Questions

1. We define

$$\partial_\alpha = \frac{\partial}{\partial\theta^\alpha}, \qquad \bar{\partial}_{\dot\alpha} = \frac{\partial}{\partial\bar{\theta}^{\dot\alpha}}, \tag{1}$$

$$\partial^\alpha = -\epsilon^{\alpha\beta}\partial_\beta, \qquad \bar{\partial}^{\dot\alpha} = -\epsilon^{\dot\alpha\dot\beta}\bar{\partial}_{\dot\beta}. \tag{2}$$

Show that $\partial_\alpha\theta^\beta = \delta_\alpha^\beta$, $\bar{\partial}_{\dot\alpha}\bar{\theta}^{\dot\beta} = \delta_{\dot\alpha}^{\dot\beta}$. Attempt to explain the rationale for the definitions of ∂^α and $\bar{\partial}^{\dot\alpha}$.

2. The Grassmann integration is defined to be the same as the derivation. Show that translation invariance and integration by parts are satisfied by Grassmann integration.

3. We define

$$d^2\theta = \frac{1}{2}d\theta^1 d\theta^2, \qquad d^2\bar{\theta} = \frac{1}{2}d\bar{\theta}^{\dot 2}d\bar{\theta}^{\dot 1}. \tag{3}$$

Show that $\int d^2\theta\theta^2 = 1$, $\int d^2\bar{\theta}\bar{\theta}^2 = 1$ and $\int d^2\theta d^2\bar{\theta}\theta^2\bar{\theta}^2 = 1$.

4. Show that

$$\int d^2\theta = \frac{1}{4}\epsilon^{\alpha\beta}\partial_\alpha\partial_\beta = \frac{1}{4}\partial^\alpha\partial_\alpha = \frac{1}{4}\partial^2, \tag{4}$$

$$\int d^2\bar{\theta} = -\frac{1}{4}\epsilon^{\dot\alpha\dot\beta}\bar{\partial}_{\dot\alpha}\bar{\partial}_{\dot\beta} = \frac{1}{4}\bar{\partial}_{\dot\alpha}\bar{\partial}^{\dot\alpha} = \frac{1}{4}\bar{\partial}^2. \tag{5}$$

Recall that $\epsilon_{\alpha\beta} = \epsilon_{\dot\alpha\dot\beta} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\epsilon^{\alpha\beta} = \epsilon^{\dot\alpha\dot\beta} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.

5. Show the analog of the Leibniz rule

$$\partial_\alpha(fg) = (\partial_\alpha f)g + (-1)^{|f|}f(\partial_\alpha g), \tag{6}$$

where $|f|$ is the parity of the function f (zero for Grassmann even and one for Grassmann odd).

6. Recall the definitions

$$Q_\alpha = -i\partial_\alpha - \sigma^\mu_{\alpha\dot\alpha}\bar{\theta}^{\dot\alpha}\partial_\mu, \qquad \bar{Q}_{\dot\alpha} = i\bar{\partial}_{\dot\alpha} + \theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu, \tag{7}$$

$$D_\alpha = \partial_\alpha + i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu, \qquad \bar{D}_{\dot\alpha} = \bar{\partial}_{\dot\alpha} + i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu. \tag{8}$$

Show that

$$\{D_\alpha, \bar{D}_{\dot\alpha}\} = 2i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu, \qquad \{D_\alpha, D_\beta\} = 0, \qquad \{\bar{D}_{\dot\alpha}, \bar{D}_{\dot\beta}\} = 0, \tag{9}$$

$$\{D_\alpha, Q_\beta\} = 0, \qquad \{D_\alpha, \bar{Q}_{\dot\alpha}\} = 0, \tag{10}$$

$$\{\bar{D}_{\dot\alpha}, Q_\beta\} = 0, \qquad \{\bar{D}_{\dot\alpha}, \bar{Q}_{\dot\alpha}\} = 0. \tag{11}$$

7. Consider a superfield $\Phi(y, \theta, \bar{\theta})$ with $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$. Compute the action of the supersymmetry covariant derivatives D_α and $\bar{D}_{\dot{\alpha}}$ on it, expressing them in $(y, \theta, \bar{\theta})$ coordinates. Solve the chirality constraint $\bar{D}_{\dot{\alpha}}\Phi = 0$. Do the same for a superfield $\bar{\Phi}(\bar{y}, \theta, \bar{\theta})$ where $\bar{y}^\mu = x^\mu - i\theta\sigma^\mu\bar{\theta}$.

8. We define

$$D^\alpha = -\epsilon^{\alpha\beta} D_\alpha, \qquad \bar{D}^{\dot{\alpha}} = -\epsilon^{\dot{\alpha}\dot{\beta}} \bar{D}_{\dot{\beta}}, \tag{12}$$

$$D^2 = D^\alpha D_\alpha, \qquad \bar{D}^2 = \bar{D}_{\dot{\alpha}} \bar{D}^{\dot{\alpha}}, \tag{13}$$

just like for ∂^α and $\bar{\partial}^{\dot{\alpha}}$. Show that

$$\{D^\alpha, \bar{D}^{\dot{\alpha}}\} = 2i\bar{\sigma}^{\mu,\dot{\alpha}\alpha}\partial_\mu. \tag{14}$$

9. Show that, up to boundary terms from total derivatives,

$$\int d^4x d^2\theta Y(x, \theta) = \frac{1}{4} \int d^4x D^2 Y(x, \theta), \tag{15}$$

$$\int d^4x d^2\bar{\theta} Y(x, \bar{\theta}) = \frac{1}{4} \int d^4x \bar{D}^2 Y(x, \bar{\theta}). \tag{16}$$

10. Show that

$$D_\alpha D_\beta D_\gamma = 0, \qquad \bar{D}_{\dot{\alpha}} \bar{D}_{\dot{\beta}} \bar{D}_{\dot{\gamma}} = 0, \tag{17}$$

$$D^2\bar{D}^2 + \bar{D}^2 D^2 - 2D^\alpha \bar{D}^2 D_\alpha = -16\Box, \tag{18}$$

$$D^2\bar{D}^2 + \bar{D}^2 D^2 - 2\bar{D}_{\dot{\alpha}} D^2 \bar{D}^{\dot{\alpha}} = -16\Box. \tag{19}$$