The purpose of this exercise is to study the most general $\mathcal{N} = 1$ supersymmetric effective action of scalars and fermions at low energy. This means that we do not impose the constraint that the action be renormalizable, but we restrict the number of derivatives to be at most two (higher-order derivatives are suppressed at low energies).

Consider the case of n chiral superfields Φ^i and their complex-conjugates $\bar{\Phi}^{\bar{\imath}}$. The most general superspace action for these fields is

$$S = \int d^4x d^2\theta d^2\bar{\theta} K(\bar{\Phi}, \Phi) + \int d^4x d^2\theta W(\Phi) + \int d^4x d^2\bar{\theta} \bar{W}(\bar{\Phi}). \tag{1}$$

Here W is a holomorphic function called superpotential and K is a real function called Kähler potential.

Using the fact that

$$S = \int d^4x \frac{1}{16} D^2 \bar{D}^2 \; K(\bar{\Phi}, \Phi)\big| + \int d^4x \frac{1}{4} \; D^2 W(\Phi)\big| + \int d^4x \frac{1}{4} \; \bar{D}^2 \bar{W}(\bar{\Phi})\big|, \tag{2}$$

and[1]

$$\Phi^i\big| = \phi^i, \qquad D_\alpha \Phi^i\big| = \psi^i_\alpha, \qquad D^2\Phi^i\big| = F^i, \tag{3}$$

$$\bar{\Phi}^{\bar{\imath}}\big| = \bar{\phi}^{\bar{\imath}}, \qquad \bar{D}_{\dot{\alpha}} \bar{\Phi}^{\bar{\imath}}\big| = \bar{\psi}^{\bar{\imath}}_{\dot{\alpha}}, \qquad \bar{D}^2\bar{\Phi}^{\bar{\imath}}\big| = \bar{F}^{\bar{\imath}}, \tag{4}$$

show that the Lagrangian density of this theory is

$$\begin{aligned} \mathcal{L} = K_{i\bar{\imath}}(\partial_\mu \phi^i)(\partial^\mu \bar{\phi}^{\bar{\imath}}) - \frac{i}{2} K_{i\bar{\imath}} (D_\mu \bar{\psi}^{\bar{\imath}}) \bar{\sigma}^\mu \psi^i + \frac{1}{4}(W_{ij} - \Gamma^k_{ij} W_k) \psi^{\alpha i} \psi^j_\alpha + \\ + \frac{1}{4}(\bar{W}_{\bar{\imath}\bar{\jmath}} - \Gamma^{\bar{k}}_{\bar{\imath}\bar{\jmath}} \bar{W}_{\bar{k}}) \bar{\psi}^{\bar{\imath}}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}\bar{\jmath}} - K^{i\bar{\imath}} W_i \bar{W}_{\bar{\imath}} + \frac{1}{16} R_{i\bar{\imath}j\bar{\jmath}} \psi^{\alpha i} \psi^j_\alpha \bar{\psi}^{\bar{\imath}}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}\bar{\jmath}}, \end{aligned} \tag{5}$$

where

$$K_{i\bar{\imath}} = \partial_i \partial_{\bar{\imath}} K, \qquad W_i = \partial_i W, \qquad \bar{W}_{\bar{\imath}} = \partial_{\bar{\imath}} \bar{W}, \qquad K_{i\bar{\jmath}} K^{\bar{\jmath}k} = \delta^k_i, \tag{6}$$

$$\Gamma^k_{ij} = K_{ij\bar{k}} K^{k\bar{k}}, \qquad \Gamma^{\bar{k}}_{\bar{\imath}\bar{\jmath}} = K_{\bar{\imath}\bar{\jmath}k} K^{k\bar{k}}, \qquad R_{i\bar{\imath}j\bar{\jmath}} = (\partial_{\bar{\imath}} \Gamma^l_{ij}) K_{l\bar{\jmath}}, \tag{7}$$

$$(D_\mu \bar{\psi})^{\bar{\imath}} = \partial_\mu \bar{\psi}^{\bar{\imath}} + \Gamma^{\bar{\imath}}_{\bar{\jmath}\bar{k}} (\partial_\mu \bar{\phi}^{\bar{\jmath}}) \bar{\psi}^{\bar{k}}. \tag{8}$$

You will need to eliminate the auxiliary fields F^i and $\bar{F}^{\bar{\imath}}$ using their equations of motion.

The terms in the space-time action have a nice geometric interpretation: ϕ^i and $\bar{\phi}^{\bar{\imath}}$ are complex coordinates on a "target space" manifold, $K_{i\bar{\imath}}$ is the metric, Γ and $\bar{\Gamma}$ are Levi-Civita connections and R is the Riemann curvature tensor.

[1] Usually the fermionic fields are defined with an extra factor of $\sqrt{2}$ i.e. $D\Phi| = \sqrt{2}\psi$ but here we used the more obvious normalization.