x4)

1. $\partial_\alpha \theta^\beta = \frac{\partial \theta^\beta}{\partial \theta^\alpha} = \delta_\alpha^\beta$, $\bar\partial_{\dot\alpha} \bar\theta^{\dot\beta} = \frac{\partial \bar\theta^{\dot\beta}}{\partial \bar\theta^{\dot\alpha}} = \delta_{\dot\alpha}^{\dot\beta}$

$$\partial^\alpha \theta_\beta = -\varepsilon^{\alpha\gamma}\partial_\gamma\left(\varepsilon_{\beta\delta}\theta^\delta\right) = -\varepsilon^{\alpha\gamma}\varepsilon_{\beta\delta}\,\delta_\gamma^\delta = \varepsilon^{\alpha\gamma}\varepsilon_{\gamma\beta} = \delta_\beta^\alpha$$

Similarly for $\bar\partial$.

2. $\int d\theta\, f(\theta)$ where $f(\theta) = f_0 + \theta f_1$

$$\Rightarrow \int d\theta\, f(\theta) = f_1 = \frac{\partial f(\theta)}{\partial \theta}$$

$$\int d\theta\, f(\theta+\zeta) = \int d\theta\left(f_0 + (\theta+\zeta)f_1\right) = \int d\theta\left(f_0 + \zeta f_1 + \theta f_1\right) = f_1$$

$\Rightarrow$ translation invariance

$$\int d\theta\, \frac{d}{d\theta}\left(f(\theta)\,g(\theta)\right) = \int d\theta\,\left(f'g + gf'\right)$$

$$= \frac{\partial^2}{\partial\theta^2}(f\,g) = 0 \quad \Rightarrow \quad \int d\theta\, f'g = -\int d\theta\, f g' \quad \text{(integration by parts formula)}$$

3. $$\int d^2\theta\, \theta^2 = \int \tfrac{1}{2}\, d\theta^1 d\theta^2\, \theta^\alpha\theta_\alpha = \int \tfrac{1}{2}\, d\theta^1 d\theta^2\, \theta^\alpha\varepsilon_{\alpha\beta}\theta^\beta =$$

$$= \int \tfrac{1}{2}\, d\theta^1 d\theta^2\; 2\theta^2\theta^1 = \frac{\partial}{\partial\theta^1}\frac{\partial}{\partial\theta^2}\,\theta^2\theta^1 = 1$$

$$\Rightarrow \int d^2\theta\, \theta^2 = 1$$

The $\bar\theta$ case works the same

4. $$\int d^2\theta = \tfrac{1}{2}\int d\theta^1 d\theta^2 = \tfrac{1}{2}\,\frac{\partial}{\partial\theta^1}\frac{\partial}{\partial\theta^2} = \tfrac{1}{4}\,\varepsilon^{\alpha\beta}\partial_\alpha\partial_\beta = \tfrac{1}{4}\,\partial^\alpha\partial_\alpha = \tfrac{1}{4}\,\partial^2$$

$\bar\theta$ case works similarly

5. Take $f = f_0 + \theta^\alpha f_{1,\alpha}$, $g = g_0 + \theta^\alpha g_{1,\alpha}$

$$f g = \left(f_0 + \theta^\alpha f_{1,\alpha}\right)\left(g_0 + \theta^\alpha g_{1,\alpha}\right) =$$

$$= f_0 g_0 + \theta^\alpha f_{1,\alpha} g_0 + f_0\theta^\alpha g_{1,\alpha} + \theta^\alpha f_{1,\alpha}\,\theta^\beta g_{1,\beta}$$

$$= f_0 g_0 + \theta^\alpha\left(f_{1,\alpha}g_0 + \underbrace{(-)^{|f_0|}}_{(-)^{|f|}} f_0 g_{1,\alpha}\right) + \underbrace{(-)^{|f_1|}}_{-(-)^{|f|}}\,\theta^\alpha\theta^\beta f_{1,\alpha}g_{1,\beta}$$

$$\partial_\alpha\left(\theta^\beta h_\beta\right) = h_\beta$$

$$\partial_\alpha\left(\theta^\beta\theta^\gamma h_{\beta\gamma}\right) = \delta_\alpha^\beta\theta^\gamma h_{\beta\gamma} - \delta_\alpha^\gamma\theta^\beta h_{\beta\gamma}$$

$$= 2\theta^\gamma h_{\alpha\gamma}$$

Take $h_{\beta\gamma} = -h_{\gamma\beta}$ since the symmetric part will not contribute anyway

$$\to \quad \partial_\alpha (fg) = f_{1,\alpha} g_0 + (-)^{|f|} f_0 g_{1,\alpha}$$

$$- (-)^{|f|} \cancel{2} \theta^\beta \tfrac{1}{\cancel{2}} (f_{1,\alpha} g_{1,\beta} - f_{1,\beta} g_{1,\alpha})$$

$$= f_{1,\alpha} (g_0 + \theta^\beta g_{1,\beta}) + (-)^{|f|} (f_0 + \theta^\beta f_{1,\beta}) g_{1,\alpha}$$

$$= (\partial_\alpha f) g + (-)^{|f|} f (\partial_\alpha g) \quad \checkmark$$

6. Straightforward computation.

7.

$$D_\alpha \phi(y, \theta, \bar\theta) = \frac{\partial y^\nu}{\partial \theta^\alpha} \frac{\partial \phi}{\partial y^\nu} + i \sigma^\mu_{\alpha\dot\alpha} \bar\theta^{\dot\alpha} \frac{\partial y^\nu}{\partial x^\mu} \frac{\partial \phi}{\partial y^\nu} + \frac{\partial \phi}{\partial \theta^\alpha}$$

$$= \left(\partial_\alpha + 2i \sigma^\mu_{\alpha\dot\alpha} \bar\theta^{\dot\alpha} \frac{\partial}{\partial y^\mu} \right) \phi(y, \theta, \bar\theta)$$

$$\bar D_{\dot\alpha} \phi(y, \theta, \bar\theta) = \frac{\partial y^\nu}{\partial \bar\theta^{\dot\alpha}} \frac{\partial \phi}{\partial y^\nu} + i \theta^\alpha \sigma^\mu_{\alpha\dot\alpha} \frac{\partial y^\nu}{\partial x^\mu} \frac{\partial \phi}{\partial y^\nu} + \frac{\partial \phi}{\partial \bar\theta^{\dot\alpha}}$$

$$= \bar\partial_{\dot\alpha} \phi$$

The chirality constraint $\bar D_{\dot\alpha} \phi = 0$ is solved by a superfield $\phi(y, \theta)$ (independent on $\bar\theta$ but with x shifted by a nilpotent quantity).

The same works for $D \leftrightarrow \bar D$ and $y \leftrightarrow \bar y$

8.

$$\{ D^\alpha, \bar D^{\dot\alpha} \} = (-\varepsilon^{\alpha\beta})(-\varepsilon^{\dot\alpha\dot\beta}) \{ D_\beta, \bar D_{\dot\beta} \}$$

$$= \varepsilon^{\alpha\beta} \varepsilon^{\dot\alpha\dot\beta} \, 2 \sigma^\mu_{\beta\dot\beta} P_\mu = \bar\sigma^{\mu, \dot\alpha\alpha} P_\mu$$

9.

$$\int d^4x \, d^2\theta \, Y = \frac{1}{4} \int d^4x \, \partial^\alpha \partial_\alpha Y$$

$$\partial_\alpha \cdot = D_\alpha \cdot - \partial_\mu (2i \sigma^\mu_{\alpha\dot\alpha} \bar\theta^{\dot\alpha} \cdot)$$

$$\partial^\alpha \cdot = D^\alpha \cdot + \partial_\mu \left(2i (\varepsilon \sigma^\mu)^\alpha{}_{\dot\alpha} \bar\theta^{\dot\alpha} \cdot \right)$$

$$\partial^\alpha \partial_\alpha = \left(D^\alpha + \partial_\mu \, 2i (\varepsilon \sigma^\mu \bar\theta)^\alpha \right) \left(D_\alpha - \partial_\nu \, 2i (\sigma^\nu \bar\theta)_\alpha \right)$$

$$= D^\alpha D_\alpha + \partial_\mu \left(2i (\varepsilon \sigma^\mu \bar\theta)^\alpha D_\alpha \right)$$

$$- \partial_\nu \left(D^\alpha 2i (\sigma^\nu \bar\theta)_\alpha \right) - \partial_\mu \left(2i (\varepsilon \sigma^\mu \bar\theta)^\alpha \partial_\nu (2i \sigma^\nu \bar\theta) \right)$$

$$= D^\alpha D_\alpha + \text{total derivative}$$

$$\Rightarrow \quad \int d^4x\, d^2\theta\, Y = \frac{1}{4}\int d^4x\, \partial^\alpha \partial_\alpha Y$$

$$= \frac{1}{4}\int d^4x \left(D^2 Y + \text{total derivative} \right)$$

$$= \frac{1}{4}\int d^4x\, D^2 Y$$ if we impose appropriate boundary conditions

10.

$$D_\alpha D_\beta D_\gamma = 0$$

$D_\alpha D_\beta D_\gamma$ is an antisymmetric rank 3 tensor but α, β, γ only take two values $\alpha, \beta, \gamma = 1, 2$ so at least two of them must coincide $\Rightarrow 0$

Same reasoning for $D \to \bar{D}$

$$D^2\bar{D}^2 + \bar{D}^2 D^2 - 2 D^\alpha \bar{D}^2 D_\alpha =$$

$$= D^\alpha D_\alpha \bar{D}_{\dot\alpha}\bar{D}^{\dot\alpha} + \bar{D}_{\dot\alpha}\bar{D}^{\dot\alpha} D^\alpha D_\alpha - 2 D^\alpha \bar{D}_{\dot\alpha}\bar{D}^{\dot\alpha} D_\alpha$$

$$= D^\alpha \left(\{D_\alpha, \bar{D}_{\dot\alpha}\} - \bar{D}_{\dot\alpha} D_\alpha \right)\bar{D}^{\dot\alpha} + \bar{D}_{\dot\alpha}\left(\{\bar{D}^{\dot\alpha}, D^\alpha\} - D^\alpha \bar{D}^{\dot\alpha} \right) D_\alpha$$
$$- 2 D^\alpha \bar{D}_{\dot\alpha}\bar{D}^{\dot\alpha} D_\alpha$$

$$= 2i\,\sigma^\mu_{\alpha\dot\alpha}\partial_\mu D^\alpha \bar{D}^{\dot\alpha} - D^\alpha \bar{D}_{\dot\alpha} D_\alpha \bar{D}^{\dot\alpha} + 2i\,\bar\sigma^{\mu,\dot\alpha\alpha}\partial_\mu \bar{D}_{\dot\alpha} D_\alpha$$
$$- \bar{D}_{\dot\alpha} D^\alpha \bar{D}^{\dot\alpha} D_\alpha - 2 D^\alpha \bar{D}_{\dot\alpha}\bar{D}^{\dot\alpha} D_\alpha$$

$$= 2i\,\sigma^\mu_{\alpha\dot\alpha}\partial_\mu D^\alpha \bar{D}^{\dot\alpha} + 2i\,\bar\sigma^{\mu,\dot\alpha\alpha}\partial_\mu \bar{D}_{\dot\alpha} D_\alpha$$
$$- D^\alpha \bar{D}_{\dot\alpha}\left(D_\alpha \bar{D}^{\dot\alpha} + \bar{D}^{\dot\alpha} D_\alpha \right) - \left(\bar{D}_{\dot\alpha} D^\alpha + D^\alpha \bar{D}_{\dot\alpha} \right)\bar{D}^{\dot\alpha} D_\alpha$$

$$= 2i\,\sigma^\mu_{\alpha\dot\alpha}\partial_\mu D^\alpha \bar{D}^{\dot\alpha} + 2i\,\bar\sigma^{\mu,\dot\alpha\alpha}\partial_\mu \bar{D}_{\dot\alpha} D_\alpha$$
$$+ D_\alpha \bar{D}_{\dot\alpha}\underbrace{\left(D^\alpha \bar{D}^{\dot\alpha} + \bar{D}^{\dot\alpha} D^\alpha \right)}_{2i\,\bar\sigma^{\mu,\dot\alpha\alpha}\partial_\mu} + \underbrace{\left(\bar{D}_{\dot\alpha} D_\alpha + D_\alpha \bar{D}_{\dot\alpha} \right)}_{2i\,\sigma^\mu_{\alpha\dot\alpha}\partial_\mu}\bar{D}^{\dot\alpha} D^\alpha$$

$$= 2i\,\sigma^\mu_{\alpha\dot\alpha}\partial_\mu \left(D^\alpha \bar{D}^{\dot\alpha} + \bar{D}^{\dot\alpha} D^\alpha \right) + 2i\,\bar\sigma^{\mu,\dot\alpha\alpha}\partial_\mu \left(D_\alpha \bar{D}_{\dot\alpha} + \bar{D}_{\dot\alpha} D_\alpha \right)$$

$$= 2i\,\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\, 2i\,\bar\sigma^{\nu,\dot\alpha\alpha}\partial_\nu + 2i\,\bar\sigma^{\mu,\dot\alpha\alpha}\partial_\mu\, 2i\,\sigma^\nu_{\alpha\dot\alpha}\partial_\nu$$

$$= -4\left(\sigma^\mu_{\alpha\dot\alpha}\bar\sigma^{\nu,\dot\alpha\alpha} + \sigma^\nu_{\alpha\dot\alpha}\bar\sigma^{\mu,\dot\alpha\alpha} \right)\partial_\mu \partial_\nu$$

$$= -4\, \mathrm{tr}\big(\underbrace{\sigma^\mu \bar\sigma^\nu + \sigma^\nu \bar\sigma^\mu}_{2\eta^{\mu\nu}\mathbb{1}_2} \big)\partial_\mu \partial_\nu$$

$$= -16\,\eta^{\mu\nu}\partial_\mu \partial_\nu = -16\,\Box \quad \checkmark$$