = 0$

$\theta^0, \bar\theta^1 \quad \Rightarrow \quad -2\bar\theta^{\dot\alpha}\sigma^\mu_{\alpha\dot\alpha}\partial\lambda^\alpha = 0 \quad \Rightarrow \quad -2\partial_\mu\lambda\sigma^\mu\bar\theta = 0 \quad \Rightarrow \quad \bar\theta\bar\sigma^\mu\partial_\mu\lambda = 0$

$\Rightarrow \boxed{\bar\sigma^\mu\partial_\mu\lambda = 0}$

$\theta^1, \bar\theta^1 \Rightarrow \quad -2\, \bar\theta^{\dot\alpha} \sigma^{\mu}_{\alpha\dot\alpha} (\varepsilon \sigma^{\rho\sigma}\theta)^{\alpha} \partial_\mu f_{\rho\sigma} = 0$

$$\Rightarrow \quad -2 \big(\bar\theta \underbrace{\sigma^{\mu,T}\varepsilon}_{\varepsilon\bar\sigma^\mu} \sigma^{\rho\sigma}\theta\big)\partial_\mu f_{\rho\sigma} = 0$$

$$\Rightarrow \quad (\bar\theta\, \bar\sigma^{\mu}\sigma^{\rho\sigma}\theta)\, \partial_\mu f_{\rho\sigma} = 0$$

$$\Rightarrow \quad (\bar\theta\, \bar\sigma^{\mu}\sigma^{\rho}\bar\sigma^{\sigma}\theta)\, \partial_\mu f_{\rho\sigma} = 0$$

Now use $\quad \bar\sigma^\mu \sigma^\rho \bar\sigma^\sigma = \eta^{\mu\rho}\bar\sigma^\sigma - \eta^{\mu\sigma}\bar\sigma^\rho + \eta^{\rho\sigma}\bar\sigma^\mu - i\varepsilon^{\mu\rho\sigma\tau}\bar\sigma_\tau$

$$\Rightarrow \quad (\bar\theta\,\bar\sigma^\sigma\theta)\partial^\mu f_{\mu\sigma} - i\,\varepsilon^{\mu\rho\sigma\tau}(\bar\theta\,\bar\sigma_\tau\theta)\,\partial_\mu f_{\rho\sigma} = 0$$

Bianchi identity

$$\Rightarrow \quad \boxed{\partial^\mu f_{\mu\sigma} = 0} \quad \text{equations of motion.}$$

(10)

$$S = \int d^4x\, d^2\theta\, W^\alpha W_\alpha + \int d^4x\, d^2\bar\theta\, \bar W_{\dot\alpha}\bar W^{\dot\alpha}$$

$$\begin{aligned}
\delta S &= \int d^4x\, d^2\theta\, W^\alpha \delta W_\alpha + \int d^4x\, d^2\bar\theta\, \bar W_{\dot\alpha}\delta \bar W^{\dot\alpha} \\
&= \int d^4x\, d^2\theta\, W^\alpha\left(-\tfrac{1}{4}\bar D^2 D_\alpha \delta V\right) + \int d^4x\, d^2\bar\theta\, \bar W_{\dot\alpha}\left(-\tfrac14 D^2 \bar D^{\dot\alpha}\delta V\right) \\
&= \int d^4x\, d^2\theta\, d^2\bar\theta\, \left(-W^\alpha D_\alpha \delta V - \bar W_{\dot\alpha}\bar D^{\dot\alpha}\delta V\right) \\
&= \int d^4x\, d^2\theta\, d^2\bar\theta\, \Big[\underbrace{D_\alpha(W^\alpha\delta V) + \bar D^{\dot\alpha}(\bar W_{\dot\alpha}\delta V)}_{\text{total derivatives}} - (D_\alpha W^\alpha + \bar D^{\dot\alpha}\bar W_{\dot\alpha})\delta V\Big] \\
&= -\int d^4x\, d^2\theta\, d^2\bar\theta\, (D_\alpha W^\alpha + \bar D^{\dot\alpha}\bar W_{\dot\alpha})\delta V
\end{aligned}$$

$\Rightarrow$ via variational principle in full superspace

$$\boxed{D_\alpha W^\alpha + \bar D^{\dot\alpha}\bar W_{\dot\alpha} = 0}$$

(11)

We have already shown that $D_\alpha W^\alpha$ contains at order $\theta\bar\theta$ a component whose real part is the equation of motion and whose imaginary part is the Bianchi identity.

$$\Rightarrow \begin{cases} \mathrm{Re}\,(D_\alpha W^\alpha) = 0 & \text{equations of motion} \\ \mathrm{Im}\,(D_\alpha W^\alpha) = 0 & \text{Bianchi identity} \end{cases}$$

The reality condition on W is $\quad (W_\alpha)^+ = -\bar W_{\dot\alpha}$

$$(D_\alpha W^\alpha)^+ = D_\alpha^+ (W^\alpha)^+ = \bar D_{\dot\alpha}(-\bar W^{\dot\alpha}) = \bar D^{\dot\alpha}\bar W_{\dot\alpha}$$

$$\mathrm{Im}\,(D_\alpha W^\alpha) = 0 \iff D_\alpha W^\alpha - \bar D^{\dot\alpha}\bar W_{\dot\alpha} = 0$$

$$\mathrm{Re}\,(D_\alpha W^\alpha) = 0 \iff D_\alpha W^\alpha + \bar D^{\dot\alpha}\bar W_{\dot\alpha} = 0$$