① $\int d^2\theta \equiv \left(\frac{\partial}{\partial\theta}\right)^2$ $[\theta] = \frac{1}{2}[x] = -\frac{1}{2}$ in mass units

$$\Rightarrow \left[\frac{\partial}{\partial\theta}\right] = \frac{1}{2} \Rightarrow \left[\frac{\partial^2}{\partial\theta}\right] = 1$$

$$\left[\int dx\right] = -1 \quad , \quad \left[\int d^4x\right] = -4$$

$$\Rightarrow \boxed{\left[\int d^4x\, d^2\theta\right] = -3} \quad , \quad \boxed{\left[\int d^4x\, d^2\theta\, d^2\bar{\theta}\right] = -2}$$

φ is a scalar field so $\boxed{[\varphi] = 1}$

ψ is a fermion field so $\boxed{[\psi] = 3/2}$

$$1 = [\Phi] = [\varphi] = \underbrace{[\theta\psi]}_{-\frac{1}{2}+\frac{3}{2}=1} = \underbrace{[\theta^2 F]}_{2\left(-\frac{1}{2}\right)+[F]} \Rightarrow \boxed{[F] = 2}$$

The action is dimensionless so

$$0 = [S] = \left[\int d^4x\, d^2\theta\, d^2\bar{\theta}\, K\right] = -2 + [K] \Rightarrow \boxed{[K] = 2}$$

$$= \left[\int d^4x\, d^2\theta\, W\right] = -3 + [W] \Rightarrow \boxed{[W] = 3}$$

②
$\varphi \Rightarrow$ 1 complex

$\psi \Rightarrow$ 2 complex

$F \Rightarrow$ 1 complex

2 complex bosonic , 2 complex fermionic

③ In a (power-counting) renormalizable theory, there are no negative (mass) dimension couplings. If we expand the superpotential W in a power series, we obtain

$$W = c_0 + c_1\phi + c_2\phi^2 + c_3\phi^3 + \dots$$

$$3 = [W] = [c_n\phi^n] = [c_n] + n[\phi] \Rightarrow [c_n] = 3-n$$

For $n > 3$ we obtain couplings with negative dimension. Hence, the most general power-counting renormalizable superpotential is

$$\boxed{W = c_0 + c_1\phi + c_2\phi^2 + c_3\phi^3}$$

④ A shift of W by a constant does not produce any observable effect since $\int d^2\theta\, c = 0$ for any constant c

⑤ The simplest real dimension 2 quantity depending on ϕ and $\bar{\Phi}$ is

$$\boxed{K = \bar{\Phi}\phi}$$

(6) A superfield G is chiral if $\bar{D}_{\dot\alpha} G = 0$. If G and H are chiral, then

$$\bar{D}_{\dot\alpha}(GH) = (\bar{D}_{\dot\alpha} G) H + (-)^{|G|} G (\bar{D}_{\dot\alpha} H) = 0$$

Hence GH is chiral.

W is chiral since

$$\bar{D}_{\dot\alpha} W = (\bar{D}_{\dot\alpha} \phi) W' = 0.$$

This can also be argued starting with the explicit expression of W above.

(7)

$$\int d^4x\, d^2\theta\, d^2\bar\theta\, \delta Y\, G \underset{\delta Y \text{ is chiral}}{=} \int d^4x\, d^2\theta\, \delta Y \left(\tfrac{1}{4} \bar{D}^2 G\right)$$

Therefore, if $\bar{D}^2 G = 0$ the integral is zero. We could take, for example,

$G = \bar{D}^{\dot\alpha} H_{\dot\alpha}$ or $G = \bar{D}^2 H$ and then the equation $\bar{D}^2 G = 0$ is automatically solved.

(8) $\int d^4x\, d^2\theta\, \delta Y G = \int d^4x\, \frac{1}{4} D^2 (\delta Y G)\big|$.

Assume δY and G are Grassmann-even

$$D^2(\delta Y G) = D^\alpha D_\alpha (\delta Y G) = D^\alpha\left[(D_\alpha \delta Y) G + \delta Y (D_\alpha G)\right]$$

$$= (D^2 \delta Y) G + 2 (D^\alpha \delta Y)(D_\alpha G) + \delta Y (D^2 G)$$

Since $D^2 \delta Y|$, $D^2 \delta Y|$ and $\delta Y|$ can be varied independently,

we obtain that

$$G| = 0, \qquad D_\alpha G| = 0, \qquad D^2 G| = 0$$

This means that all the terms in the Grassmann expansion vanish, so $\boxed{G = 0}$

(9)

$$S = \int d^4x\, d^2\theta\, d^2\bar\theta\, \bar\Phi \phi + \left(\int d^4x\, d^2\theta \left(c_1 \phi + c_2 \phi^2 + c_3 \phi^3\right) + \text{c.c.}\right)$$

$$\delta_\phi S = \int d^4x\, d^2\theta\, d^2\bar\theta\, \bar\Phi \delta\phi + \int d^4x\, d^2\theta \left(c_1 + 2c_2\phi + 3c_3\phi^2\right)\delta\phi$$

$$= \int d^4x\, d^2\theta \left(\underbrace{\tfrac{1}{4}\bar{D}^2\bar\Phi + c_1 + 2c_2\phi + 3c_3\phi^2}_{\text{chiral superfield}}\right)\delta\phi$$

$$\rightarrow \boxed{\tfrac{1}{4}\bar{D}^2\bar\Phi + c_1 + 2c_2\phi + 3c_3\phi^2 = 0}$$

(10) Clearly $\boxed{\varphi = \phi|}$

$$D_\alpha \phi| = \frac{\partial}{\partial\theta^\alpha}\phi| = (\psi_\alpha + 2\theta_\alpha F)| = \psi_\alpha$$

$$\boxed{D_\alpha \phi| = \psi_\alpha}$$

$$D^2\phi| = \partial^\alpha\partial_\alpha\phi| = 2F \quad \Rightarrow \quad \boxed{D^2\phi| = 2F}$$

(11)
$$D_\alpha\left(\frac{1}{4}\bar{D}^2\bar{\Phi} + c_1 + 2c_2\phi + 3c_3\phi^2\right) = 0$$

$$D_\alpha\bar{D}^2\bar{\Phi} = D_\alpha\bar{D}^{\dot\alpha}\bar{D}_{\dot\alpha}\bar{\Phi} = -D_\alpha\bar{D}_{\dot\alpha}\bar{D}^{\dot\alpha}\bar{\Phi} = -(2i\partial_{\alpha\dot\alpha} - \bar{D}_{\dot\alpha}D_\alpha)\bar{D}^{\dot\alpha}\bar{\Phi}$$
$$= -2i\partial_{\alpha\dot\alpha}\bar{D}^{\dot\alpha}\bar{\Phi} - \bar{D}^{\dot\alpha}D_\alpha\bar{D}_{\dot\alpha}\bar{\Phi}$$
$$= -4i\partial_{\alpha\dot\alpha}\bar{D}^{\dot\alpha}\bar{\Phi}$$

$$\frac{1}{4}D_\alpha\bar{D}^2\bar{\Phi}| = -i\partial_{\alpha\dot\alpha}\bar{D}^{\dot\alpha}\bar{\Phi}| = -i\partial_{\alpha\dot\alpha}\bar{\Psi}^{\dot\alpha}$$

$$D_\alpha\phi^2| = 2(D_\alpha\phi)\phi| = 2\psi_\alpha\varphi$$

$$\Rightarrow \quad \boxed{-i\partial_{\alpha\dot\alpha}\bar{\Psi}^{\dot\alpha} + 2c_2\psi_\alpha + 6c_3\psi_\alpha\varphi = 0}$$

Dirac kinetic term

mass term

=> Yukawa interaction term

$\psi\psi\varphi$ term in the action

We can also act with D^2 on the superfield equations of motion

$$\Rightarrow \quad D^2\left(\frac{1}{4}\bar{D}^2\bar{\Phi} + c_1 + 2c_2\phi + 3c_3\phi^2\right) = 0$$

Use the identity $D^2\bar{D}^2 + \bar{D}^2D^2 - 2D^\alpha\bar{D}^2D_\alpha = -16\Box$ from a previous exercise =>

$$D^2\bar{D}^2\bar{\Phi} = \underbrace{(-\bar{D}^2D^2 + 2D^\alpha\bar{D}^2D_\alpha)\bar{\Phi}}_{0 \text{ by chirality}} - 16\Box\bar{\Phi}$$

$$\Rightarrow \quad -4\Box\bar{\Phi} + 2c_2D^2\phi + 3c_3(D^2\phi^2) = 0$$

$$D^2\phi^2 = D^\alpha D_\alpha(\phi^2) = D^\alpha(2(D_\alpha\phi)\phi) = 2(D^2\phi)\phi - 2(D_\alpha\phi)(D^\alpha\phi)$$

Conjugation of the superfield equation of motion is

$$\frac{1}{4}D^2\phi + c_1^* + 2c_2^*\bar{\Phi} + 3c_3^*\bar{\Phi}^2 = 0$$

$$\Rightarrow \quad D^2\phi = -4(c_1^* + 2c_2^*\bar{\Phi} + 3c_3^*\bar{\Phi}^2)$$

Plugging back, we obtain

$$-4\,\Box\bar{\Phi} + 2c_2(-4)\left(c_1^* + 2c_2^*\bar{\Phi} + 3c_3^*\bar{\Phi}^2\right)$$
$$+ 3c_3 \cdot 2(-4)\left(c_1^* + 2c_2^*\bar{\Phi} + 3c_3^*\bar{\Phi}^2\right)\Phi$$
$$- 6c_3(D_\alpha\Phi)(D^\alpha\Phi) = 0$$

Projecting to the bottom component

$$\Rightarrow \quad \Box\bar{\varphi} + 2c_2\left(c_1^* + 2c_2^*\bar{\varphi} + 3c_3^*\bar{\varphi}^2\right)$$
$$+ 6c_3\left(c_1^* + 2c_2^*\bar{\varphi} + 3c_3^*\bar{\varphi}^2\right)\varphi$$
$$+ \frac{3}{2}c_3\psi_\alpha\psi^\alpha = 0$$

The fields φ, $\bar{\varphi}$ can be shifted so that the linear term c_1 in the superpotential vanishes. Setting $c_1 = 0$ from now on,

$$\Box\bar{\varphi} + 4|c_2|^2\bar{\varphi} + 6c_2c_3^*\bar{\varphi}^2 + 12c_3c_2^*\varphi\bar{\varphi}$$
$$+ 18|c_3|^2\bar{\varphi}^2\varphi + \frac{3}{2}c_3\psi_\alpha\psi^\alpha = 0$$

$$\Rightarrow \boxed{\Box\bar{\varphi} + \frac{\partial}{\partial\varphi}\underbrace{\left|2c_2\varphi + 3c_3\varphi^2\right|^2}_{\text{potential}} + \frac{3}{2}c_3\psi_\alpha\psi^\alpha = 0}$$

(12)

$$\int d^4x\, d^2\theta\, d^2\bar\theta\ \bar\Phi\,\Phi = \int d^4x \underbrace{\left(\frac{1}{4}\bar D^2\,\bar\Phi\right)}_{\frac{1}{2}\bar F}\underbrace{\left(\frac{1}{4}D^2\Phi\right)}_{\frac{1}{2}F} \quad + \text{kinetic terms for } \varphi, \psi$$

$$\int d^4x\, d^2\theta \left(c_2\Phi^2 + c_3\Phi^3\right) = \int d^4x\, \frac{1}{4} D^2\left(c_2\Phi^2 + c_3\Phi^3\right)$$

$$\left(\begin{aligned} D^2\left(c_2\Phi^2 + c_3\Phi^3\right) &= D^\alpha\left(2c_2(D_\alpha\Phi)\Phi + 3c_3(D_\alpha\Phi)\Phi^2\right)\\ &= 2c_2(D^2\Phi)\Phi - 2c_2(D_\alpha\Phi)(D^\alpha\Phi) + 3c_3(D^2\Phi)\Phi^2\\ &\quad - 3c_3(D_\alpha\Phi)\,2(D^\alpha\Phi)\Phi \end{aligned}\right)$$

$$= \int d^4x\, \frac{1}{4}\left(2c_2\cdot\underline{2F}\varphi - 2c_2\psi_\alpha\psi^\alpha + 3c_3\cdot\underline{2F}\varphi^2 - 6c_3\psi_\alpha\psi^\alpha\varphi\right)$$

$$= \int d^4x\left[F\left(c_2\varphi + \frac{3c_3}{2}\varphi^2\right) - \frac{c_2}{2}\psi_\alpha\psi^\alpha - \frac{3}{2}c_3\psi_\alpha\psi^\alpha\varphi\right]$$

Next, we need to integrate out $F, \bar F$, using their equations of motion. The equations of motion are

$$\frac{1}{4}\bar F + c_2\varphi + \frac{3c_3}{2}\varphi^2 = 0$$

$$\frac{1}{4}F + c_2^*\bar\varphi + \frac{3c_3^*}{2}\bar\varphi^2 = 0$$

Substituting back in the action we find

$$\frac{1}{4}\bar F F + F\left(-\frac{1}{4}\bar F\right) + \bar F\left(-\frac{1}{4}F\right) = -\frac{1}{4}\bar F F$$

$$= -\left|2c_2\varphi + 3c_3\varphi^2\right|^2$$

Let us finish the computation of the kinetic terms

$$\int d^4x\, d^2\theta\, d^2\bar\theta\ \bar\Phi\Phi = \int d^2x\, \frac{1}{4}D^2\frac{1}{4}\bar D^2\left(\bar\Phi\Phi\right)\Big|$$

$$= \frac{1}{16}\int d^4x\, D^2\left(\left(\bar D^2\bar\Phi\right)\Phi\right)\Big| = \frac{1}{16}\int d^4x\left[\left(D^2\bar D^2\bar\Phi\right)\Phi\Big| + 2\left(D^\alpha\bar D^2\bar\Phi\right)D_\alpha\Phi\Big|\right.$$

$$\left. + \underbrace{\left(\bar D^2\bar\Phi\right)\left(D^2\Phi\right)\Big|}_{F\text{-terms, already computed.}}\right]$$

But we have seen above that

$$D^2\bar D^2\bar\Phi = -16\,\Box\bar\Phi$$

$$D^\alpha\bar D^2\bar\Phi = -4i\,\partial^{\dot\alpha\alpha}\bar D_{\dot\alpha}\bar\Phi$$

$$\Rightarrow \text{kinetic terms} = \int d^4x\left(-(\Box\bar\varphi)\varphi - \frac{i}{2}\left(\partial^{\dot\alpha\alpha}\bar\psi_{\dot\alpha}\right)\psi_\alpha\right)$$

Putting it all together, we find

$$
\begin{aligned}
S = \int d^4x \Big(& (\partial_\mu \bar{\varphi})(\partial^\mu \varphi) + \frac{i}{2} \bar{\psi} \bar{\sigma}^\mu \partial_\mu \psi \\
& - \frac{c_2}{2} \psi_\alpha \psi^\alpha - \frac{3}{2} c_3 \psi_\alpha \psi^\alpha \varphi \\
& - \frac{c_2^*}{2} \bar{\psi}^{\dot\alpha} \bar{\psi}_{\dot\alpha} - \frac{3}{2} c_3^* \bar{\psi}^{\dot\alpha} \psi_{\dot\alpha} \bar{\varphi} \\
& - |2 c_2 \varphi + 3 c_3 \varphi^2|^2 \Big)
\end{aligned}
$$

(13) No, since it is an absolute value squared.