Introduction

In this problem we will study the Higgs sector of the supersymmetric Standard Model. The Higgs sector contains two chiral superfields H_u and H_d transforming as doublets under the SU(2) term of the $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ Standard Model gauge group. Under the U(1) term H_u has charge $-\frac{1}{2}$ while H_d has charge $+\frac{1}{2}$. When written out the two entries of the doublets are

$$H_u = \begin{pmatrix} H_u^+ \\ H_u^0 \end{pmatrix}, \qquad H_d = \begin{pmatrix} H_d^0 \\ H_d^- \end{pmatrix}, \tag{1}$$

where the superscripts indicate the electromagnetic charge.

We denote by V' the vector superfield corresponding to the U(1) factor of the gauge group and by V the vector superfield corresponding to the SU(2) factor. The Higgs fields do not have strong interactions so we do not need to consider the SU(3) factor. The action, restricted to the Higgs fields, is

$$\begin{aligned} S = \int d^4x d^2\theta d^2\bar{\theta} \left(\bar{H}_d e^{2V - V'} H_d + \bar{H}_u e^{2V+V'} H_u \right) + \\ + \frac{1}{32\pi} \Im \int d^4 x d^2\theta \left(2\tau \operatorname{tr}(W^\alpha W_\alpha) + \tau' W'^\alpha W'_\alpha \right) + \\ + \left(\int d^4 x d^2\theta \mu H_u^\intercal \epsilon H_d + \mathrm{cc} \right), \end{aligned} \tag{2}$$

where $\tau = \frac{4\pi i}{g^2}$, $\tau' = \frac{4\pi i}{g'^{,2}}$ and $\Im$ is the imaginary part.

We introduce some soft supersymmetry breaking terms by promoting some of the coupling constants to superfields; we replace $\mu \to \mu + m_{ud}^2 \theta^2$ and we multiply the Kähler potential of H_d by $1 - \theta^2\bar{\theta}^2 m_d^2$ and the Kähler potential of H_u by $1 - \theta^2\bar{\theta}^2 m_u^2$, i.e.

$$\bar{H}_d e^{2V-V'} H_d \to (1 - \theta^2\bar{\theta}^2 m_d^2) \bar{H}_d e^{2V-V'} H_d, \tag{3}$$

$$\bar{H}_u e^{2V+V'} H_u \to (1 - \theta^2\bar{\theta}^2 m_u^2) \bar{H}_u e^{2V+V'} H_u, \tag{4}$$

with $m_u^2, m_d^2 \in \mathbb{R}$ and $m_{ud}^2 \in \mathbb{C}$.

Questions

1. Compute the D-terms and the F-terms of the action with soft breaking terms.
2. If h_u is the lowest component in the Grassmann expansion of H_u and similarly for h_d, show that the scalar potential is

$$\begin{aligned} V(h_u, h_d) = (|\mu|^2 + m_u^2) h_u^\dagger h_u + (|\mu|^2 + m_d^2) h_d^\dagger h_d + \\ + (m_{ud}^2 h_u^\intercal \epsilon h_d + \mathrm{cc}) + \frac{1}{2} g^2 |h_u^\dagger d_d|^2 + \frac{1}{8}(g^2 + g'^2)(h_u^\dagger h_u - h_d^\dagger h_d)^2, \end{aligned} \tag{5}$$

where $\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

3. The fields h_u and h_d can be written in components as

$$h_u = \begin{pmatrix} h_u^+ \\ h_u^0 \end{pmatrix}, \qquad h_d = \begin{pmatrix} h_d^0 \\ h_d^- \end{pmatrix}. \tag{6}$$

   Let us for now set $h_u^+ = 0$ and $h_d^- = 0$. If these charged fields acquired a vacuum expectation value, then the electromagnetic gauge group would be broken and the photon would become massive.

   In order to have symmetry breaking, the potential of the neutral fields $V(h_u^0, h_d^0)$ should be unstable at the origin, which means that the matrix of second-order derivatives should not be positive definite. Show that in order for this to happen we need to have

$$|m_{ud}^2|^2 \geq (|\mu|^2 + m_u^2)(|\mu|^2 + m_d^2). \tag{7}$$

4. If $h_d^0 = \bar{h}_u^0 e^{-i\psi}$ where ψ is the phase of m_{ud}, then the quartic terms in the potential of the neutral fields $V(h_u^0, h_d^0)$ vanish. Show that in this case the condition for stability at large values of the fields is

$$2|m_{ud}^2| \leq 2|\mu|^2 + m_u^2 + m_d^2. \tag{8}$$

5. Assume that the fields h_u^0 and h_d^0 acquire a vacuum expectation value

$$\langle h_u^0 \rangle = \frac{v_u}{\sqrt{2}}, \qquad \langle h_d^0 \rangle = \frac{v_d}{\sqrt{2}}. \tag{9}$$

   Show that the relation between these vacuum expectation values and the parameters of the potential is

$$|\mu|^2 + m_u^2 = m_{ud}^2 \frac{v_d}{v_u} - \frac{1}{8}(g^2 + g'^{,2})(v_u^2 - v_d^2), \tag{10}$$

$$|\mu|^2 + m_d^2 = m_{ud}^2 \frac{v_u}{v_d} + \frac{1}{8}(g^2 + g'^{,2})(v_u^2 - v_d^2). \tag{11}$$

6. Mass matrices are of two types. A real mass matrix arises out of quadratic terms in the action of the form $\frac{1}{2}(m^2)_{ij}\phi^i\phi^j$ where the fields ϕ^i are real. Complex mass matrices arise out of terms in the action of the form $(m^2)_{i\bar{\imath}}\phi^i\bar{\phi}^{\bar{\imath}}$, where ϕ^i is complex and $\bar{\phi}^{\bar{\imath}}$ is its complex conjugate.[1]

   Going back to the full potential $V(h_u, h_d)$, show that the matrix of second order derivatives evaluated at the minimum of the potential $V(h_d, h_u)$ can be decomposed into several blocks. The first block is a real mass term

$$\frac{1}{2}\begin{pmatrix} \frac{\partial^2 V}{\partial(\Im h_u^0)^2} & \frac{\partial^2 V}{\partial \Im h_u^0 \partial \Im h_d^0} \\ \frac{\partial^2 V}{\partial \Im h_u^0 \partial \Im h_d^0} & \frac{\partial^2 V}{\partial(\Im h_d^0)^2} \end{pmatrix} = m_{ud}^2 \begin{pmatrix} \frac{v_d}{v_u} & 1 \\ 1 & \frac{v_u}{v_d} \end{pmatrix}. \tag{12}$$

[1] In principle one can always decompose the fields in their real and imaginary parts. However, besides doubling the size of the matrices, this is also physically awkward since some fields are naturally complex, as for example any fields transforming in a complex representation of the gauge group.

The second block corresponds to complex fields

$$\begin{pmatrix} \frac{\partial^2 V}{\partial h_u^+ \partial \overline{h_u^+}} & \frac{\partial^2 V}{\partial h_u^+ \partial h_d^-} \\ \frac{\partial^2 V}{\partial \overline{h_u^+} \partial \overline{h_d^-}} & \frac{\partial^2 V}{\partial h_d^- \partial \overline{h_d^-}} \end{pmatrix} = \begin{pmatrix} \frac{1}{4}g^2 v_d^2 + m_{ud}^2 \frac{v_d}{v_u} & m_{ud}^2 + \frac{1}{4}g^2 v_u v_d \\ m_{ud}^2 + \frac{1}{4}g^2 v_u v_d & \frac{1}{4}g^2 v_u^2 + m_{ud}^2 \frac{v_u}{v_d} \end{pmatrix}. \tag{13}$$

Finally, the third block corresponds to real fields

$$\frac{1}{2}\begin{pmatrix} \frac{\partial^2 V}{\partial(\Re h_u^0)^2} & \frac{\partial^2 V}{\partial \Re h_u^0 \partial \Re h_d^0} \\ \frac{\partial^2 V}{\partial \Re h_u^0 \partial \Re h_d^0} & \frac{\partial^2 V}{\partial(\Re h_d^0)^2} \end{pmatrix} = \begin{pmatrix} m_{ud}^2 \frac{v_d}{v_u} + \frac{1}{4}(g^2 + g'^2) v_u^2 & -m_{ud}^2 - \frac{1}{4}(g^2 + g'^2) v_u v_d \\ -m_{ud}^2 - \frac{1}{4}(g^2 + g'^2) v_u v_d & m_{ud}^2 \frac{v_u}{v_d} + \frac{1}{4}(g^2 + g'^2) v_d^2 \end{pmatrix}. \tag{14}$$

7. Find the eigenvalues of the mass matrices above. How many zero eigenvalues are there? How many Higgs fields should we expect in the supersymmetric Standard Model with spontaneously broken electro-weak symmetry?

8. The mass of the Standard Model Z boson is $M_Z^2 = \frac{1}{4}(g^2 + g'^2)(v_u^2 + v_d^2)$. How does the mass of the lightest Higgs compare to the mass of the Z boson?