(1)

$$[P_\mu, P^2] = 0$$

$$[M_{\mu\nu}, P^2] = 0 \quad \text{since } P^2 \text{ is a scalar}$$

$$[P_\mu, W_\nu] = \frac{1}{2}\varepsilon_{\nu\rho\sigma\tau}[P_\mu, M^{\rho\sigma}P^\tau] = \frac{1}{2}\varepsilon_{\nu\rho\sigma\tau}[P_\mu, M^{\rho\sigma}]P^\tau$$

$$= \frac{1}{2}\varepsilon_{\nu\rho\sigma\tau}\left(i\delta_\mu^\rho P^\sigma - i\delta_\mu^\sigma P^\rho\right)P^\tau = 0 \quad \text{by antisymmetry of } \varepsilon$$

(2)

$$[D, P^2] = 2iP^2$$

$$[K_\mu, P^2] = [K_\mu, P_\nu]P^\nu + P_\nu[K_\mu, P^\nu]$$

$$= -2i(\eta_{\mu\nu}D + M_{\mu\nu})P^\nu - 2iP_\nu(\delta_\mu^\nu D + M_\mu{}^\nu)$$

$\Rightarrow$ P^2 is not a Casimir for the conformal group.

Let us compute

$$K_\mu|p\rangle = K_\mu\frac{P^2}{m^2}|p\rangle = \frac{1}{m^2}\left([K_\mu, P^2] + P^2K_\mu\right)|p\rangle$$

$$= -\frac{2i}{m^2}\left((\eta_{\mu\nu}D + M_{\mu\nu})P^\nu + P_\nu(\delta_\mu^\nu D + M_\mu{}^\nu)\right)|p\rangle + \frac{1}{m^2}P^2K_\mu|p\rangle$$

assume for simplicity that the "particle" is of spin 0 so $M_{\mu\nu}|p\rangle = 0$

and also $D|p\rangle = d|p\rangle$

$$\Rightarrow K_\mu|p\rangle = -\frac{2i}{m^2}(2dP_\mu)|p\rangle + \frac{1}{m^2}P^2K_\mu|p\rangle$$

Therefore, $K_\mu|p\rangle$ is not an eigenstate of P^2 and it does not have a well-defined mass quantum number.

(3)

$$[J_{\mu\nu}, J_{\rho d}] = [M_{\mu\nu}, \tfrac{1}{2}(P_\rho - K_\rho)] = \frac{1}{2}i(\eta_{\nu\rho}(P_\mu - K_\mu) - (\mu \leftrightarrow \nu))$$

$$= i(\eta_{\nu\rho}J_{\mu d} - \eta_{\mu\rho}J_{\nu d})$$

Similarly,

$$[J_{\mu\nu}, J_{\rho,d+1}] = i(\eta_{\nu\rho}J_{\mu,d+1} - \eta_{\mu\rho}J_{\nu,d+1})$$

Finally,

$$[J_{\mu,d}, J_{\nu,d+1}] = \frac{1}{4}[P_\mu - K_\mu, P_\nu + K_\nu] = \frac{1}{4}[P_\mu, K_\nu] + \frac{1}{4}[P_\nu, K_\mu]$$

$$= \frac{1}{4}(-2i)(\eta_{\mu\nu}D + M_{\mu\nu}) + \frac{1}{4}(-2i)(\eta_{\mu\nu}D + M_{\nu\mu})$$

$$= -i\eta_{\mu\nu}D = -i\eta_{\mu\nu}J_{d,d+1}$$

④

$$[Q_\alpha, W^\mu] = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}[Q_\alpha, P_\nu M_{\rho\sigma}] = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} P_\nu [Q_\alpha, M_{\rho\sigma}]$$
$$= \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} P_\nu \left(-i(\sigma_{\rho\sigma})_\alpha{}^\beta Q_\beta\right) = (\sigma^{\mu\nu})_\alpha{}^\beta P_\nu Q_\beta$$

since $\sigma^{\mu\nu} = -\frac{i}{2}\varepsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma}$

$$[Q_\alpha, \frac{1}{4}\bar{Q}_{\dot\beta}\bar\sigma^{\mu,\dot\beta\beta}Q_\beta] = \frac{1}{4}\bar\sigma^{\mu,\dot\beta\beta}\{Q_\alpha, \bar{Q}_{\dot\beta}\}Q_\beta$$
$$= \frac{1}{4}\bar\sigma^{\mu,\dot\beta\beta}\, 2\sigma^\nu_{\alpha\dot\beta} P_\nu Q_\beta = \frac{1}{2}\sigma^\nu_{\alpha\dot\beta}\bar\sigma^{\mu,\dot\beta\beta}P_\nu Q_\beta$$
$$= \frac{1}{2}(\sigma^\nu\bar\sigma^\mu)_\alpha{}^\beta P_\nu Q_\beta = \frac{1}{2}(\eta^{\mu\nu}\mathbb{1} + 2\sigma^{\nu\mu})_\alpha{}^\beta P_\nu Q_\beta$$

$$\Rightarrow \quad [Q_\alpha, \tilde{W}^\mu] = \frac{1}{2}P^\mu Q_\alpha$$

Next,
$$[Q_\alpha, P_\mu\tilde{W}_\nu - P_\nu\tilde{W}_\mu] = P_\mu[Q_\alpha, \tilde{W}_\nu] - (\mu\leftrightarrow\nu)$$
$$= \frac{1}{2}P_\mu P_\nu Q_\alpha - (\mu\leftrightarrow\nu) = 0$$

Similarly, $[P_\rho, P_\mu\tilde{W}_\nu - P_\nu\tilde{W}_\mu] = 0$

Finally, $C = (P_\mu\tilde{W}_\nu - P_\nu\tilde{W}_\mu)^2$ is a scalar so $[M_{\mu\nu}, C] = 0$. It follows that C commutes with Q, P, M so it is a Casimir element for the super-Poincaré algebra.

⑤ Suppose there are no extra generators besides M, K, P, D, Q. Then, we are forced to set $[K_\mu, Q^i_\alpha] = 0$, $[K_\mu, \bar{Q}_{\dot\alpha i}] = 0$

Now consider
$$[K_\mu, \{Q^i_\alpha, \bar{Q}_{\dot\alpha j}\}] = [K_\mu, 2\sigma^\nu_{\alpha\dot\alpha}\delta^i_j P_\nu]$$
$$= 2\sigma^\nu_{\alpha\dot\alpha}(-2i)(\eta_{\mu\nu}D + M_{\mu\nu})$$

This can also be computed in a different way
$$[K_\mu, \{Q^i_\alpha, \bar{Q}_{\dot\alpha j}\}] = \{[K_\mu, Q^i_\alpha], \bar{Q}_{\dot\alpha j}\} + \{Q^i_\alpha, [K_\mu, \bar{Q}_{\dot\alpha j}]\}$$
$$= 0 + 0 = 0$$

So we obtain a contradiction.

A different way to see why this can not work is to recall that the algebra of M, K, P, D is the algebra of a six-dimensional orthogonal group and the spinor representations of this algebra are larger than the ones of the four-dimensional algebra.

(6) Check QQS

$$[Q^i_\alpha, \{Q^j_\beta, S^\gamma_k\}] = [\{Q^i_\alpha, \cancel{Q^j_\beta}\}, S^\gamma_k] - [Q^j_\beta, \{Q^i_\alpha, S^\gamma_k\}]$$

$$[Q^i_\alpha, \{Q^j_\beta, S^\gamma_k\}] = [Q^i_\alpha, -2\delta^j_k (\sigma^{\mu\nu})_\beta{}^\gamma M_{\mu\nu} - 4i\delta_\beta^\gamma I^j_k + 2\delta_\beta^\gamma \delta^j_k D + \frac{2(4-\mathcal{N})}{\mathcal{N}} i \delta_\beta^\gamma \delta^j_k R]$$

$$= 2\delta^j_k (\sigma^{\mu\nu})_\beta{}^\gamma \; i(\sigma_{\mu\nu})_\alpha{}^\delta Q^i_\delta$$
$$+ 4i\delta_\beta^\gamma \left(\delta^i_k Q^j_\alpha - \frac{1}{\mathcal{N}} \delta^j_k Q^i_\alpha\right)$$
$$- 2\delta_\beta^\gamma \delta^j_k \frac{i}{2} Q^i_\alpha$$
$$+ \frac{2(4-\mathcal{N})}{\mathcal{N}} i \delta_\beta^\gamma \delta^j_k \left(\frac{1}{2} Q^i_\alpha\right)$$

This RHS + RHS with $\begin{pmatrix} \alpha \leftrightarrow \beta \\ i \leftrightarrow j \end{pmatrix}$ should yield 0

Grouping the δ^j_k and δ^i_k, we find

$$2i(\sigma^{\mu\nu})_\beta{}^\gamma (\sigma_{\mu\nu})_\alpha{}^\delta Q^i_\delta + 4i\delta_\alpha^\gamma Q^i_\beta - 4i/\mathcal{N}\, \delta_\beta^\gamma Q^i_\alpha$$
$$- 2\delta_\beta^\gamma \frac{i}{2} Q^i_\alpha + \frac{2i(4-\mathcal{N})}{2\mathcal{N}} \delta_\beta^\gamma Q^i_\alpha = 0$$

$$\Rightarrow \quad (\sigma^{\mu\nu})_\beta{}^\gamma (\sigma_{\mu\nu})_\alpha{}^\delta + 2\delta_\alpha^\gamma \delta_\beta^\delta - \frac{2}{\mathcal{N}} \delta_\beta^\gamma \delta_\alpha^\delta - \frac{1}{2}\delta_\beta^\gamma \delta_\alpha^\delta$$
$$+ \frac{4-\mathcal{N}}{2\mathcal{N}} \delta_\beta^\gamma \delta_\alpha^\delta = 0$$

$$\Rightarrow \quad \delta_\beta^\gamma \delta_\alpha^\delta \left(1 - \frac{2}{\mathcal{N}} - \frac{1}{2} + \frac{4-\mathcal{N}}{2\mathcal{N}}\right) = 0$$

where we have used the identity

$$(\sigma^{\mu\nu})_\beta{}^\gamma (\sigma_{\mu\nu})_\alpha{}^\delta + 2\delta_\alpha^\gamma \delta_\beta^\delta = \delta_\alpha^\delta \delta_\beta^\gamma$$

(see the following for a proof).

In order to find the Fierz identity for $(\sigma^{\mu\nu})_\alpha{}^\beta(\sigma_{\mu\nu})_\gamma{}^\delta$ we decompose $\delta_\alpha^\delta \delta_\gamma^\beta$ over a basis δ_γ^δ and $(\sigma_{12})_\gamma{}^\delta$, $(\sigma_{23})_\gamma{}^\delta$, $(\sigma_{31})_\gamma{}^\delta$

$$\Rightarrow \quad \delta_\alpha^\delta \delta_\gamma^\beta = \delta_\gamma^\delta M_\alpha{}^\beta + \sum_{ij \in \{(12),(23),(31)\}} (\sigma_{ij})_\gamma{}^\delta (M^{ij})_\alpha{}^\beta$$

trace over (γ, δ)

$$\Rightarrow \quad \delta_\alpha^\beta = 2 M_\alpha{}^\beta \quad \Rightarrow \quad M_\alpha{}^\beta = \frac{1}{2}\delta_\alpha^\beta$$

Recall that

$$tr(\sigma_{12}\sigma_{12}) = -\frac{1}{2}$$

$$tr(\sigma_{23}\sigma_{23}) = -\frac{1}{2}$$

$$tr(\sigma_{31}\sigma_{31}) = -\frac{1}{2}$$

while the rest of the traces vanish.

$$\Rightarrow \quad (\sigma^{12})_\alpha{}^\beta = -\frac{1}{2}(M^{12})_\alpha{}^\beta \quad \text{etc}$$

$$\Rightarrow \quad M^{ij} = -2\sigma^{ij}$$

Then,

$$\delta_\alpha^\delta \delta_\gamma^\beta = \frac{1}{2}\delta_\gamma^\delta \delta_\alpha^\beta - 2\sum_{(ij)\in\{(12),(23),(31)\}} (\sigma_{ij})_\gamma{}^\delta (\sigma^{ij})_\alpha{}^\beta$$

$$= \frac{1}{2}\delta_\gamma^\delta \delta_\alpha^\beta - \sum_{i,j\in\{1,2,3\}} (\sigma_{ij})_\gamma{}^\delta (\sigma^{ij})_\alpha{}^\beta$$

$$= \frac{1}{2}\delta_\gamma^\delta \delta_\alpha^\beta - \frac{1}{2}(\sigma_{\mu\nu})_\alpha{}^\beta (\sigma^{\mu\nu})_\gamma{}^\delta$$

Finally,

$$\boxed{(\sigma_{\mu\nu})_\alpha{}^\beta (\sigma^{\mu\nu})_\gamma{}^\delta = \delta_\alpha^\beta \delta_\gamma^\delta - 2\delta_\alpha^\delta \delta_\gamma^\beta}$$

Let us check the $\bar{Q}QS$ Jacobi identity

$$[\bar{Q}_{\dot\alpha i}, \{Q^j_\alpha, S^\beta_k\}] = [\{\bar{Q}_{\dot\alpha i}, Q^j_\alpha\}, S^\beta_k] - [Q^j_\alpha, \{\bar{Q}_{\dot\alpha i}, S^\beta_k\}]$$

$$\text{LHS} = [\bar{Q}_{\dot\alpha i}, \{Q^j_\alpha, S^\beta_k\}] = [\bar{Q}_{\dot\alpha i}, -2(\sigma^{\mu\nu})_\alpha{}^\beta M_{\mu\nu}\delta^j_k - 4i\delta^\beta_\alpha I_k{}^j + 2\delta^\beta_\alpha\delta^j_k D + \frac{2i(4-\mathcal{N})}{\mathcal{N}}\delta^\beta_\alpha\delta^j_k R]$$

$$= 2(\sigma^{\mu\nu})_\alpha{}^\beta\delta^j_k\left(-i\,\bar{Q}_{\dot\beta i}(\bar\sigma_{\mu\nu})^{\dot\beta}{}_{\dot\alpha}\right)$$
$$+4i\,\delta^\beta_\alpha\left(-\delta^j_i\bar{Q}_{\dot\alpha k} + \frac{1}{\mathcal{N}}\delta^j_k\bar{Q}_{\dot\alpha i}\right)$$
$$-2\delta^\beta_\alpha\delta^j_k\left(\frac{i}{2}\bar{Q}_{\dot\alpha i}\right)$$
$$-\frac{2i(4-\mathcal{N})}{\mathcal{N}}\delta^\beta_\alpha\delta^j_k\left(\frac{1}{2}\bar{Q}_{\dot\alpha i}\right)$$

The contraction $(\sigma^{\mu\nu})_\alpha{}^\beta(\bar\sigma_{\mu\nu})^{\dot\alpha}{}_{\dot\beta}$ vanishes since the $\sigma^{\mu\nu}$ and $\bar\sigma_{\mu\nu}$ have different duality properties

$$(\sigma^{\mu\nu})_\alpha{}^\beta(\bar\sigma_{\mu\nu})^{\dot\alpha}{}_{\dot\beta} = \left[-\frac{i}{2}\varepsilon^{\mu\nu\mu'\nu'}(\sigma_{\mu'\nu'})_\alpha{}^\beta\right](\bar\sigma_{\mu\nu})^{\dot\alpha}{}_{\dot\beta}$$
$$= (\sigma_{\mu'\nu'})_\alpha{}^\beta\left[-\frac{i}{2}\varepsilon^{\mu'\nu'\mu\nu}(\bar\sigma_{\mu\nu})^{\dot\alpha}{}_{\dot\beta}\right] = -(\sigma_{\mu'\nu'})_\alpha{}^\beta(\bar\sigma^{\mu'\nu'})^{\dot\alpha}{}_{\dot\beta}$$

$$\Rightarrow \boxed{(\sigma^{\mu\nu})_\alpha{}^\beta(\bar\sigma_{\mu\nu})^{\dot\alpha}{}_{\dot\beta} = 0}$$

$$\Rightarrow \text{LHS} = \delta^\beta_\alpha\left\{\delta^j_k\bar{Q}_{\dot\alpha i}\left(\frac{4i}{\mathcal{N}} - i - \frac{i(4-\mathcal{N})}{\mathcal{N}}\right) + (-4i)\delta^j_i\bar{Q}_{\dot\alpha k}\right\}$$

$$\Rightarrow \text{LHS} = -4i\,\delta^j_i\bar{Q}_{\dot\alpha k}\delta^\beta_\alpha$$

$$\text{RHS} = [2\sigma^\mu_{\alpha\dot\alpha}\delta^j_i P_\mu, S^\beta_k] = 2\sigma^\mu_{\alpha\dot\alpha}\delta^j_i\left(i(\varepsilon\sigma_\mu)^\beta{}_{\dot\beta}\bar{Q}^{\dot\beta}_k\right)$$

Using the contraction of the First identity from the first problem set,

$$\sigma^\mu_{\alpha\dot\alpha}\sigma_{\mu,\beta\dot\beta} = 2\varepsilon_{\alpha\beta}\varepsilon_{\dot\alpha\dot\beta} - 2\underbrace{(\sigma^{\mu\nu}\varepsilon)_{\alpha\beta}(\varepsilon\bar\sigma_{\mu\nu})_{\dot\alpha\dot\beta}}_{0}$$
$$= 2\varepsilon_{\alpha\beta}\varepsilon_{\dot\alpha\dot\beta}$$

$$\Rightarrow \sigma^\mu_{\alpha\dot\alpha}(\varepsilon\sigma_\mu)^\beta{}_{\dot\beta} = -2\delta^\beta_\alpha\varepsilon_{\dot\alpha\dot\beta}$$

$$\Rightarrow \text{RHS} = -4i\,\delta^j_i\delta^\beta_\alpha\bar{Q}_{\dot\alpha k}$$

$$\Rightarrow \text{LHS} = \text{RHS} \quad \checkmark$$

(7) For any Lie algebra, the algebra generated by commutators is a Lie subalgebra. It is called the derived algebra. It is clearly closed under Lie bracket. Since it's a subalgebra, its Jacobi identities follow from the Jacobi identities of the algebra it is embedded in.

In our case, we are dealing with a superalgebra but the same holds in this case. Therefore, we can remove the generator R since it does not appear in any commutator (it is not in the derived algebra).

(8) The question is if we can consistently assign an R-charge to the fields. Assume that ϕ has charge q i.e. $[R, \phi] = q\phi$

Then,

$$[R, \underbrace{[Q, \phi]}_{\lambda}] = (q - \tfrac{1}{2})\,[Q, \phi] \qquad (Q \text{ has charge } -\tfrac{1}{2})$$

$$\Rightarrow \quad [R, \lambda] = (q - \tfrac{1}{2})\,\lambda$$

Next,

$$[R, \{Q, \lambda\}] = (q-1)\,\{Q, \lambda\}$$

$$\Rightarrow \quad [R, [\phi, \phi]] = (q-1)\,[\phi, \phi] = 2q\,[\phi, \phi]$$

This implies $q = -1$

However,

$$[R, [D, \phi]] = q\,[D, \phi]$$
$$= [R, \{Q, \bar{\lambda}\}] = q\,\{Q, \bar{\lambda}\}$$

implies that $\{Q, \bar{\lambda}\}$ has charge -1 so $\bar{\lambda}$ has charge $-\tfrac{1}{2}$

$\bar{\lambda} \sim [Q, A]$ implies A has charge 0 which implies that $F_{\mu\nu}$ also has charge 0. However, the supersymmetry algebra implies that $F_{\mu\nu}$ and $[\phi, \phi]$ should both have charge -2, since they appear in the RHS of $\{Q, \lambda\}$