Solutions 1

1. $$\{\gamma^\mu,\gamma^\nu\} = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0\end{pmatrix}\begin{pmatrix} 0 & \sigma^\nu \\ \bar\sigma^\nu & 0\end{pmatrix} + (\mu \leftrightarrow \nu)$$
$$= \begin{pmatrix} \sigma^\mu\bar\sigma^\nu + \sigma^\nu\bar\sigma^\mu & 0 \\ 0 & \bar\sigma^\mu\sigma^\nu + \bar\sigma^\nu\sigma^\mu \end{pmatrix} = 2\eta^{\mu\nu}\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}$$

2. Can be checked explicitly.
$$\bar\sigma^\mu = -\varepsilon\sigma^{\mu,T}\varepsilon$$
$$\bar\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} \overset{?}{=} -\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix} = -\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} \checkmark$$
$$\bar\sigma^1 = \begin{pmatrix} 0 & -1 \\ -1 & 0\end{pmatrix} \overset{?}{=} -\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0\end{pmatrix} \checkmark$$
$$\bar\sigma^2 = \begin{pmatrix} 0 & i \\ -i & 0\end{pmatrix} \overset{?}{=} -\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\begin{pmatrix} 0 & i \\ -i & 0\end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix} = -\begin{pmatrix} -i & 0 \\ 0 & -i\end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix} = \begin{pmatrix} 0 & i \\ -i & 0\end{pmatrix} \checkmark$$
$$\bar\sigma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1\end{pmatrix} \overset{?}{=} -\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix} = -\begin{pmatrix} 0 & -1 \\ -1 & 0\end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1\end{pmatrix} \checkmark$$

3. $$\mathrm{tr}(\sigma^\mu\bar\sigma^\nu) = \mathrm{tr}(\sigma^\mu(-\varepsilon\sigma^{\nu,T}\varepsilon)) = -\mathrm{tr}((\varepsilon\sigma^\mu\varepsilon)\sigma^{\nu,T})$$
$$= -\mathrm{tr}(\sigma^\nu(\varepsilon\sigma^{\mu,T}\varepsilon)) = \mathrm{tr}(\sigma^\nu\bar\sigma^\mu)$$
Now, take the trace of the relations $\sigma^\mu\bar\sigma^\nu + \sigma^\nu\bar\sigma^\mu = 2\eta^{\mu\nu}\mathbb{1}$
$$\Rightarrow \quad \mathrm{tr}(\sigma^\mu\bar\sigma^\nu) + \mathrm{tr}(\sigma^\nu\bar\sigma^\mu) = 2\eta^{\mu\nu}\,\mathrm{tr}(\mathbb{1}) = 4\eta^{\mu\nu}$$
$$\Rightarrow \quad 2\,\mathrm{tr}(\sigma^\mu\bar\sigma^\nu) = 4\eta^{\mu\nu}$$
$$\Rightarrow \quad \mathrm{tr}(\sigma^\mu\bar\sigma^\nu) = 2\eta^{\mu\nu} \checkmark$$

4. $$\mathrm{tr}(\sigma^{\mu\nu}) = \frac{1}{4}\left(\mathrm{tr}(\sigma^\mu\bar\sigma^\nu) - \mathrm{tr}(\sigma^\nu\bar\sigma^\mu)\right) = 0,$$
using the equation above. Similarly for $\bar\sigma^{\mu\nu}$

5. $$(\sigma^\mu\bar\sigma^\nu\varepsilon)^T = -\varepsilon\bar\sigma^{\nu,T}\sigma^{\mu,T} = -\varepsilon(-\varepsilon\sigma^{\nu,T}\varepsilon)^T\sigma^{\mu,T}$$
$$= \varepsilon^2\sigma^\nu\varepsilon\sigma^{\mu,T} = -\sigma^\nu\varepsilon\sigma^{\mu,T}$$
$$= -\sigma^\nu(\bar\sigma^\mu\varepsilon) = -\sigma^\nu\bar\sigma^\mu\varepsilon$$
$$(\sigma^\mu\bar\sigma^\nu\varepsilon - \sigma^\nu\bar\sigma^\mu\varepsilon)^T = -\sigma^\nu\bar\sigma^\mu\varepsilon + \sigma^\mu\bar\sigma^\nu\varepsilon$$
$$\Rightarrow \quad (\sigma^{\mu\nu}\varepsilon)^T = \sigma^{\mu\nu}\varepsilon \checkmark$$
similarly, $(\bar\sigma^{\mu\nu}\varepsilon)^T = \bar\sigma^{\mu\nu}\varepsilon$

6.

$$\sigma^{12} = \frac{1}{4}(\sigma^1\bar\sigma^2 - \sigma^2\bar\sigma^1)$$
$$= \frac{1}{4}(-\sigma^1\sigma^2 + \sigma^2\sigma^1)$$

Recall that $\sigma^1\sigma^2 = i\sigma^3$, $\sigma^2\sigma^1 = -\sigma^1\sigma^2$

$$\Rightarrow \sigma^{12} = \frac{1}{4}(-i\sigma^3 - i\sigma^3) = -\frac{i}{2}\sigma^3.$$

$$\sigma^{23} = \frac{1}{4}(\sigma^2\bar\sigma^3 - \sigma^3\bar\sigma^2) = \frac{1}{4}(-\sigma^2\sigma^3 + \sigma^3\sigma^2)$$
$$= -\frac{i}{2}\sigma^1$$

$$\sigma^{31} = \frac{1}{4}(\sigma^3\bar\sigma^1 - \sigma^1\bar\sigma^3) = -\frac{i}{2}\sigma^2$$

$$\sigma^{0i} = \frac{1}{4}(\sigma^0\bar\sigma^i - \sigma^i\bar\sigma^0) = -\frac{1}{2}\sigma^i$$

Therefore,
$$\sigma^{12} = -\frac{i}{2}\sigma^3 = i\sigma^{03}$$
$$\sigma^{23} = -\frac{i}{2}\sigma^1 = i\sigma^{01}$$
$$\sigma^{31} = -\frac{i}{2}\sigma^2 = i\sigma^{02}$$

To write these constraints in a more covariant way introduce $\varepsilon^{\mu\nu\rho\sigma}$ Levi-Civita totally antisymmetric tensor, $\varepsilon^{0123} = 1$, $\varepsilon_{0123} = -1$

$$\sigma^{\mu\nu} = -\frac{i}{2}\varepsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma}$$

For the other matrices we have

$$\bar\sigma^{\mu\nu} = \frac{i}{2}\varepsilon^{\mu\nu\rho\sigma}\bar\sigma_{\rho\sigma}$$

7. To compute $\sigma^\mu\bar\sigma^\nu\sigma^\rho$ we decompose it over a basis of σ^τ

$$\sigma^\mu\bar\sigma^\nu\sigma^\rho = \sum_\tau C^{\mu\nu\rho\tau}\sigma^\tau$$

Contract with $\bar\sigma^{\tau'}$

$$\Rightarrow \quad tr(\sigma^\mu\bar\sigma^\nu\sigma^\rho\bar\sigma^\tau) = 2C^{\mu\nu\rho\tau}$$

So we only need to compute this trace.

$$tr(\sigma^\mu\bar\sigma^\nu\sigma^\rho\bar\sigma^\tau) = tr((2\eta^{\mu\nu} - \sigma^\nu\bar\sigma^\mu)\sigma^\rho\bar\sigma^\tau)$$
$$= 4\eta^{\mu\nu}\eta^{\rho\tau} - tr(\sigma^\nu\bar\sigma^\mu\sigma^\rho\bar\sigma^\tau)$$
$$= 4\eta^{\mu\nu}\eta^{\rho\tau} - tr(\sigma^\nu(2\eta^{\mu\rho} - \bar\sigma^\rho\sigma^\mu)\bar\sigma^\tau)$$

$$= 4\eta^{\mu\nu}\eta^{\rho\tau} - 4\eta^{\mu\rho}\eta^{\nu\tau} + tr(\sigma^\nu\bar\sigma^\rho\sigma^\mu\bar\sigma^\tau)$$

$$= 4\eta^{\mu\nu}\eta^{\rho\tau} - 4\eta^{\mu\rho}\eta^{\nu\tau} + tr(\sigma^\nu\bar\sigma^\rho(2\eta^{\mu\tau} - \sigma^\tau\bar\sigma^\mu))$$

$$= 4\eta^{\mu\nu}\eta^{\rho\tau} - 4\eta^{\mu\rho}\eta^{\nu\tau} + 4\eta^{\nu\rho}\eta^{\mu\tau} - tr(\sigma^\nu\bar\sigma^\rho\sigma^\tau\bar\sigma^\mu)$$

$$\Rightarrow \quad tr(\sigma^\mu\bar\sigma^\nu\sigma^\rho\bar\sigma^\tau) + tr(\bar\sigma^\mu\sigma^\nu\bar\sigma^\rho\sigma^\tau)$$
$$= 4\eta^{\mu\nu}\eta^{\rho\tau} - 4\eta^{\mu\rho}\eta^{\nu\tau} + 4\eta^{\nu\rho}\eta^{\mu\tau}$$

Now look at the difference

$$T^{\mu\nu\rho\tau} = tr(\sigma^\mu\bar\sigma^\nu\sigma^\rho\bar\sigma^\tau) - tr(\bar\sigma^\mu\sigma^\nu\bar\sigma^\rho\sigma^\tau)$$

Obviously, $T^{\mu\nu\rho\tau} = -T^{\nu\rho\tau\mu}$

Similarly, $T^{\mu\nu\rho\tau} = -T^{\nu\mu\rho\tau}$

It follows that $T^{\mu\nu\rho\tau}$ is completely antisymmetric so $T^{\mu\nu\rho\tau} = c\,\varepsilon^{\mu\nu\rho\tau}$

To compute c, consider $\mu=0,\ \nu=1,\ \rho=2,\ \tau=3$

$$tr(\sigma^0\bar\sigma^1\sigma^2\bar\sigma^3) - tr(\bar\sigma^0\sigma^1\bar\sigma^2\sigma^3) = c$$

$$= 2\,tr(\sigma^1\sigma^2\sigma^3) = 2\,tr(i\sigma^3\sigma^3) = 4i$$

$$\Rightarrow \quad c = 4i$$

We conclude that $T^{\mu\nu\rho\tau} = 4i\,\varepsilon^{\mu\nu\rho\tau}$

Finally,

$$\boxed{tr(\sigma^\mu\bar\sigma^\nu\sigma^\rho\bar\sigma^\tau) = 2\eta^{\mu\nu}\eta^{\rho\tau} - 2\eta^{\mu\rho}\eta^{\nu\tau} + 2\eta^{\nu\rho}\eta^{\mu\tau} + 2i\,\varepsilon^{\mu\nu\rho\tau}}$$

$$\boxed{tr(\bar\sigma^\mu\sigma^\nu\bar\sigma^\rho\sigma^\tau) = 2\eta^{\mu\nu}\eta^{\rho\tau} - 2\eta^{\mu\rho}\eta^{\nu\tau} + 2\eta^{\nu\rho}\eta^{\mu\tau} - 2i\,\varepsilon^{\mu\nu\rho\tau}}$$

Going back,

$$\boxed{\sigma^\mu\bar\sigma^\nu\sigma^\rho = \eta^{\mu\nu}\sigma^\rho - \eta^{\mu\rho}\sigma^\nu + \eta^{\nu\rho}\sigma^\mu + i\,\varepsilon^{\mu\nu\rho\tau}\sigma_\tau}$$

8. We will decompose $\sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\beta}$ over a basis

$$\underbrace{\varepsilon_{\alpha\beta}}_{\text{anti-symmetric}}, \quad \underbrace{(\sigma^{12}\varepsilon)_{\alpha\beta}, \quad (\sigma^{23}\varepsilon)_{\alpha\beta}, \quad (\sigma^{31}\varepsilon)_{\alpha\beta}}_{\text{symmetric}}$$

$$\Rightarrow (*) \quad \sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\beta} = \varepsilon_{\alpha\beta} M_{\dot\alpha\dot\beta} + \sum_{(ij)\in\{(12),(23),(31)\}} (\sigma^{ij}\varepsilon)_{\alpha\beta} M^{ij}_{\dot\alpha\dot\beta}$$

This formula is not manifestly invariant since the range of the sums is not $0,\ldots,3$ as usual.

$$\sum_{(ij)\in\{(12),(23),(31)\}} (\sigma^{ij}\varepsilon)_{\alpha\beta} M^{ij}_{\dot\alpha\dot\beta} = \frac{1}{2}\sum_{i,j=1}^{3} (\sigma^{ij}\varepsilon)_{\alpha\beta} M^{ij}_{\dot\alpha\dot\beta}$$

where $M^{ij}_{\dot\alpha\dot\beta} = -M^{ji}_{\dot\alpha\dot\beta}$

Next, we can extend M^{ij} to have the same duality properties as $(\sigma^{\mu\nu}\varepsilon)_{\alpha\beta}$. Doing so, we can extend the sum if we introduce at the same time a compensating factor of $1/2$.

$$\Rightarrow \sum_{(ij)\in\{(12),(23),(31)\}} (\sigma^{ij}\varepsilon)_{\alpha\beta} M^{ij}_{\dot\alpha\dot\beta} = \frac{1}{4}(\sigma_{\mu\nu}\varepsilon)_{\alpha\beta} M^{\mu\nu}_{\dot\alpha\dot\beta}$$

Eq $*$ therefore becomes

$$(**) \quad \sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\beta} = \varepsilon_{\alpha\beta} M_{\dot\alpha\dot\beta} + \frac{1}{4}(\sigma_{\mu'\nu'}\varepsilon)_{\alpha\beta} M^{\mu'\nu'}_{\dot\alpha\dot\beta}$$

The task is now to compute $M_{\dot\alpha\dot\beta}$ and $M^{\mu'\nu'}_{\dot\alpha\dot\beta}$.

Contract $(**)$ with $\varepsilon^{\alpha\beta}$ $\Rightarrow$

$$(\sigma^{\mu,T}\varepsilon\sigma^\nu)_{\dot\alpha\dot\beta} = +2M_{\dot\alpha\dot\beta}$$

$$= (\varepsilon\bar\sigma^\mu\sigma^\nu)_{\dot\alpha\dot\beta} = [\varepsilon(\eta^{\mu\nu} + 2\bar\sigma^{\mu\nu})]_{\dot\alpha\dot\beta}$$

$$\Rightarrow \boxed{M_{\dot\alpha\dot\beta} = +\frac{1}{2}\varepsilon_{\dot\alpha\dot\beta}\eta^{\mu\nu} + (\varepsilon\bar\sigma^{\mu\nu})_{\dot\alpha\dot\beta}}$$

9 Contract with $(\varepsilon\sigma^{\rho\tau})^{\alpha\beta}$

In the LHS of (**) we obtain after contraction

$$\left(\sigma^{\mu,T}\varepsilon\,\sigma^{\rho\tau}\sigma^{\nu}\right)_{\dot\alpha\dot\beta} = \left(\varepsilon\,\bar\sigma^{\mu}\sigma^{\rho\tau}\sigma^{\nu}\right)_{\dot\alpha\dot\beta}$$

Recall that

$$\bar\sigma^{\mu}\sigma^{\nu}\bar\sigma^{\rho} = \eta^{\mu\nu}\bar\sigma^{\rho} - \eta^{\mu\rho}\bar\sigma^{\nu} + \eta^{\nu\rho}\bar\sigma^{\mu} - i\varepsilon^{\mu\nu\rho\tau}\bar\sigma_{\tau}$$

$$\Rightarrow \bar\sigma^{\mu}\sigma^{\nu\rho} = \frac{1}{4}\Big(\eta^{\mu\nu}\bar\sigma^{\rho} - \eta^{\mu\rho}\bar\sigma^{\nu} + \cancel{\eta^{\nu\rho}\bar\sigma^{\mu}} - i\varepsilon^{\mu\nu\rho\tau}\bar\sigma_{\tau}$$
$$- \eta^{\mu\rho}\bar\sigma^{\nu} + \eta^{\mu\nu}\bar\sigma^{\rho} - \cancel{\eta^{\nu\rho}\bar\sigma^{\mu}} - i\varepsilon^{\mu\nu\rho\tau}\bar\sigma_{\tau}\Big)$$
$$= \frac{1}{2}\left(\eta^{\mu\nu}\bar\sigma^{\rho} - \eta^{\mu\rho}\bar\sigma^{\nu} - i\varepsilon^{\mu\nu\rho\tau}\bar\sigma_{\tau}\right)$$

$$\Rightarrow \bar\sigma^{\mu}\sigma^{\rho\tau}\sigma^{\nu} = \frac{1}{2}\left(\eta^{\mu\rho}\bar\sigma^{\tau} - \eta^{\mu\tau}\bar\sigma^{\rho} - i\varepsilon^{\mu\rho\tau\varepsilon}\bar\sigma_{\varepsilon}\right)\sigma^{\nu}$$
$$= \frac{1}{2}\Big[\eta^{\mu\rho}\left(\eta^{\tau\nu} + 2\bar\sigma^{\tau\nu}\right) - \eta^{\mu\tau}\left(\eta^{\rho\nu} + 2\bar\sigma^{\rho\nu}\right)$$
$$- i\varepsilon^{\mu\rho\tau}{}_{\varepsilon}\left(\eta^{\varepsilon\nu} + 2\bar\sigma^{\varepsilon\nu}\right)\Big]$$

$$\Rightarrow \left(\sigma^{\mu,T}\varepsilon\,\sigma^{\rho\tau}\sigma^{\nu}\right)_{\dot\alpha\dot\beta} = \frac{1}{2}\left(\eta^{\mu\rho}\eta^{\tau\nu} - \eta^{\mu\tau}\eta^{\rho\nu} - i\varepsilon^{\mu\rho\tau\nu}\right)\varepsilon_{\dot\alpha\dot\beta}$$
$$+ \eta^{\mu\rho}(\varepsilon\bar\sigma^{\tau\nu})_{\dot\alpha\dot\beta} - \eta^{\mu\tau}(\varepsilon\bar\sigma^{\rho\nu})_{\dot\alpha\dot\beta}$$
$$- i\varepsilon^{\mu\rho\tau}{}_{\varepsilon}(\varepsilon\bar\sigma^{\varepsilon\nu})_{\dot\alpha\dot\beta}$$

In the RHS of (**) we obtain

$$\frac{1}{4}\mathrm{tr}\left(\sigma_{\mu'\nu'}\varepsilon\,\varepsilon\,\sigma^{\rho\tau}\right)M^{\mu'\nu'}_{\dot\alpha\dot\beta} = -\frac{1}{4}\mathrm{tr}\left(\sigma_{\mu'\nu'}\sigma^{\rho\tau}\right)M^{\mu'\nu'}_{\dot\alpha\dot\beta}$$

Now, $\mathrm{tr}\left(\sigma_{\mu'\nu'}\sigma^{\rho\tau}\right) = \frac{1}{16}\mathrm{tr}\left(\sigma_{\mu'}\bar\sigma_{\nu'}\sigma^{\rho}\bar\sigma^{\tau}\right) - \left(\begin{smallmatrix}\mu'\leftrightarrow\nu' \\ \text{and} \\ \rho\leftrightarrow\tau\end{smallmatrix}\right)$

$$= \frac{1}{16}\Big(\cancel{2\eta_{\mu'\nu'}\eta^{\rho\tau}} - 2\delta^{\rho}_{\mu'}\delta^{\tau}_{\nu'} + 2\delta^{\tau}_{\mu'}\delta^{\rho}_{\nu'} + 2i\varepsilon_{\mu'\nu'}{}^{\rho\tau}$$
$$- \cancel{2\eta_{\mu'\nu'}\eta^{\rho\tau}} + 2\delta^{\rho}_{\nu'}\delta^{\tau}_{\mu'} - 2\delta^{\tau}_{\nu'}\delta^{\rho}_{\mu'} + 2i\varepsilon_{\mu'\nu'}{}^{\rho\tau}\Big) - (\rho\leftrightarrow\tau)$$
$$= \frac{1}{4}\left(\delta^{\tau}_{\mu'}\delta^{\rho}_{\nu'} - \delta^{\tau}_{\nu'}\delta^{\rho}_{\mu'} + i\varepsilon_{\mu'\nu'}{}^{\rho\tau}\right) - (\rho\leftrightarrow\tau)$$
$$= \frac{1}{2}\left(\delta^{\tau}_{\mu'}\delta^{\rho}_{\nu'} - \delta^{\tau}_{\nu'}\delta^{\rho}_{\mu'} + i\varepsilon_{\mu'\nu'}{}^{\rho\tau}\right)$$

$$\Rightarrow -\frac{1}{4}\,\mathrm{tr}\left(\sigma_{\mu'\nu'}\sigma^{\rho\tau}\right) M^{\mu'\nu'}{}_{\dot\alpha\dot\beta} = -\frac{1}{4}\,\frac{1}{2}\left(\delta^{\tau}_{\mu'}\delta^{\rho}_{\nu'} - \delta^{\tau}_{\nu'}\delta^{\rho}_{\mu'} + i\,\varepsilon_{\mu'\nu'}{}^{\rho\tau}\right) M^{\mu'\nu'}{}_{\dot\alpha\dot\beta}$$

$$= \frac{1}{4}\left(M^{\rho\tau}{}_{\dot\alpha\dot\beta} + \frac{i}{2}\,\varepsilon^{\rho\tau}{}_{\mu'\nu'} M^{\mu'\nu'}{}_{\dot\alpha\dot\beta}\right)$$

$$= \frac{1}{2} M^{\rho\tau}{}_{\dot\alpha\dot\beta}$$, where we used the duality condition on $M^{\mu\nu}$, i.e.

$$M^{\rho\tau} = \frac{i}{2}\,\varepsilon^{\rho\tau}{}_{\mu'\nu'} M^{\mu'\nu'}$$

So in the end,

$$M^{\rho\tau}{}_{\dot\alpha\dot\beta} = \left(\eta^{\mu\rho}\eta^{\tau\nu} - \eta^{\mu\tau}\eta^{\rho\nu} - i\,\varepsilon^{\mu\rho\tau\nu}\right)\varepsilon_{\dot\alpha\dot\beta} + 2\eta^{\mu\rho}\left(\varepsilon\bar\sigma^{\tau\nu}\right)_{\dot\alpha\dot\beta} - 2\eta^{\mu\tau}\left(\varepsilon\bar\sigma^{\rho\nu}\right)_{\dot\alpha\dot\beta} - 2i\,\varepsilon^{\mu\rho\tau}{}_{\varepsilon}\left(\varepsilon\bar\sigma^{\varepsilon\nu}\right)_{\dot\alpha\dot\beta}$$

The contraction in the RHS becomes

$$\frac{1}{4}\left(\sigma_{\mu'\nu'}\varepsilon\right)_{\alpha\beta}\Big[\left(\eta^{\mu\mu'}\eta^{\nu'\nu} + \eta^{\mu\nu'}\eta^{\mu'\nu} - i\,\varepsilon^{\mu\mu'\nu'\nu}\right)\varepsilon_{\dot\alpha\dot\beta} + 2\eta^{\mu\mu'}\left(\varepsilon\bar\sigma^{\nu'\nu}\right)_{\dot\alpha\dot\beta} - 2\eta^{\mu\nu'}\left(\varepsilon\bar\sigma^{\mu'\nu}\right)_{\dot\alpha\dot\beta} - 2i\,\varepsilon^{\mu\mu'\nu'}{}_{\varepsilon}\left(\varepsilon\bar\sigma^{\varepsilon\nu}\right)_{\dot\alpha\dot\beta}\Big]$$

$$= \frac{1}{2}\left(\sigma^{\mu\nu}\cancel{\varepsilon}\right)_{\alpha\beta}\varepsilon_{\dot\alpha\dot\beta} + \frac{1}{2}\left(\sigma^{\mu\nu}\cancel{\varepsilon}\right)_{\alpha\beta}\varepsilon_{\dot\alpha\dot\beta}$$
$$+ \frac{1}{2}\left(\sigma^{\mu}{}_{\nu'}\varepsilon\right)_{\alpha\beta}\left(\varepsilon\bar\sigma^{\nu'\nu}\right)_{\dot\alpha\dot\beta} - \frac{1}{2}\left(\sigma_{\mu'}{}^{\mu}\varepsilon\right)_{\alpha\beta}\left(\varepsilon\bar\sigma^{\mu'\nu}\right)_{\dot\alpha\dot\beta}$$
$$+ \left(\sigma^{\mu}{}_{\varepsilon}\varepsilon\right)_{\alpha\beta}\left(\varepsilon\bar\sigma^{\varepsilon\nu}\right)_{\dot\alpha\dot\beta}$$

$$= \left(\sigma^{\mu\nu}\varepsilon\right)_{\alpha\beta}\varepsilon_{\dot\alpha\dot\beta} + 2\left(\sigma^{\mu}{}_{\rho}\varepsilon\right)_{\alpha\beta}\left(\varepsilon\bar\sigma^{\rho\nu}\right)_{\dot\alpha\dot\beta}$$

Finally,

$$\sigma^{\mu}_{\alpha\dot\alpha}\sigma^{\nu}_{\beta\dot\beta} = +\frac{1}{2}\eta^{\mu\nu}\varepsilon_{\alpha\beta}\varepsilon_{\dot\alpha\dot\beta} + \left(\varepsilon\bar\sigma^{\mu\nu}\right)_{\dot\alpha\dot\beta}\varepsilon_{\alpha\beta} + \left(\sigma^{\mu\nu}\varepsilon\right)_{\alpha\beta}\varepsilon_{\dot\alpha\dot\beta} + 2\underbrace{\left(\sigma^{\mu}{}_{\rho}\varepsilon\right)_{\alpha\beta}\left(\varepsilon\bar\sigma^{\rho\nu}\right)_{\dot\alpha\dot\beta}}_{\text{is actually symmetric in } (\mu\leftrightarrow\nu)}$$