Introduction

The Wess-Zumino model has the superspace action

$$S = \int d^4x d^2\theta d^2\bar{\theta}\bar{\Phi}\Phi + \left(\int d^4x d^2\theta\Big(\frac{1}{2}m\Phi^2 + \frac{1}{3}\lambda\Phi^3\Big) + \text{c.c.}\right). \tag{1}$$

The chiral superfield $\Phi(y,\theta)$ has the Grassmann expansion

$$\Phi(y,\theta) = \phi(x) + \theta\psi(y) + \theta^2 F(y), \tag{2}$$

where $y^\mu = x^\mu + i(\theta\sigma^\mu\bar{\theta})$. The anti-chiral superfield $\bar{\Phi}(\bar{y},\bar{\theta})$ depends on the complex conjugate variable $\bar{y}^\mu = x^\mu - i(\theta\sigma^\mu\bar{\theta})$.

In the rest of the exercise we will find the action of supersymmetry on the component fields of Φ. Recall that we have shown in class that

$$[\mathrm{Q}_\alpha, Y(x,\theta,\bar{\theta})] = -(-i\partial_\alpha - (\sigma^\mu\bar{\theta})_\alpha\partial_\mu)Y(x,\theta,\bar{\theta}), \tag{3}$$
$$[\bar{\mathrm{Q}}_{\dot{\alpha}}, Y(x,\theta,\bar{\theta})] = -(i\partial_{\dot{\alpha}} + (\theta\sigma^\mu)_{\dot{\alpha}}\partial_\mu)Y(x,\theta,\bar{\theta}). \tag{4}$$

Questions

1. Find the action of the differential operators in the right hand side of the commutation relations above, on chiral supefields $\Phi(y,\theta)$.
2. Find the action of Q and $\bar{\mathrm{Q}}$ on the component fields, including the auxiliary field F.
3. Write the F-dependent part of the action and compute the value of F when its equations of motion are satisfied.
4. Replace the on-shell value of F in the commutation relations of Q and $\bar{\mathrm{Q}}$ with ϕ and ψ. Notice that the right hand side of these commutation relations contains terms *non-linear* in the fields.
5. Check if the supersymmetry algebra is satisfied when acting on the fields ϕ and ψ. In other words, check that

$$\{\bar{\mathrm{Q}}_{\dot{\alpha}}, [\mathrm{Q}_\alpha, \phi]\} = [\{\mathrm{Q}_\alpha, \bar{\mathrm{Q}}_{\dot{\alpha}}\}, \phi] - \{\mathrm{Q}_\alpha, [\bar{\mathrm{Q}}_{\dot{\alpha}}, \phi]\}, \tag{5}$$
$$[\bar{\mathrm{Q}}_{\dot{\alpha}}, \{\mathrm{Q}_\alpha, \psi_\beta\}] = [\{\mathrm{Q}_\alpha, \bar{\mathrm{Q}}_{\dot{\alpha}}\}, \psi_\beta] - [\mathrm{Q}_\alpha, \{\bar{\mathrm{Q}}_{\dot{\alpha}}, \psi_\beta\}], \tag{6}$$

   given that $\{\mathrm{Q}_\alpha, \bar{\mathrm{Q}}_{\dot{\alpha}}\} = 2\sigma^\mu_{\alpha\dot{\alpha}}\mathrm{P}_\mu$ and $[\mathrm{P}_\mu, Y] = i\partial_\mu Y$ for any field Y.
6. In the superspace action keep only the terms which contain at least one ψ field. Integrate the odd parts to find the space-time action involving fermions.
7. Find the equation of motion for the fermions and compare with the relation obtained from the super-Jacobi identity of Q, $\bar{\mathrm{Q}}$ and ψ. Does supersymmetry close on-shell?