(1) Take $M = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$, $M' = \left(\begin{array}{c|c} A' & B' \\ \hline C' & D' \end{array}\right)$

$$\mathrm{tr}(MM') = \mathrm{tr}(AA' + BC') + \mathrm{tr}(CB' + DD')$$
$$\mathrm{str}(MM') = \mathrm{tr}(AA' + BC') - \mathrm{tr}(CB' + DD')$$

Since A, D, A', D' are even, we have

$$\mathrm{tr}(AA') = \mathrm{tr}(A'A) , \quad \mathrm{tr}(DD') = \mathrm{tr}(D'D)$$

Since B, C, B', C' are odd, we have

$$\mathrm{tr}(BC') = -\mathrm{tr}(C'B) , \quad \mathrm{tr}(CB') = -\mathrm{tr}(B'C)$$

$$\Rightarrow \mathrm{str}(MM') = \mathrm{tr}(A'A) - \mathrm{tr}(C'B) + \mathrm{tr}(B'C) - \mathrm{tr}(D'D)$$
$$= \mathrm{tr}(A'A + B'C) - \mathrm{tr}(C'B + D'D)$$
$$= \mathrm{str}(M'M)$$

(it's the same expression as for $\mathrm{str}(MM')$ but with primes and non-primes reversed)

$$\mathrm{tr}(MM') = \mathrm{tr}(A'A) - \mathrm{tr}(C'B) - \mathrm{tr}(B'C) + \mathrm{tr}(D'D)$$
$$\mathrm{tr}(M'M) = \mathrm{tr}(A'A) + \mathrm{tr}(B'C) + \mathrm{tr}(C'B) + \mathrm{tr}(D'D) \neq \mathrm{tr}(MM')$$

(2)
$$MM' = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)\left(\begin{array}{c|c} A' & B' \\ \hline C' & D' \end{array}\right) = \left(\begin{array}{c|c} AA' + BC' & AB' + BD' \\ \hline CA' + DC' & CB' + DD' \end{array}\right)$$

$$(MM')^T = \left(\begin{array}{c|c} (AA' + BC')^T & (CA' + DC')^T \\ \hline (AB' + BD')^T & (CB' + DD')^T \end{array}\right)$$
$$= \left(\begin{array}{c|c} A'^T A^T - C'^T B^T & A'^T C^T + C'^T D^T \\ \hline B'^T A^T + D'^T B^T & -B'^T C^T + D'^T D^T \end{array}\right)$$

Notice that for odd matrices there is an extra sign when transposing, i.e. $(BC')^T = -C'^T B^T$

$$M'^T M^T = \left(\begin{array}{c|c} A'^T & C'^T \\ \hline B'^T & D'^T \end{array}\right)\left(\begin{array}{c|c} A^T & C^T \\ \hline B^T & D^T \end{array}\right) = \left(\begin{array}{c|c} A'^T A^T + C'^T B^T & A'^T C^T + C'^T D^T \\ \hline B'^T A^T + D'^T B^T & B'^T C^T + D'^T D^T \end{array}\right) \neq (MM')^T$$

Next, look at the supertransposition

$$(MM')^{ST} = \left(\begin{array}{c|c} (AA' + BC')^T & -(CA' + DC')^T \\ \hline (AB' + BD')^T & (CB' + DD')^T \end{array}\right) = \left(\begin{array}{c|c} A'^T A^T - C'^T B^T & -A'^T C^T - C'^T D^T \\ \hline B'^T A^T + D'^T B^T & -B'^T C^T + D'^T D^T \end{array}\right)$$

$$M'^{ST} M^{ST} = \left(\begin{array}{c|c} A'^T & -C'^T \\ \hline B'^T & D'^T \end{array}\right)\left(\begin{array}{c|c} A^T & -C^T \\ \hline B^T & D^T \end{array}\right) = \left(\begin{array}{c|c} A'^T A^T - C'^T B^T & -A'^T C^T - C'^T D^T \\ \hline B'^T A^T + D'^T B^T & -B'^T C^T + D'^T D^T \end{array}\right) = (MM')^{ST}$$

(3.) If $M = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$, $M^{ST} = \left(\begin{array}{c|c} A^T & -C^T \\ \hline B^T & D^T \end{array}\right)$

$(M^{ST})^{ST} = \left(\begin{array}{c|c} A & -B \\ \hline -C & D \end{array}\right) \Rightarrow (((M^{ST})^{ST})^{ST})^{ST} = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = M$

(4) If M is a diagonal matrix $M = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$

$\Rightarrow \det M = \lambda_1 \cdots \lambda_n$

$\ln M = \mathrm{diag}(\ln\lambda_1, \dots, \ln\lambda_n)$

$\mathrm{tr} \ln M = \ln\lambda_1 + \dots + \ln\lambda_n$

$\exp(\mathrm{tr} \ln M) = \lambda_1 \cdots \lambda_n = \det M$

If M is diagonalizable, $M = UDU^{-1}$ where D is diagonal

We set $f(M) = U f(D) U^{-1}$ in general so

$\ln M = U \ln D U^{-1}$

$\mathrm{tr} \ln M = \mathrm{tr}(U \ln D U^{-1}) = \mathrm{tr} \ln D = \ln\lambda_1 + \dots + \ln\lambda_n$

Similarly $\det M = \det(UDU^{-1}) = \lambda_1 \cdots \lambda_n = \exp(\mathrm{tr} \ln M)$

If M is close to identity, we have

$\det(1 + \varepsilon M) = \exp(\mathrm{tr} \ln(1 + \varepsilon M)) = \exp(\mathrm{tr}(\varepsilon M + \mathcal{O}(\varepsilon^2)))$

$= 1 + \varepsilon \,\mathrm{tr} M + \mathcal{O}(\varepsilon^2)$

(5) We define by analogy

$\mathrm{sdet}(1 + \varepsilon M) = 1 + \varepsilon \,\mathrm{str} M + \mathcal{O}(\varepsilon^2)$

$\mathrm{sdet}^p(1 + \varepsilon M) = (1 + \varepsilon \,\mathrm{str} M + \mathcal{O}(\varepsilon^2))^p = 1 + p\varepsilon \,\mathrm{str} M + \mathcal{O}(\varepsilon^2)$

$= \mathrm{sdet}(1 + p\varepsilon M) + \mathcal{O}(\varepsilon) = \mathrm{sdet}(1 + \varepsilon M)^p + \mathcal{O}(\varepsilon^2)$

(6) If we take $\varepsilon = \frac{1}{p}$ and $\lim_{p\to\infty}$ we find

$\lim_{p\to\infty} \mathrm{sdet}\left(1 + \frac{1}{p} M\right)^p = \lim_{p\to\infty}\left(1 + \frac{1}{p}\,\mathrm{str} M\right)^p$

$\Rightarrow \mathrm{sdet}\, e^M = e^{\mathrm{str} M}$

(Strictly speaking, we need to show that sdet is continuous before taking the limit this way.)

(7)

$$\mathrm{sdet}\,(e^M e^N) = \mathrm{sdet}\,(e^{M+N+\text{commutators}})$$

$$= e^{\mathrm{str}\,(M+N+\text{commutators})} = e^{\mathrm{str}\,M + \mathrm{str}\,N} \quad (\text{since } \mathrm{str}([\cdot,\cdot])=0)$$

$$= e^{\mathrm{str}\,M}\, e^{\mathrm{str}\,N} = \mathrm{sdet}\,e^M\ \mathrm{sdet}\,e^N$$

(8)

$$\int d^n x\, d^n \bar{x}\, d^m \theta\, d^m \bar{\theta}\, \exp\left(-\bar{x}Ax - \bar{\theta}Cx - \bar{x}B\theta - \bar{\theta}D\theta\right)$$

Make a change of variables

$$\bar{x} = \bar{y} - \bar{\theta}CA^{-1}, \qquad x = y - A^{-1}B\theta$$

$$\Rightarrow \quad \bar{x}Ax + \bar{\theta}Cx + \bar{x}B\theta + \bar{\theta}D\theta = (\bar{y} - \bar{\theta}CA^{-1})A(y - A^{-1}B\theta) +$$

$$+ \bar{\theta}C(y - A^{-1}B\theta) + (\bar{y} - \bar{\theta}CA^{-1})B\theta + \bar{\theta}D\theta$$

$$= \bar{y}Ay - \cancel{\bar{y}B\theta} - \cancel{\bar{\theta}Cy} + \bar{\theta}CA^{-1}B\theta + \cancel{\bar{\theta}Cy} - \bar{\theta}CA^{-1}B\theta$$

$$+ \cancel{\bar{y}B\theta} - \bar{\theta}CA^{-1}B\theta + \bar{\theta}D\theta$$

$$= \bar{y}Ay + \bar{\theta}(D - CA^{-1}B)\theta$$

$$\Rightarrow \int d^n x\, d^n \bar{x}\, d^m \theta\, d^m \bar{\theta}\, \exp\left(-\bar{x}Ax - \bar{\theta}Cx - \bar{x}B\theta - \bar{\theta}D\theta\right)$$

$$= \int d^n y\, d^n \bar{y}\, d^m \theta\, d^m \bar{\theta}\, \exp(-\bar{y}Ay)\, \exp\left(\bar{\theta}(D - CA^{-1}B)\theta\right)$$

$$= \frac{\pi^n}{\det A}\,(-)^m \det(D - CA^{-1}B)$$

$$\Rightarrow \quad \mathrm{sdet}\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \frac{\det A}{\det(D - CA^{-1}B)}$$

To show the other version, translate the θ's instead of x's.

(9) No. This is different from the case of matrices where the determinant of singular matrices can certainly be defined.

(10)

$$\mathrm{sdet}\, M^{st} = \mathrm{sdet}\left(\begin{array}{c|c} A^T & -C^T \\ \hline B^T & D^T \end{array}\right) = \frac{\det A^T}{\det\left(D^T - B^T D^{T,-1}(-C^T)\right)}$$

$$= \frac{\det A}{\det\left(D - CD^{-1}B\right)^T} = \mathrm{sdet}\, M$$