Definitions

The matrices σ^μ, $\bar\sigma^\nu$ for $\mu = 0, 1, 2, 3$ are called Pauli matrices for the metric $\eta^{\mu\nu}$ if

$$\sigma^\mu\bar\sigma^\nu + \sigma^\nu\bar\sigma^\mu = 2\eta^{\mu\nu}\mathbf{1}, \tag{1}$$
$$\bar\sigma^\mu\sigma^\nu + \bar\sigma^\nu\sigma^\mu = 2\eta^{\mu\nu}\mathbf{1}. \tag{2}$$

A representation of these matrices for mostly minus signature $\eta = \mathrm{diag}(+---)$ is

$$\sigma^0 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \qquad \sigma^1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \tag{3}$$
$$\bar\sigma^0 = \sigma^0, \qquad \bar\sigma^i = -\sigma^i, \quad \text{for } i = 1, 2, 3. \tag{4}$$

Questions

1. Show that
$$\gamma^\mu = \begin{pmatrix}0 & \sigma^\mu\\ \bar\sigma^\mu & 0\end{pmatrix} \tag{5}$$
are Dirac matrices, i.e. they satisfy $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbf{1}$.

2. Show that $\bar\sigma^\mu = -\epsilon\sigma^{\mu,\intercal}\epsilon$, where $\epsilon = i\sigma^2 = \left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$.

3. Show that $\mathrm{tr}(\sigma^\mu\bar\sigma^\nu) = 2\eta^{\mu\nu}$.

4. We define
$$\sigma^{\mu\nu} = \frac{1}{4}(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu), \qquad \bar\sigma^{\mu\nu} = \frac{1}{4}(\bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu). \tag{6}$$
Show that $\mathrm{tr}(\sigma^{\mu\nu}) = 0$ and $\mathrm{tr}(\bar\sigma^{\mu\nu}) = 0$.

5. Show that $(\sigma^{\mu\nu}\epsilon)^\intercal = \sigma^{\mu\nu}\epsilon$ and $(\bar\sigma^{\mu\nu}\epsilon)^\intercal = \bar\sigma^{\mu\nu}\epsilon$.

6. The six matrices $\sigma^{\mu\nu}\epsilon$ are symmetric, but there are only three linearly independent 2×2 symmetric matrices. Therefore, there must be three independent linear relations between the $\sigma^{\mu\nu}\epsilon$ or equivalently between the $\sigma^{\mu\nu}$. Find them.

7. Compute $\sigma^\mu\bar\sigma^\nu\sigma^\rho$ and $\bar\sigma^\mu\sigma^\nu\bar\sigma^\rho$.

8. (Fierz identities) Find identities of type
$$\sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\beta} \pm \sigma^\nu_{\alpha\dot\alpha}\sigma^\mu_{\beta\dot\beta} = \sum_l M^l_{\alpha\beta}N^l_{\dot\alpha\dot\beta}. \tag{7}$$