Introduction

The Grassmann expansion of a vector superfield is

$$\begin{aligned}V(x,\theta,\bar\theta) = C(x) + i\theta\chi(x) - i\bar\theta\bar\chi(x) + (\theta\sigma^\mu\bar\theta)v_\mu(x) + \\ + \frac{i}{2}\theta^2(M(x)+iN(x)) - \frac{i}{2}\bar\theta^2(M(x)-iN(x)) + \\ + i\theta^2\bar\theta\left(\bar\lambda(x) + \frac{i}{2}\bar\sigma^\mu\partial_\mu\chi(x)\right) - i\bar\theta^2\theta\left(\lambda(x)+\frac{i}{2}\sigma^\mu\partial_\mu\bar\chi(x)\right) + \\ + \frac{1}{2}\theta^2\bar\theta^2\left(D - \frac{1}{2}\Box C\right).\end{aligned} \tag{1}$$

The following formulas will be useful

$$\theta^\alpha\theta^\beta = -\frac{1}{2}\epsilon^{\alpha\beta}\theta^2, \qquad \bar\theta^{\dot\alpha}\bar\theta^{\dot\beta} = \frac{1}{2}\epsilon^{\dot\alpha\dot\beta}\bar\theta^2, \qquad \bar\sigma^{\mu,\dot\alpha\alpha} = \epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}\sigma^\mu_{\beta\dot\beta}, \qquad \chi\sigma^\mu\bar\psi = -\bar\psi\bar\sigma^\mu\chi. \tag{2}$$

Questions

1. Show that if C, D, M, N and v_μ are real and $\chi^\dagger = \bar\chi$, $\lambda^\dagger = \bar\lambda$, then $V = V^\dagger$.
2. Count the number of bosonic and fermionic degrees of freedom in V.
3. If we take the dimension of V to be zero, compute the dimensions of the component fields. Which component fields have dimensions corresponding to physical fields?
4. If Λ is a chiral field and $\bar\Lambda$ is an anti-chiral field defined by

   $$\Lambda = \frac{1}{2}C + i\theta\chi + \frac{i}{2}\theta^2(M+iN) \qquad \bar\Lambda = \frac{1}{2}C - i\bar\theta\bar\chi - \frac{i}{2}\bar\theta^2(M-iN), \tag{3}$$

   show that

   $$V(x,\theta,\bar\theta) = \Lambda(y,\theta) + \bar\Lambda(\bar y,\bar\theta) + (\theta\sigma^\mu\bar\theta)v_\mu(x) + i\theta^2\bar\theta\bar\lambda(x) - i\bar\theta^2\theta\lambda(x) + \frac{1}{2}\theta^2\bar\theta^2 D, \tag{4}$$

   where $y^\mu = x^\mu + i\theta\sigma^\mu\bar\theta$, $\bar y^\mu = x^\mu - i\theta\sigma^\mu\bar\theta$.
5. The gauge transformations of the vector multiplet superfield V are $\delta V = \Lambda + \bar\Lambda$ where Λ is chiral and $\bar\Lambda$ is anti-chiral. The previous question established the existence of a gauge where

   $$V_{WZ} = (\theta\sigma^\mu\bar\theta)v_\mu(x) + i\theta^2\bar\theta\bar\lambda(x) - i\bar\theta^2\theta\lambda(x) + \frac{1}{2}\theta^2\bar\theta^2 D. \tag{5}$$

   This is called the Wess-Zumino gauge[1]. However, we haven't performed the most general gauge transformation since the bottom component of Λ and of $\bar\Lambda$ is restricted to be real. In other words, the Wess-Zumino gauge is a *partial* gauge condition and it doesn't fix the gauge completely.

[1]Named after Julius Wess and Bruno Zumino, two of the pioneers of supersymmetry.

How do the component fields transform under a gauge transformation where $\Lambda(y,\theta) = i\phi(y)$, $\bar{\Lambda}(\bar{y},\bar{\theta}) = -i\phi(\bar{y})$, where $\phi(x)$ is real when x is real. Does such a transformation preserve the Wess-Zumino gauge?

6. How many degrees of freedom are there in an off-shell vector field? How many bosonic and fermionic degrees of freedom are there in the component fields in the Wess-Zumino gauge? Do you expect to be able to close the supersymmetry transformations on the fields v_μ, λ and D?

7. We define $W_\alpha = -\frac{1}{4}\bar{D}^2 D_\alpha V$ and $\bar{W}_{\dot{\alpha}} = -\frac{1}{4}D^2\bar{D}_{\dot{\alpha}}V$. Show that W_α and $\bar{W}_{\dot{\alpha}}$ are gauge-invariant. Since they are gauge invariant we can compute them in the Wess-Zumino gauge. What is the bottom component of W_α and $\bar{W}_{\dot{\alpha}}$?

8. Using the definition, show that

$$\bar{D}_{\dot{\alpha}}W_\alpha = 0, \qquad D_\alpha\bar{W}_{\dot{\alpha}} = 0, \qquad D^\alpha W_\alpha = \bar{D}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}. \tag{6}$$

9. The Grassmann expansion of W_α is

$$W_\alpha(y,\theta) = -i\lambda_\alpha + \theta_\alpha D + i(\sigma^{\mu\nu}\theta)_\alpha f_{\mu\nu} + \theta^2(\sigma^\mu\partial_\mu\bar{\lambda})_\alpha. \tag{7}$$

Show that $D^\alpha W_\alpha = 0$ implies the equations of motion for the gauge field and the fermion λ.

10. Starting from the action

$$S = \int d^4x d^2\theta W^\alpha W_\alpha + \int d^4x d^2\bar{\theta}\bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}, \tag{8}$$

find the superspace equations of motion by taking the variation with respect to V.

11. The field strength $f_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu$ of a gauge field v_μ satisfies a Bianchi identity $\epsilon^{\mu\nu\rho\sigma}\partial_\mu f_{\nu\rho} = 0$, where the indices μ, ν, ρ are completely anti-symmetrized. The equations of motion for the field strength are $\partial^\mu f_{\mu\nu} = 0$. Show that the supersymmetric analog of the Bianchi identity is $D^\alpha W_\alpha - \bar{D}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}} = 0$ while the analog of the equations of motion is $D^\alpha W_\alpha + \bar{D}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}} = 0$.