I.

$$S = \int d^4x\, d^2\theta\, d^2\bar\theta\; K(\bar\Phi, \phi) + \left(\int d^4x\, d^2\theta\, W(\phi) + c.c. \right)$$

$$= \int d^4x\, \frac{1}{16} D^2 \bar D^2 K(\bar\Phi, \phi) + \left(\int d^4x\, \frac{1}{4} D^2 W(\phi) + c.c. \right)$$

$$D^2 W = D^\alpha \left((D_\alpha \phi^i) W_i \right) = (D^2 \phi^i) W_i - (D_\alpha \phi^i)(D^\alpha \phi^j) W_{ij}$$

$$\bar D^2 K = \bar D_{\dot\alpha} \left((\bar D^{\dot\alpha} \bar\Phi^{\bar\imath}) K_{\bar\imath} \right) = (\bar D^2 \bar\Phi^{\bar\imath}) K_{\bar\imath} - (\bar D^{\dot\alpha} \bar\Phi^{\bar\imath})(\bar D_{\dot\alpha} \bar\Phi^{\bar\jmath}) K_{\bar\imath\bar\jmath}$$

$$D^2 \bar D^2 K = D^2 \left((\bar D^2 \bar\Phi^{\bar\imath}) K_{\bar\imath} - (\bar D^{\dot\alpha} \bar\Phi^{\bar\imath})(\bar D_{\dot\alpha} \bar\Phi^{\bar\jmath}) K_{\bar\imath\bar\jmath} \right)$$

$$= D^\alpha D_\alpha (\ldots)$$

$$\begin{aligned} = D^\alpha \Big(& (D_\alpha \bar D^2 \bar\Phi^{\bar\imath}) K_{\bar\imath} + (\bar D^2 \bar\Phi^{\bar\imath})(D_\alpha \phi^i) K_{i\bar\imath} \\ & - (D_\alpha \bar D^{\dot\alpha} \bar\Phi^{\bar\imath})(\bar D_{\dot\alpha} \bar\Phi^{\bar\jmath}) K_{\bar\imath\bar\jmath} \\ & + (\bar D^{\dot\alpha} \bar\Phi^{\bar\imath})(D_\alpha \bar D_{\dot\alpha} \bar\Phi^{\bar\jmath}) K_{\bar\imath\bar\jmath} - (\bar D^{\dot\alpha} \bar\Phi^{\bar\imath})(\bar D_{\dot\alpha} \bar\Phi^{\bar\jmath})(D_\alpha \phi^k) K_{\bar\imath\bar\jmath k} \Big) \end{aligned}$$

$$\begin{aligned} \Big(D_\alpha \bar D^2 \bar\Phi^{\bar\imath} &= D_\alpha \bar D_{\dot\alpha} \bar D^{\dot\alpha} \bar\Phi^{\bar\imath} = (2i\partial_{\alpha\dot\alpha} - \bar D_{\dot\alpha} D_\alpha) \bar D^{\dot\alpha} \bar\Phi^{\bar\imath} \\ &= 2i\partial_{\alpha\dot\alpha} \bar D^{\dot\alpha} \bar\Phi^{\bar\imath} + \bar D^{\dot\alpha} (2i\partial_{\alpha\dot\alpha} - \bar D_{\dot\alpha} \cancel{D_\alpha}) \bar\Phi^{\bar\imath} \\ &= 4i\partial_{\alpha\dot\alpha} \bar D^{\dot\alpha} \bar\Phi^{\bar\imath} \end{aligned}$$

$$D_\alpha \bar D_{\dot\alpha} \bar\Phi^{\bar\jmath} = 2i\partial_{\alpha\dot\alpha} \bar\Phi^{\bar\jmath}$$

$$\begin{aligned} \Rightarrow D^2 \bar D^2 K = D^\alpha \Big(& 4i\partial_{\alpha\dot\alpha} (\bar D^{\dot\alpha} \bar\Phi^{\bar\imath}) K_{\bar\imath} + (\bar D^2 \bar\Phi^{\bar\imath})(D_\alpha \phi^i) K_{i\bar\imath} \\ & + 2i(\partial_{\alpha\dot\alpha} \bar\Phi^{\bar\imath})(\bar D^{\dot\alpha} \bar\Phi^{\bar\jmath}) K_{\bar\imath\bar\jmath} + (\bar D^{\dot\alpha} \bar\Phi^{\bar\imath})(2i\partial_{\alpha\dot\alpha} \bar\Phi^{\bar\jmath}) K_{\bar\imath\bar\jmath} \\ & - (\bar D^{\dot\alpha} \bar\Phi^{\bar\imath})(\bar D_{\dot\alpha} \bar\Phi^{\bar\jmath})(D_\alpha \phi^k) K_{\bar\imath\bar\jmath k} \Big) \end{aligned}$$

$$\begin{aligned} = D^\alpha \Big(& 4i\partial_{\alpha\dot\alpha} (\bar D^{\dot\alpha} \bar\Phi^{\bar\imath}) K_{\bar\imath} + (\bar D^2 \bar\Phi^{\bar\imath})(D_\alpha \phi^i) K_{i\bar\imath} \\ & + 4i(\partial_{\alpha\dot\alpha} \bar\Phi^{\bar\imath})(\bar D^{\dot\alpha} \bar\Phi^{\bar\jmath}) K_{\bar\imath\bar\jmath} - (\bar D^{\dot\alpha} \bar\Phi^{\bar\imath})(\bar D_{\dot\alpha} \bar\Phi^{\bar\jmath})(D_\alpha \phi^k) K_{\bar\imath\bar\jmath k} \Big) \end{aligned}$$

$$\begin{aligned} = & -8(\partial_{\alpha\dot\alpha} \partial^{\dot\alpha\alpha} \bar\Phi^{\bar\imath}) K_{\bar\imath} - 4i\partial_{\alpha\dot\alpha} (\bar D^{\dot\alpha} \bar\Phi^{\bar\imath})(D^\alpha \phi^j) K_{\bar\imath j} \\ & + (D^\alpha \bar D^2 \bar\Phi^{\bar\imath})(D_\alpha \phi^i) K_{i\bar\imath} + (\bar D^2 \bar\Phi^{\bar\imath})(D^2 \phi^i) K_{i\bar\imath} \\ & - (\bar D^2 \bar\Phi^{\bar\imath})(D_\alpha \phi^i)(D^\alpha \phi^j) K_{ij\bar\imath} \\ & + 4i(\partial_{\alpha\dot\alpha} \bar\Phi^{\bar\imath})(D^\alpha \bar D^{\dot\alpha} \bar\Phi^{\bar\jmath}) K_{\bar\imath\bar\jmath} - 4i(\partial_{\alpha\dot\alpha} \bar\Phi^{\bar\imath})(\bar D^{\dot\alpha} \bar\Phi^{\bar\jmath})(D^\alpha \phi^k) K_{\bar\imath\bar\jmath k} \\ & - (D^\alpha \bar D^{\dot\alpha} \bar\Phi^{\bar\imath})(\bar D_{\dot\alpha} \bar\Phi^{\bar\jmath})(D_\alpha \phi^k) K_{\bar\imath\bar\jmath k} + (\bar D^{\dot\alpha} \bar\Phi^{\bar\imath})(D^\alpha \bar D_{\dot\alpha} \bar\Phi^{\bar\jmath})(D_\alpha \phi^k) K_{\bar\imath\bar\jmath k} \\ & - (\bar D^{\dot\alpha} \bar\Phi^{\bar\imath})(\bar D_{\dot\alpha} \bar\Phi^{\bar\jmath})(D^2 \phi^k) K_{\bar\imath\bar\jmath k} + (\bar D^{\dot\alpha} \bar\Phi^{\bar\imath})(\bar D_{\dot\alpha} \bar\Phi^{\bar\jmath})(D_\alpha \phi^k)(D^\alpha \phi^l) K_{\bar\imath\bar\jmath kl} \end{aligned}$$

$$\begin{aligned} \Big(D^\alpha \bar D^2 \bar\Phi^{\bar\imath} &= D^\alpha \bar D_{\dot\alpha} \bar D^{\dot\alpha} \bar\Phi^{\bar\imath} = -D^\alpha \bar D^{\dot\alpha} \bar D_{\dot\alpha} \bar\Phi^{\bar\imath} = -(2i\partial^{\dot\alpha\alpha} - \bar D^{\dot\alpha} D^\alpha) \bar D_{\dot\alpha} \bar\Phi^{\bar\imath} \\ &= -2i\partial^{\dot\alpha\alpha} \bar D_{\dot\alpha} \bar\Phi^{\bar\imath} - \bar D_{\dot\alpha} D^\alpha \bar D^{\dot\alpha} \bar\Phi^{\bar\imath} = -4i\partial^{\dot\alpha\alpha} \bar D_{\dot\alpha} \bar\Phi^{\bar\imath} \end{aligned}$$

$$D^\alpha \bar D^{\dot\alpha} \bar\Phi^{\bar\jmath} = 2i\partial^{\dot\alpha\alpha} \bar\Phi^{\bar\jmath}$$

Introduce the notations

$$\phi^i| = \varphi^i \,, \quad D_\alpha \phi^i| = \psi^i_\alpha \,, \quad D^2 \phi^i| = F^i$$

$$\bar\Phi^{\bar\imath}| = \bar\varphi^{\bar\imath} \,, \quad \bar D_{\dot\alpha} \bar\Phi^{\bar\imath}| = \bar\psi^{\bar\imath}_{\dot\alpha} \,, \quad \bar D^2 \bar\Phi^{\bar\imath}| = \bar F^{\bar\imath}$$

We will also keep the notation $K| = K$ for brevity.

$$\Rightarrow \quad D^2 \bar D^2 K| = \boxed{-8\left(\partial_{\alpha\dot\alpha}\partial^{\dot\alpha\alpha}\bar\varphi^{\bar\imath}\right)K_{\bar\imath}} - \underline{4i\left(\partial_{\alpha\dot\alpha}\bar\psi^{\dot\alpha\bar\imath}\right)\psi^{\alpha j}} K_{\bar\imath j}$$

$$- \underline{4i\left(\partial^{\dot\alpha\alpha}\bar\psi^{\bar\imath}_{\dot\alpha}\right)\psi^i_\alpha} K_{i\bar\imath} + \bar F^{\bar\imath} F^i K_{i\bar\imath} - \bar F^{\bar\imath}\psi^i_\alpha\psi^{\alpha j} K_{\bar\imath ij}$$

$$\boxed{-8\left(\partial_{\alpha\dot\alpha}\bar\varphi^{\bar\imath}\right)\left(\partial^{\dot\alpha\alpha}\bar\varphi^{\bar\jmath}\right)K_{\bar\imath\bar\jmath}} - \underline{\underline{4i\left(\partial_{\alpha\dot\alpha}\bar\varphi^{\bar\imath}\right)\bar\psi^{\dot\alpha\bar\jmath}\psi^{\alpha k}}} K_{\bar\imath\bar\jmath k}$$

$$- \underline{\underline{2i\left(\partial^{\dot\alpha\alpha}\bar\varphi^{\bar\imath}\right)\bar\psi^{\bar\jmath}_{\dot\alpha}\psi^k_\alpha K_{\bar\imath\bar\jmath k}}} - \underline{\underline{2i\,\bar\psi^{\bar\imath}_{\dot\alpha}\left(\partial^{\dot\alpha\alpha}\bar\varphi^{\bar\jmath}\right)\psi^k_\alpha K_{\bar\imath\bar\jmath k}}}$$

$$- \bar\psi^{\dot\alpha\bar\imath}\bar\psi^{\bar\jmath}_{\dot\alpha} F^k K_{\bar\imath\bar\jmath k} + \bar\psi^{\dot\alpha\bar\imath}\bar\psi^{\bar\jmath}_{\dot\alpha}\psi^k_\alpha\psi^{\alpha l} K_{\bar\imath\bar\jmath kl}$$

$\boxed{}$ terms $= -8\left(\partial_{\alpha\dot\alpha}\partial^{\dot\alpha\alpha}\bar\varphi^{\bar\imath}\right)K_{\bar\imath} - 8\left(\partial_{\alpha\dot\alpha}\bar\varphi^{\bar\imath}\right)\left(\partial^{\dot\alpha\alpha}\bar\varphi^{\bar\jmath}\right)K_{\bar\imath\bar\jmath}$

$$= -8\,\partial_{\alpha\dot\alpha}\left(\left(\partial^{\dot\alpha\alpha}\bar\varphi^{\bar\imath}\right)K_{\bar\imath}\right) + 8\left(\partial^{\dot\alpha\alpha}\bar\varphi^{\bar\imath}\right)\left(\partial_{\alpha\dot\alpha}\varphi^i\right)K_{i\bar\imath}$$

$\underline{}$ terms $= -8i\left(\partial^{\dot\alpha\alpha}\bar\psi^{\bar\imath}_{\dot\alpha}\right)\psi^i_\alpha K_{i\bar\imath}$

$\underline{\underline{}}$ terms $= -8i\left(\partial^{\dot\alpha\alpha}\bar\varphi^{\bar\imath}\right)\bar\psi^{\bar\jmath}_{\dot\alpha}\psi^k_\alpha K_{\bar\imath\bar\jmath k}$

Next, we need to integrate out the auxiliary fields F. They appear in the action in the form

$$\frac{1}{16}\left(\bar F^{\bar\imath} F^i K_{i\bar\imath} - \bar\psi^{\dot\alpha\bar\imath}\bar\psi^{\bar\jmath}_{\dot\alpha} F^k K_{\bar\imath\bar\jmath k} - \bar F^{\bar\imath}\psi^j_\alpha\psi^{\alpha k} K_{\bar\imath jk}\right)$$

$$+ \frac{1}{4} F^i W_i + \frac{1}{4}\bar F^{\bar\imath}\bar W_{\bar\imath}$$

The equations of motion for F^i, $\bar F^{\bar\imath}$ are

$$F^i K_{i\bar\imath} - \psi^j_\alpha\psi^{\alpha k} K_{\bar\imath jk} + 4\bar W_{\bar\imath} = 0$$

$$\Rightarrow \quad F^i_{sol} = \left(\psi^j_\alpha\psi^{\alpha k} K_{\bar\imath jk} - 4\bar W_{\bar\imath}\right)K^{\bar\imath i}$$

$$\bar F^{\bar\imath} K_{i\bar\imath} - \bar\psi^{\dot\alpha\bar\jmath}\bar\psi^{\bar k}_{\dot\alpha} K_{\bar\jmath\bar k i} + 4 W_i = 0$$

$$\Rightarrow \quad \bar F^{\bar\imath}_{sol} = \left(\bar\psi^{\dot\alpha\bar\jmath}\bar\psi^{\bar k}_{\dot\alpha} K_{\bar\jmath\bar k i} - 4 W_i\right)K^{i\bar\imath}$$

Replacing these solutions back in the action, yields

$$-\frac{1}{16}\bar F^{\bar\imath}_{sol} K_{\bar\imath i} F^i_{sol} = -\frac{1}{16}\left(\bar\psi^{\dot\alpha\bar\jmath}\bar\psi^{\bar k}_{\dot\alpha} K_{\bar\jmath\bar k i} - 4W_i\right)\left(\psi^j_\alpha\psi^{\alpha k} K_{\bar\imath jk} - 4\bar W_{\bar\imath}\right)K^{\bar\imath i}$$

Let us discuss some of the terms in the action

$$* \quad \frac{8}{16}\left(\partial^{\dot\alpha\alpha}\bar\varphi^{\bar\imath}\right)\left(\partial_{\alpha\dot\alpha}\varphi^{i}\right)K_{i\bar\imath} = \left(\partial_\mu\bar\varphi^{\bar\imath}\right)\left(\partial^\mu\varphi^{i}\right)K_{i\bar\imath}$$

kinetic term for the bosons

$$* \quad -\frac{8i}{16}\left[\left(\partial^{\dot\alpha\alpha}\bar\psi^{\bar\imath}_{\dot\alpha}\right)\psi^{i}_{\alpha}K_{i\bar\imath} + \left(\partial^{\dot\alpha\alpha}\bar\varphi^{\bar\imath}\right)\bar\psi^{\bar\jmath}_{\dot\alpha}\psi^{k}_{\alpha}K_{\bar\imath\bar\jmath k}\right]$$

$$= -\frac{i}{2}\left(\partial^{\dot\alpha\alpha}\bar\psi^{\bar\imath}_{\dot\alpha} + \left(\partial^{\dot\alpha\alpha}\bar\varphi^{\bar\jmath}\right)\bar\psi^{\bar k}_{\dot\alpha}\underbrace{\Gamma^{\bar\imath}_{\bar\jmath\bar k}}_{K_{\bar\jmath\bar k l}K^{l\bar\imath}}\right)\psi^{i}_{\alpha}K_{i\bar\imath}$$

$$= -\frac{i}{2}\underbrace{\left(\partial_\mu\bar\psi^{\bar\imath} + \Gamma^{\bar\imath}_{\bar\jmath\bar k}\partial_\mu\bar\varphi^{\bar\jmath}\bar\psi^{\bar k}\right)}_{(D_\mu\bar\psi)^{\bar\imath}}\bar\sigma^\mu\psi^{i}K_{i\bar\imath}$$

$$= -\frac{i}{2}\left(D_\mu\bar\psi\right)^{\bar\imath}\bar\sigma^\mu\psi^{i}K_{i\bar\imath}$$

* terms quadratic in the fermions

$$-\frac{1}{4}\psi^{i}_{\alpha}\psi^{\alpha j}W_{ij} - \frac{1}{4}\bar\psi^{\dot\alpha\bar\imath}\bar\psi^{\bar\jmath}_{\dot\alpha}\bar W_{\bar\imath\bar\jmath}$$

} from the superpotential

$$+\frac{1}{4}\psi^{j}_{\alpha}\psi^{\alpha k}W_{i}\Gamma^{i}_{jk} + \frac{1}{4}\bar\psi^{\dot\alpha\bar\jmath}\bar\psi^{\bar k}_{\dot\alpha}\bar W_{\bar\imath}\Gamma^{\bar\imath}_{\bar\jmath\bar k}$$

} from the F-terms

$$= -\frac{1}{4}\psi^{i}_{\alpha}\psi^{\alpha j}\left(W_{ij} - W_{k}\Gamma^{k}_{ij}\right) - \frac{1}{4}\bar\psi^{\dot\alpha\bar\imath}\bar\psi^{\bar\jmath}_{\dot\alpha}\left(\bar W_{\bar\imath\bar\jmath} - \bar W_{\bar k}\Gamma^{\bar k}_{\bar\imath\bar\jmath}\right)$$

* Potential for the scalars.

$$-W_{i}\bar W_{\bar\imath}K^{i\bar\imath}$$

* quartic terms for the fermions

$$-\frac{1}{16}\bar\psi^{\dot\alpha\bar\jmath}\bar\psi^{\bar k}_{\dot\alpha}\psi^{j}_{\alpha}\psi^{\alpha k}K_{\bar\jmath\bar k i}K_{\bar\imath jk}K^{i\bar\imath}$$

$$+\frac{1}{16}\bar\psi^{\dot\alpha\bar\imath}\bar\psi^{\bar\jmath}_{\dot\alpha}\psi^{k}_{\alpha}\psi^{\alpha l}K_{\bar\imath\bar\jmath kl}$$

$$= \frac{1}{16}\bar\psi^{\dot\alpha\bar\imath}\bar\psi^{\bar\jmath}_{\dot\alpha}\psi^{k}_{\alpha}\psi^{\alpha l}\left(\underbrace{K_{\bar\imath\bar\jmath kl} - K_{\bar\imath\bar\jmath i}K_{\bar k kl}K^{i\bar k}}_{R_{k\bar\imath l\bar\jmath}}\right)$$

$R_{k\bar\imath l\bar\jmath}$ the Riemann curvature of the manifold whose coordinates are the scalar fields the full Lagrangian is

$$\mathcal{L} = K_{i\bar{\imath}} (\partial_\mu \varphi^i)(\partial^\mu \bar{\varphi}^{\bar{\imath}}) - \frac{i}{2} (D_\mu \psi)^{\bar{\imath}} \bar{\sigma}^\mu \psi^i K_{i\bar{\imath}}$$

$$+ \frac{1}{4} \left(W_{ij} - \Gamma^k_{ij} W_k \right) \psi^{\alpha i} \psi^j_\alpha + \frac{1}{4} \left(\bar{W}_{\bar{\imath}\bar{\jmath}} - \Gamma^{\bar{k}}_{\bar{\imath}\bar{\jmath}} \bar{W}_{\bar{k}} \right) \bar{\psi}^{\bar{\imath}}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha} \bar{\jmath}}$$

$$- K^{i\bar{\imath}} W_i \bar{W}_{\bar{\imath}} + \frac{1}{16} R_{i\bar{\imath} j\bar{\jmath}} \psi^{\alpha i} \psi^j_\alpha \bar{\psi}^{\bar{\imath}}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha} \bar{\jmath}}$$

where

$K_{i\bar{\imath}} = \partial_i \partial_{\bar{\imath}} K$, $W_i = \partial_i W$, $\bar{W}_{\bar{\imath}} = \partial_{\bar{\imath}} \bar{W}$, $K^{i\bar{\imath}}$ is the inverse of $K_{i\bar{\imath}}$

$\Gamma_{ij}{}^k = K_{ij\bar{k}} K^{k\bar{k}}$

$R_{i\bar{\imath} j\bar{\jmath}} = \left(\partial_{\bar{\imath}} \Gamma^l_{ij} \right) K_{l\bar{\jmath}}$

Kähler manifolds* $\quad z^i, \bar{z}^{\bar{i}} \in \mathbb{C} \quad i = 1, \dots, n \quad$ coordinates

Metric $g_{i\bar{\jmath}}$ is hermitian, positive-definite and invertible.

Covariant derivative must respect the analytic structure

$$\nabla_i V_j = \partial_i V_j - \Gamma_{ij}^{k} V_k$$
$$\nabla_{\bar{\imath}} V_j = \partial_{\bar{\imath}} V_j - \Gamma_{\bar{\imath} j}^{k} V_k$$

Compatibility with the metric

$$\nabla_k g_{i\bar{\jmath}} = 0 \quad , \quad \nabla_{\bar{k}} g_{i\bar{\jmath}} = 0$$

No torsion

$$\Gamma_{ij}^{k} = \Gamma_{ji}^{k}$$

Also set $\Gamma_{\bar{\imath} j}^{k} = 0$

The only non-vanishing components of the Levi-Civita connection are Γ_{jk}^{i}, $\Gamma_{\bar{\jmath}\bar{k}}^{\bar{\imath}}$.

$$\Rightarrow \quad \nabla_k g_{i\bar{\jmath}} = \partial_k g_{i\bar{\jmath}} - \Gamma_{ki}^{l} g_{l\bar{\jmath}} = 0$$

since $\Gamma_{ik}^{l} = \Gamma_{ki}^{l}$

$$\Rightarrow \quad \partial_k g_{i\bar{\jmath}} = \partial_i g_{k\bar{\jmath}}$$

Similarly, $\partial_{\bar{\imath}} g_{j\bar{k}} = \partial_{\bar{k}} g_{j\bar{\imath}}$

$$\Rightarrow \quad g_{i\bar{\jmath}} = \frac{\partial}{\partial z^i} \frac{\partial}{\partial \bar{z}^{\bar{\jmath}}} \underbrace{K(z, \bar{z})}_{\text{locally defined}} \leftarrow \text{Kähler potential.}$$

Curvature

$$[\nabla_i, \nabla_j] V_k = R_{ijk}{}^{l} V_l \qquad (R_{ijk}{}^{\bar{l}} = 0)$$
$$[\nabla_i, \nabla_{\bar{\jmath}}] V_k = R_{i\bar{\jmath}k}{}^{l} V_l \qquad (R_{i\bar{\jmath}k}{}^{\bar{l}} = 0)$$

$$[\nabla_i, \nabla_j] V_k = \nabla_i (\nabla_j V_k) - (i \leftrightarrow j) = \partial_i (\nabla_j V_k) - \cancel{\Gamma_{ij}^{l} (\nabla_l V_k)} - \Gamma_{ik}^{l} (\nabla_j V_l) - (i \leftrightarrow j)$$
$$= \partial_i (\partial_j \cancel{V_k} - \Gamma_{jk}^{l} V_l) - \Gamma_{ik}^{l} (\partial_j V_l - \Gamma_{jl}^{m} V_m) - (i \leftrightarrow j)$$
$$= -(\partial_i \Gamma_{jk}^{l}) V_l - \Gamma_{jk}^{l} (\partial_i V_l) - \Gamma_{ik}^{l} (\partial_j V_l) + \Gamma_{ik}^{l} \Gamma_{jl}^{m} V_m - (i \leftrightarrow j)$$
$$= -(\partial_i \Gamma_{jk}^{l}) V_l + \Gamma_{ik}^{l} \Gamma_{jl}^{m} V_m - (i \leftrightarrow j)$$

$$\Gamma_{ki}^{l} = + (\partial_k g_{i\bar{\jmath}}) g^{\bar{\jmath} l}$$

$$\partial_i \Gamma_{jk}^{l} = \partial_i \left((\partial_j g_{k\bar{\imath}}) g^{\bar{\imath} l} \right) = (\partial_i \partial_j g_{k\bar{\imath}}) g^{\bar{\imath} l} + (\partial_j g_{k\bar{\imath}})(\partial_i g^{\bar{\imath} l})$$
$$= (\partial_i \partial_j g_{k\bar{\imath}}) g^{\bar{\imath} l} - (\partial_j g_{k\bar{\imath}}) g^{\bar{\imath} m} (\partial_i g_{m\bar{k}}) g^{\bar{k} l} = (\partial_i \partial_j \cdots) - \Gamma_{jk}^{m} \Gamma_{im}^{l}$$

$$\Rightarrow \quad [\nabla_i, \nabla_j] V_k = R_{ijk}{}^{\ell} V_\ell \underset{\forall \text{ vector } V}{=0} \quad \Rightarrow \quad R_{ijk}{}^{\ell} = 0$$

$$[\nabla_i, \nabla_{\bar{J}}] V_k = \nabla_i \nabla_{\bar{J}} V_k - \nabla_{\bar{J}} \nabla_i V_k$$

$$\nabla_{\bar{J}} V_k = \partial_{\bar{J}} V_k$$

$$\nabla_i (\nabla_{\bar{J}} V_k) = \partial_i (\nabla_{\bar{J}} V_k) - \Gamma^{\ell}_{ik} (\nabla_{\bar{J}} V_\ell) = \partial_i \partial_{\bar{J}} V_k - \Gamma^{\ell}_{ik} \partial_{\bar{J}} V_\ell$$

$$\nabla_i V_k = \partial_i V_k - \Gamma^{\ell}_{ik} V_\ell$$

$$\nabla_{\bar{J}} (\nabla_i V_k) = \partial_{\bar{J}} (\nabla_i V_k) = \partial_{\bar{J}} (\partial_i V_k - \Gamma^{\ell}_{ik} V_\ell) = \partial_i \partial_{\bar{J}} V_k - \partial_{\bar{J}} (\Gamma^{\ell}_{ik} V_\ell)$$

$$\Rightarrow \quad [\nabla_i, \nabla_{\bar{J}}] V_k = \cancel{\partial_i \partial_{\bar{J}} V_k} - \Gamma^{\ell}_{ik} \partial_{\bar{J}} V_\ell - \cancel{\partial_i \partial_{\bar{J}} V_k} + \partial_{\bar{J}} (\Gamma^{\ell}_{ik} V_\ell)$$

$$= (\partial_{\bar{J}} \Gamma^{\ell}_{ik}) V_\ell$$

$$= R_{i\bar{J}k}{}^{\ell} V_\ell$$

$$\Rightarrow \quad \boxed{R_{i\bar{J}k}{}^{\ell} = \partial_{\bar{J}} \Gamma^{\ell}_{ik}}$$

<u>Symmetry properties</u>

The Riemann tensor of a Riemannian manifold has some symmetries. What becomes of these symmetries in the case of a Kähler manifold?

$$R_{i\bar{J}k\bar{\ell}} = (\partial_{\bar{J}} \Gamma^{\ell}_{ik}) g_{\ell\bar{\ell}}$$ Symmetry in $i \leftrightarrow k$ is obvious.

The usual symmetries of the Riemann tensor imply

$$R_{\bar{J}ik\bar{\ell}} = - R_{i\bar{J}k\bar{\ell}}$$

$$R_{i\bar{J}\bar{\ell}k} = - R_{i\bar{J}k\bar{\ell}}$$

$$R_{\bar{J}i\bar{\ell}k} = R_{i\bar{J}k\bar{\ell}}$$

Since $\begin{cases} R_{i\bar{J}k\bar{\ell}} = R_{k\bar{J}i\bar{\ell}} \\ R_{\bar{J}i\bar{\ell}k} = R_{\bar{\ell}i\bar{J}k} \end{cases} \Rightarrow \quad R_{i\bar{\ell}k\bar{J}} = R_{i\bar{J}k\bar{\ell}}$

Another identity satisfied by a general Riemann tensor is the Bianchi identity

$$R_{\mu\nu\rho}{}^{\sigma} + \text{cyclic}(\mu\nu\rho) = 0$$

$$\Rightarrow \quad R_{i\bar{J}k}{}^{\ell} + \underbrace{R_{\bar{J}ki}{}^{\ell}}_{-R_{k\bar{J}i}{}^{\ell}} + \underbrace{R_{ki\bar{J}}{}^{\ell}}_{0} = 0$$

$$\Rightarrow \quad R_{i\bar{J}k\bar{\ell}} = R_{k\bar{J}i\bar{\ell}}$$ which is the identity we noticed above.