Definitions

A supermatrix is a block matrix

$$M = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right), \tag{1}$$

where A and D are square matrices with Grassmann even entries and B and C are matrices with Grassmann odd entries.

The purpose of this exercise is to extend to these matrices the usual notions of transposition, trace and determinant.

Questions

1. For a matrix M as above we define the trace and the supertrace

$$\mathrm{tr}(M) = \mathrm{tr}(A) + \mathrm{tr}(D), \qquad \mathrm{str}(M) = \mathrm{tr}(A) - \mathrm{tr}(D). \tag{2}$$

   Show that $\mathrm{tr}(MM') \neq \mathrm{tr}(M'M)$, but $\mathrm{str}(MM') = \mathrm{str}(M'M)$.

2. We define

$$M^{\mathsf{T}} = \left(\begin{array}{c|c} A^{\mathsf{T}} & C^{\mathsf{T}} \\ \hline B^{\mathsf{T}} & D^{\mathsf{T}} \end{array}\right), \qquad M^{\mathsf{ST}} = \left(\begin{array}{c|c} A^{\mathsf{T}} & -C^{\mathsf{T}} \\ \hline B^{\mathsf{T}} & D^{\mathsf{T}} \end{array}\right). \tag{3}$$

   Show that $(MM')^{\mathsf{T}} \neq M'^{,\mathsf{T}} M^{\mathsf{T}}$ and $(MM')^{\mathsf{ST}} = M'^{\mathsf{ST}} M^{\mathsf{ST}}$.

3. Show that $(((M^{\mathsf{ST}})^{\mathsf{ST}})^{\mathsf{ST}})^{\mathsf{ST}} = M$, while obviously $(M^{\mathsf{T}})^{\mathsf{T}} = M$.

4. Show that for a diagonal matrix M we have $\det M = \exp(\mathrm{tr}(\ln M))$. Show that this extends to a diagonalizable matrix. It can be shown to be true in general. Conclude that for a matrix close to identity we have $\det(1 + \epsilon M) = 1 + \epsilon\,\mathrm{tr}(M) + \mathcal{O}(\epsilon^2)$.

5. For a supermatrix M which is close to identity we define[1] $\mathrm{sdet}(1 + \epsilon M) = 1 + \epsilon\,\mathrm{str}(M) + \mathcal{O}(\epsilon^2)$. Use this to show that $\mathrm{sdet}^p(1 + \epsilon M) = \mathrm{sdet}(1 + \epsilon M)^p + \mathcal{O}(\epsilon^2)$.

6. In the previous formula, take $\epsilon = \frac{1}{p}$ and $p \to \infty$ and conclude that $\mathrm{sdet}\, e^M = \exp(\mathrm{str}(M))$.

7. The Baker-Campbell-Hausdorff formula states that $e^M e^N = \exp(M + N + \text{commutators})$. Use this to show that $\mathrm{sdet}(e^M e^N) = \mathrm{sdet}(e^M)\,\mathrm{sdet}(e^N)$. It follows that in general $\mathrm{sdet}(MN) = \mathrm{sdet}\, M\, \mathrm{sdet}\, N$.

[1]The superdeterminant is often called Berezinian in honor of Felix Alexandrovich Berezin, who found the formula for the superdeterminant in terms of the determinants of the component block matrices.

8. An alternative way to define the superdeterminant goes as follows. Consider

$$Z = \begin{pmatrix} x \\ \theta \end{pmatrix}, \qquad \bar{Z} = (\bar{x}, \bar{\theta}). \tag{4}$$

Then,

$$\bar{Z}MZ = \bar{x}Ax + \bar{\theta}Cx + \bar{x}B\theta + \bar{\theta}D\theta. \tag{5}$$

Assuming that the integrals converge, we have

$$\int d^n x d^n \bar{x} \exp(-\bar{x}Ax) = \frac{\pi^n}{\det A}, \qquad \int d^m \theta d^m \bar{\theta} \exp(-\bar{\theta}D\theta) = (-1)^m \det D. \tag{6}$$

Using these formulas, show that the superdeterminant can be computed from

$$\int d^n x d^n \bar{x} d^m \theta d^m \bar{\theta} \exp(-\bar{Z}MZ) = \frac{\pi^n (-1)^m}{\operatorname{sdet} M}, \tag{7}$$

with the result

$$\operatorname{sdet} M = \frac{\det A}{\det(D - CA^{-1}B)} = \frac{\det(A - BD^{-1}C)}{\det D}. \tag{8}$$

Hint: translate the coordinates x and $\bar{x}$ to write the quadratic form in the exponent as a sum of bosonic and a femionic part. To obtain the second formula, translate θ and $\bar{\theta}$.

9. What happens if either A or D is not invertible? Can the superdeterminant be defined?

10. Using the explicit formulas above, show that $\operatorname{sdet} M^{\mathsf{ST}} = \operatorname{sdet} M$.