Wess - Zumino model

$$S = \int d^4x\, d^2\theta\, d^2\bar\theta\; \bar\Phi \phi + \left(\int d^4x\, d^2\theta \left(\frac{1}{2} m \phi^2 + \frac{1}{3} \lambda \phi^3 \right) + c.c. \right)$$

$$\phi(y,\theta) = \varphi(y) + \theta\psi(y) + \theta^2 F(y)$$

$$[Q_\alpha , Y(x,\theta,\bar\theta)] = -(-i\partial_\alpha - \sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu) Y(x,\theta,\bar\theta)$$

$$[\bar Q_{\dot\alpha} , Y(x,\theta,\bar\theta)] = -(i\partial_{\dot\alpha} + \theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu) Y(x,\theta,\bar\theta)$$

Action on a chiral superfield.

$$y^\nu = x^\nu + i\theta\sigma^\nu\bar\theta$$

$$\partial_\alpha = \partial_\alpha + \frac{\partial y^\nu}{\partial\theta^\alpha}\frac{\partial}{\partial y^\nu}$$

$$= \partial_\alpha + i(\sigma^\nu\bar\theta)_\alpha\, \partial/\partial y^\nu$$

$$\Rightarrow -i\partial_\alpha - (\sigma^\mu\bar\theta)_\alpha\partial_\mu = -i\partial_\alpha + \cancel{(\sigma^\mu\bar\theta)_\alpha}\, \partial/\partial y^\mu - \cancel{(\sigma^\mu\bar\theta)_\alpha}\, \partial/\partial y^\mu = -i\partial_\alpha$$

$$\Rightarrow \boxed{[Q_\alpha , \phi(y,\theta)] = +i\partial_\alpha\phi}$$

$$\partial_{\dot\alpha} = \partial_{\dot\alpha} + \frac{\partial y^\nu}{\partial\bar\theta^{\dot\alpha}}\frac{\partial}{\partial y^\nu} = \partial_{\dot\alpha} - i(\theta\sigma^\nu)_{\dot\alpha}\partial_\nu$$

$$\Rightarrow i\partial_{\dot\alpha} + (\theta\sigma^\nu)_{\dot\alpha}\partial_\nu = i\partial_{\dot\alpha} + 2(\theta\sigma^\mu)_{\dot\alpha}\partial_\mu$$

$$\Rightarrow \boxed{[\bar Q_{\dot\alpha} , \phi(y,\theta)] = -2(\theta\sigma^\mu)_{\dot\alpha}\partial_\mu\phi(y,\theta)}$$

$$\partial_\alpha\phi = \psi_\alpha + 2\theta_\alpha F$$

$$[Q_\alpha , \phi] = [Q_\alpha , \varphi] + [Q_\alpha , \theta\psi] + [Q_\alpha , \theta^2 F]$$

$$= i(\psi_\alpha + 2\theta_\alpha F)$$

$$\Rightarrow \boxed{[Q_\alpha , \varphi] = i\psi_\alpha}$$

$$[Q_\alpha , \theta\psi] = [Q_\alpha , \theta^\beta\psi_\beta] = -\theta^\beta\{Q_\alpha , \psi_\beta\} = 2i\theta_\alpha F$$

$$= +\varepsilon_{\alpha\beta}\theta^\beta\, 2iF$$

$$\Rightarrow \boxed{\{Q_\alpha , \psi_\beta\} = -2i\,\varepsilon_{\alpha\beta} F}$$

Finally $\boxed{[Q_\alpha , F] = 0}$

The commutators involving $\bar{Q}$ are

$$[\bar{Q}_{\dot{\alpha}}, \phi(y,\theta)] = [\bar{Q}_{\dot{\alpha}}, \varphi] + [\bar{Q}_{\dot{\alpha}}, \theta\psi] + [\bar{Q}_{\dot{\alpha}}, \theta^2 F]$$

$$= -2(\theta\sigma^\mu)_{\dot{\alpha}}\, \partial_\mu \Phi$$

$$\Rightarrow \quad \boxed{[\bar{Q}_{\dot{\alpha}}, \varphi] = 0}$$

$$[\bar{Q}_{\dot{\alpha}}, \theta\psi] = -2(\theta\sigma^\mu)_{\dot{\alpha}}\, \partial_\mu \varphi$$

$$\Rightarrow \quad \boxed{\{\bar{Q}_{\dot{\alpha}}, \psi_\alpha\} = +2\sigma^\mu_{\alpha\dot{\alpha}}\, \partial_\mu \varphi}$$

$$[\bar{Q}_{\dot{\alpha}}, \theta^2 F] = -2(\theta\sigma^\mu)_{\dot{\alpha}}\, \partial_\mu(\theta\psi) = -2\,\theta^\alpha \sigma^\mu_{\alpha\dot{\alpha}}\, \theta^\beta \partial_\mu \psi_\beta$$

$$= -2\left(-\tfrac{1}{2}\varepsilon^{\alpha\beta}\theta^2\right)\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\psi_\beta$$

$$= +(\partial_\mu\psi\sigma^\mu)_{\dot{\alpha}}\,\theta^2$$

$$\Rightarrow \quad \boxed{[\bar{Q}_{\dot{\alpha}}, F] = +(\partial_\mu\psi\sigma^\mu)_{\dot{\alpha}}}$$

Solution for F

The terms involving F in the action are

$$S = \int d^4x\left(\bar{F}F + (m\varphi + \lambda\varphi^2)F + (\bar{m}\bar{\varphi} + \bar{\lambda}\bar{\varphi}^2)\bar{F}\right)$$

the equations of motion for F are

$$\bar{F} = -(m\varphi + \lambda\varphi^2)$$

$$F = -(\bar{m}\bar{\varphi} + \bar{\lambda}\bar{\varphi}^2)$$

On-shell,

$$[Q_\alpha, \varphi] = +i\psi_\alpha \qquad [\bar{Q}_{\dot{\alpha}}, \varphi] = 0$$

$$\{Q_\alpha, \psi_\beta\} = 2i\,\varepsilon_{\alpha\beta}\,\overline{(m\varphi + \lambda\varphi^2)}$$

$$\{\bar{Q}_{\dot{\alpha}}, \psi_\alpha\} = 2\sigma^\mu_{\alpha\dot{\alpha}}\,\partial_\mu\varphi$$

Let us now study explicitly the supersymmetry algebra when realised on φ, ψ fields

$$\{\bar{Q}_{\dot\alpha}, [Q_\alpha, \varphi]\} = [\underbrace{\{Q_\alpha, \bar{Q}_{\dot\alpha}\}}_{2\sigma^\mu_{\alpha\dot\alpha} P_\mu}, \varphi] - \{Q_\alpha, \underbrace{[\bar{Q}_{\dot\alpha}, \varphi]}_{0}\}$$

$$= \{\bar{Q}_{\dot\alpha}, i\psi_\alpha\} = i\, 2\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\varphi$$

$$\Rightarrow \quad [P_\mu, \varphi] = i\partial_\mu\varphi \quad \checkmark$$

$$[\bar{Q}_{\dot\alpha}, \{Q_\alpha, \psi_\beta\}] = [\underbrace{\{Q_\alpha, \bar{Q}_{\dot\alpha}\}}_{2\sigma^\mu_{\alpha\dot\alpha} P_\mu}, \psi_\beta] - [Q_\alpha, \{\bar{Q}_{\dot\alpha}, \psi_\beta\}]$$

$$\text{LHS} = [\bar{Q}_{\dot\alpha}, 2i\,\varepsilon_{\alpha\beta}(m\varphi + \lambda\varphi^2)]$$

$$= 2i\,\varepsilon_{\alpha\beta}\underbrace{[Q_\alpha^+, (m\varphi + \lambda\varphi^2)^+]}$$

$$-[Q_\alpha, m\varphi + \lambda\varphi^2]^+ = -\left(m(+i\psi_\alpha) + 2\lambda\varphi(+i\psi_\alpha)\right)^+$$

$$= +i(\bar{m}\,\bar\psi_{\dot\alpha} + 2\bar\lambda\,\bar\varphi\,\bar\psi_{\dot\alpha})$$

$$\text{RHS} = 2i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\psi_\beta - [Q_\alpha, 2\sigma^\mu_{\beta\dot\alpha}\partial_\mu\varphi]$$

$$= 2i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\psi_\beta - 2\sigma^\mu_{\beta\dot\alpha}\partial_\mu(+i\psi_\alpha)$$

Contract LHS and RHS by $\varepsilon^{\alpha\beta}$

$$\Rightarrow \quad 4(\bar{m}\,\bar\psi_{\dot\alpha} + 2\bar\lambda\,\bar\varphi\,\bar\psi_{\dot\alpha}) = 4i\,\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\psi^\alpha$$

So we need to check if

$$\boxed{\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\psi^\alpha = -i(\bar{m}\,\bar\psi_{\dot\alpha} + 2\bar\lambda\,\bar\varphi\,\bar\psi_{\dot\alpha})}$$

The terms in the action involving $\psi, \bar\psi$

$$S = \int d^4x\, d^2\theta\, d^2\bar\theta\, (\bar\theta\bar\psi(\bar{y}))(\theta\psi)(y) + \left(\int d^4x\, d^2\theta\left(\tfrac{1}{2}m(\theta\psi)^2 + \lambda\varphi(\theta\psi)^2\right) + \text{c.c}\right)$$

$$\bar\theta\bar\psi(\bar{y})\,\theta\psi(y) = \left(\bar\theta\bar\psi(x) - i(\theta\sigma^\mu\bar\theta)\bar\theta\partial_\mu\bar\psi(x)\right)\times$$

$$\left(\theta\psi(x) + i(\theta\sigma^\mu\bar\theta)\theta\partial_\mu\psi(x)\right)$$

$$= \bar\theta\bar\psi(x)\,\theta\psi(x) + i(\bar\theta\bar\psi)(\theta\sigma^\mu\bar\theta)\theta\partial_\mu\psi$$

$$- i(\theta\sigma^\mu\bar\theta)(\bar\theta\partial_\mu\bar\psi)(\theta\psi)$$

The only terms which survive the full superspace integral are

$$i(\bar\theta\bar\Psi)(\theta\sigma^\mu\bar\theta)(\theta\partial_\mu\Psi) - i(\theta\sigma^\mu\bar\theta)(\bar\theta\partial_\mu\bar\Psi)(\theta\Psi)$$

$$= 2i(\bar\theta\bar\Psi)(\theta\sigma^\mu\bar\theta)(\theta\partial_\mu\Psi) - \partial_\mu\Big(\underbrace{i(\bar\theta\bar\Psi)(\theta\Psi)(\theta\sigma^\mu\bar\theta)}_{\text{drop the total derivative}}\Big)$$

$$= 2i\,\bar\theta_{\dot\alpha}\bar\Psi^{\dot\alpha}\,\theta^\beta\sigma^\mu_{\beta\dot\beta}\bar\theta^{\dot\beta}\,\theta^\alpha\partial_\mu\Psi_\alpha$$

$$= +2i\underbrace{(\bar\theta_{\dot\alpha}\bar\theta^{\dot\beta})}_{\frac{1}{2}\delta_{\dot\alpha}^{\dot\beta}\bar\theta^2}\underbrace{(\theta^\beta\theta^\alpha)}_{-\frac{1}{2}\varepsilon^{\beta\alpha}\theta^2}\bar\Psi^{\dot\alpha}\sigma^\mu_{\beta\dot\beta}\partial_\mu\Psi_\alpha$$

$$= -\frac{i}{2}\theta^2\bar\theta^2\,\bar\Psi^{\dot\beta}\sigma^\mu_{\beta\dot\beta}\partial_\mu\Psi^\beta$$

$$= \frac{i}{2}\theta^2\bar\theta^2(\partial_\mu\Psi)\sigma^\mu\bar\Psi = -\frac{i}{2}\theta^2\bar\theta^2\,\bar\Psi\bar\sigma^\mu(\partial_\mu\Psi)$$

The superpotential contributes

$$\frac{1}{2}m(\theta\Psi)^2 + \lambda\varphi(\theta\Psi)^2$$

$$(\theta\Psi)^2 = \theta^\alpha\Psi_\alpha\theta^\beta\Psi_\beta = -\theta^\alpha\theta^\beta\Psi_\alpha\Psi_\beta = \frac{1}{2}\varepsilon^{\alpha\beta}\theta^2\Psi_\alpha\Psi_\beta$$

$$= -\frac{1}{2}\theta^2\Psi^\beta\Psi_\beta = -\frac{1}{2}\theta^2\Psi^2$$

Finally, the terms that contribute to the equation of motion for Ψ are

$$-\frac{i}{2}\bar\Psi\bar\sigma^\mu\partial_\mu\Psi + \left[\left(\frac{1}{2}m+\lambda\varphi\right)\left(-\frac{1}{2}\Psi^2\right) + c.c\right]$$

Variation wrt $\bar\Psi_{\dot\alpha}$

$$\Rightarrow\ -\frac{i}{2}(\bar\sigma^\mu\partial_\mu\Psi)^{\dot\alpha} + \overline{\left(\frac{1}{2}m+\lambda\varphi\right)}\left(-\frac{1}{\cancel{2}}\right)(\cancel{2}\bar\Psi^{\dot\alpha}) = 0$$

$$\Rightarrow\ \boxed{(\bar\sigma^\mu\partial_\mu\Psi)^{\dot\alpha} = i(\bar m + 2\bar\lambda\bar\varphi)\bar\Psi^{\dot\alpha}}$$

This is the same as the condition found before if we use $\bar\sigma_\mu^{\dot\alpha\alpha} = \varepsilon^{\dot\alpha\dot\beta}\varepsilon^{\alpha\beta}\sigma_{\mu,\beta\dot\beta}$