1. The Poincaré algebra is defined by the commutation relations

$$[\mathrm{P}_\mu, \mathrm{P}_\nu] = 0, \qquad [\mathrm{M}_{\mu\nu}, \mathrm{P}_\rho] = i(\eta_{\nu\rho}\mathrm{P}_\mu - \eta_{\mu\rho}\mathrm{P}_\nu), \tag{1}$$

$$[\mathrm{M}_{\mu\nu}, \mathrm{M}_{\rho\sigma}] = i(\eta_{\nu\rho}\mathrm{M}_{\mu\sigma} - \eta_{\mu\rho}\mathrm{M}_{\nu\sigma} + \eta_{\mu\sigma}\mathrm{M}_{\mu\rho} - \eta_{\nu\sigma}\mathrm{M}_{\mu\rho}), \tag{2}$$

Confirm that the quadratic and quartic Casimirs P^2 and W^2 where $\mathrm{W}_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\mathrm{M}^{\nu\rho}\mathrm{P}^\sigma$ commute with all the elements of the Poincaré algebra.

2. The conformal algebra contains all the generators of the Poincaré algebra and also the generators D and K_μ. The extra commutation relations are

$$[\mathrm{D}, \mathrm{P}_\mu] = i\mathrm{P}_\mu, \qquad [\mathrm{D}, \mathrm{K}_\mu] = -i\mathrm{K}_\mu, \qquad [\mathrm{D}, \mathrm{M}_{\mu\nu}] = 0, \qquad [\mathrm{K}_\mu, \mathrm{K}_\nu] = 0, \tag{3}$$

$$[\mathrm{M}_{\mu\nu}, \mathrm{K}_\rho] = i(\eta_{\nu\rho}\mathrm{K}_\mu - \eta_{\mu\rho}\mathrm{K}_\nu), \qquad [\mathrm{P}_\mu, \mathrm{K}_\nu] = -2i(\eta_{\mu\nu}\mathrm{D} + M_{\mu\nu}). \tag{4}$$

What are the commutators of the Poincaré Casimirs with the conformal algebra? What happens to a massive particle state $|p\rangle$ ($\mathrm{P}^2|p\rangle = m^2|p\rangle$) under the action of a conformal boost K_μ?

3. If $\mu, \nu = 0, \ldots, d-1$ then we define $\mathrm{J}_{\mu\nu} = \mathrm{M}_{\mu\nu}$, $\mathrm{J}_{\mu,d} = \frac{1}{2}(\mathrm{P}_\mu - \mathrm{K}_\mu)$, $\mathrm{J}_{\mu,d+1} = \frac{1}{2}(\mathrm{P}_\mu + \mathrm{K}_\mu)$, $\mathrm{J}_{d,d+1} = \mathrm{D}$. Show that the commutators of these generators are the same as for a $d+2$-dimensional orthogonal group, with metric signature $(2, d)$.

4. Compute the commutation relation between Q_α and the Pauli-Lubański vector $\mathrm{W}^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\mathrm{P}_\nu\mathrm{M}_{\rho\sigma}$. Show that for

$$\tilde{\mathrm{W}}^\mu = \mathrm{W}^\mu + \frac{1}{4}\bar{\mathrm{Q}}_{\dot\alpha}\bar\sigma^{\mu,\dot\alpha\alpha}\mathrm{Q}_\alpha, \tag{5}$$

we have $[\mathrm{Q}_\alpha, \tilde{\mathrm{W}}_\mu] = \frac{1}{2}\mathrm{P}_\mu\mathrm{Q}_\alpha$. Using this, conclude that the square of $\mathrm{P}_\mu\tilde{\mathrm{W}}_\nu - \mathrm{P}_\nu\tilde{\mathrm{W}}_\mu$ is a sextic Casimir of the super-Poincaré group. Unlike for the Poincaré group, there are no quartic Casimirs.

5. The $\mathcal{N}$-extended superconformal algebra contains, besides the generators in the conformal algebra, also the supersymmetry generators Q^i_α and $\bar{\mathrm{Q}}_{\dot\alpha i}$ for $i = 1, \ldots, \mathcal{N}$ and the conformal supersymmetry generators $\mathrm{S}_{\alpha i}$ and $\bar{\mathrm{S}}^i_{\dot\alpha}$ and some $\mathrm{u}(\mathcal{N})$ generators ${\mathrm{I}_i}^j$ and R which rotate the Q and S supercharges.

   The extra (anti-)commutation relations are

$$\{\mathrm{S}_{\alpha j}, \bar{\mathrm{S}}^i_{\dot\alpha}\} = 2\delta^i_j\sigma^\mu_{\alpha\dot\alpha}\mathrm{K}_\mu, \tag{6}$$

$$\{\mathrm{Q}^i_\alpha, \mathrm{S}^\beta_j\} = -2\delta^i_j{(\sigma^{\mu\nu})_\alpha}^\beta\mathrm{M}_{\mu\nu} - 4i\delta^\beta_\alpha I^i_j + 2\delta^\beta_\alpha\delta^i_j D + \frac{2i(4-\mathcal{N})}{\mathcal{N}}\delta^\beta_\alpha\delta^i_j\mathrm{R}, \tag{7}$$

$$[\mathrm{Q}^i_\alpha, \mathrm{K}_\mu] = -i(\sigma_\mu)_{\alpha\dot\alpha}\bar{\mathrm{S}}^{\dot\alpha i}, \qquad [\bar{\mathrm{Q}}_{\dot\alpha i}, \mathrm{K}_\mu] = -i(\sigma_\mu)_{\alpha\dot\alpha}\mathrm{S}^\alpha_i, \tag{8}$$

$$[\mathrm{S}_{\alpha i}, \mathrm{P}_\mu] = -i(\sigma_\mu)_{\alpha\dot\alpha}\bar{\mathrm{Q}}^{\dot\alpha}_i, \qquad [\bar{\mathrm{S}}^i_{\dot\alpha}, \mathrm{P}_\mu] = -i(\sigma_\mu)_{\alpha\dot\alpha}\mathrm{Q}^{\alpha i}, \tag{9}$$

$$[\mathrm{D}, \mathrm{Q}] = \tfrac{i}{2}\mathrm{Q}, \qquad [\mathrm{D}, \bar{\mathrm{Q}}] = \tfrac{i}{2}\bar{\mathrm{Q}}, \qquad [\mathrm{D}, \mathrm{S}] = -\tfrac{i}{2}\mathrm{S}, \qquad [\mathrm{D}, \bar{\mathrm{S}}] = -\tfrac{i}{2}\bar{\mathrm{S}}, \tag{10}$$

$$[\mathrm{R}, \mathrm{Q}] = -\tfrac{1}{2}\mathrm{Q}, \qquad [\mathrm{R}, \bar{\mathrm{Q}}] = \tfrac{1}{2}\bar{\mathrm{Q}}, \qquad [\mathrm{R}, \mathrm{S}] = \tfrac{1}{2}\mathrm{S}, \qquad [\mathrm{R}, \bar{\mathrm{S}}] = -\tfrac{1}{2}\bar{\mathrm{S}}, \tag{11}$$

$$[{\mathrm{I}_j}^i, \mathrm{Q}^k] = \delta^k_j Q^i - \frac{1}{\mathcal{N}}\delta^i_j Q^k, \qquad [{\mathrm{I}_j}^i, \bar{\mathrm{Q}}_k] = -\delta^i_k\bar{Q}_j + \frac{1}{\mathcal{N}}\delta^i_j\bar{Q}_k. \tag{12}$$

Can you close the algebra without the extra generators S? (Hint: try to set $[\mathrm{K}, \mathrm{Q}] = 0$ and see what happens.)

6. Check a few of the Jacobi identities to confirm the superconformal algebra is consistent.

7. Something special happens when $\mathcal{N} = 4$; the generator R does not appear in the RHS of any of the commutators. If we remove it from the algebra are the Jacobi identities still satisfied?

8. There is a field theory with $\mathcal{N} = 4$ superconformal symmetry. It contains six scalars ϕ^I, four Weyl fermions $\lambda_{\alpha i}$ and $\bar{\lambda}^i_{\dot{\alpha}}$ and a gauge field A_μ with field strength $F_{\mu\nu}$. The supersymmetry transformations are, schematically

$$[\mathrm{Q}, \phi] \sim \lambda, \qquad \{\mathrm{Q}, \lambda\} \sim F_{\mu\nu}\sigma^{\mu\nu}_{\alpha\beta} + \epsilon_{\alpha\beta}[\phi^I, \phi^J], \tag{13}$$

$$\{\mathrm{Q}, \bar{\lambda}\} \sim \sigma^\mu_{\alpha\dot{\alpha}}[\nabla_\mu, \phi^I], \qquad [\mathrm{Q}, A] \sim \bar{\lambda}. \tag{14}$$

Is there a consistent action of the generator R on the fields ϕ, λ and F?