Constraining the symbol from singularities

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Introduction

- In perturbative quantum field theory we study *Feynman integrals*.
- *Feynman rules*: Associate to a graph G (with loops) an integral.
- Feynman integrals can be computed in many cases in terms of *iterated integrals*.

My talk: how do we make the transformation to iterated integrals.
Work with Hannesdóttir, McLeod & Schwartz.

Iterated integrals

Consider a complex manifold X and a piecewise continuous path $\gamma\colon [0,1] \to X$ with $\gamma(1) = z$. Take $\omega_1, \ldots, \omega_n$ meromorphic differential one-forms such that $d\omega_i = 0$ outside their polar locus. Then, define the *iterated integral* along γ:

$$I_\gamma(z) = \int_\gamma \omega_1 \circ \cdots \circ \omega_n = \int_\Delta (\gamma^*\omega_1)(t_1) \cdots (\gamma^*\omega_n)(t_n), \qquad (1)$$

where $\Delta = \{(t_1, \ldots, t_n) \in [0,1]^n \mid t_1 \leq t_2 \cdots \leq t_n\}$.

Iterated integrals

In general we have linear combinations $\sum_{i_1,\dots,i_n} c_{i_1\cdots i_n}(z)\int_\gamma \omega_{i_1}\circ\cdots\circ\omega_{i_n}$, where c's are <u>algebraic</u> functions of z.
The *symbol* is the formal linear combination $\sum c_{i_1\cdots i_n}(z)[\omega_{i_1}|\cdots|\omega_{i_n}]$.
The length of the iterated integral is called *transcendental weight*.[1]

[1] Only well-defined conjecturally.

Symbol

At the beginning the notion of symbol seemed like a mathematical trick, far removed from physics.
As time passed, it became clear that there is a strong connection to physics, in particular through singularities of Feynman integrals.
At first, the connection to physical singularities was through:

- "first entry" condition,
- "last entry" condition (Caron-Huot),
- Steinmann relations,
- other (more ad-hoc) constraints such as near collinear limits.

Later, Spradlin, Volovich et al. studied the Landau equations in momentum twistor space.
We worked on extensions to *Pham relations*, which I will briefly describe in this talk.

Goal: rewrite Feynman integrals as iterated integrals

- what are the symbol entries (letters)?
- in what order do they appear?

Do it purely by imposing *physical* constraints such as Steinmann-Pham compatibility constraints.

Philosophy: If some constraints remain undetermined, it's because there's something remaining to be understood about the physics.

Why write Feynman integrals as iterated integrals?

- (roughly) half the number of integrations (even fewer in odd dimensions).
- identities/cancelations between iterated integrals much easier than cancelations between Feynman integrals.
- as a stepping stone for new discoveries: connections to cluster algebras, antipodal duality, cosmic Galois group, etc.

Today, computing a Feynman integral analytically has become synonymous to writing it as an iterated integral (when possible).

Symbol alphabet

What are the symbol entries?
First guess: since the function defined by a given symbol has singularities where the symbol entries vanish (where the corresponding differential one-forms have poles), symbol entries should be Landau singularities.
Not correct! There are also square root singularities, but the conversion to momentum twistors confuses the issue by rationalizing a number of square roots.
The symbol entries get deformed to accommodate these square root singularities.

Location of singularities

The location of singularities of a Feynman integral can be obtained by solving Landau equations (Landau 1959, Leray 1959, see also Pham).

Given a subgraph H of G ($H \rightarrowtail G$), the Landau equations read

$$\begin{aligned} q_e^2 &= m_e^2, && \forall e \in E(H), \\ \sum_{e \in \Gamma} \alpha_e q_e &= 0, && \Gamma \text{ any cycle in } H, \end{aligned}$$

for α_e not all vanishing.

Theorem (Landau, Leray)

When the external momenta p are such that there exists a subgraph H of G for which the Landau equations have a solution, then the integral has a potential singularity there. If no such subgraph exists for a given p, the integral is non-singular at p.

Discontinuities across branch cuts

The Landau loci where the integrals are singular are typically of complex codimension one in the space of complexified kinematics. In a neighborhood of a Landau locus which does not contain other singularities we have:

Theorem (Cutkosky)

The monodromy around such a Landau locus corresponding to a subgraph H is given by

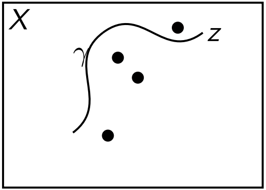
$$\int \prod_{i=1}^{l} d^d k_i \prod_{e \in E(H)} (-2\pi i)\delta(q_e^2 - m_e^2) \prod_{e \in E(G) \setminus E(H)} \frac{1}{q_e^2 - m_e^2}.$$

Same transcendental weight but higher power of π. Fits with what is expected from unipotent monodromy.

Singularities and monodromies for iterated integrals

[Goncharov: 2001] Monodromy of the iterated integral $I_\gamma(z)$ around a polar locus ω_a:

$$(1 - M_a) I_\gamma(z) = \int_{\gamma_i} \omega_1 \circ \cdots \circ \omega_{a-1}$$
$$(2\pi i \operatorname{Res}_a \omega_a) \int_{\gamma_f} \omega_{a+1} \circ \cdots \circ \omega_n. \quad (1)$$



Lots of possible singularities, but most of them have to be canceled to match the physical expectations.
Role of the initial point of γ.

Example: bubble integral in two dimensions

$$\int \frac{d^2 q_1}{(q_1^2 - m_1^2)(q_2^2 - m_2^2)} = \frac{\pi}{\sqrt{p^2 - (m_1+m_2)^2}\sqrt{p^2 - (m_1-m_2)^2}} \times \log \frac{\sqrt{p^2 - (m_1+m_2)^2} + \sqrt{p^2 - (m_1-m_2)^2}}{\sqrt{p^2 - (m_1+m_2)^2} - \sqrt{p^2 - (m_1-m_2)^2}}. \quad (2)$$

- Logarithmic singularities at $m_1 = 0$ and $m_2 = 0$,
- $p^2 = (m_1 + m_2)^2$: threshold visible in the physical region,
- $p^2 = (m_1 - m_2)^2$: threshold invisible in the physical region, but visible after taking monodromies around $m_i^2 = 0$.

The two square root singularities are very *different* from a physics point of view.

Factorization and reducibility of Landau loci

Need to distinguish between the singularities in the physical region and the rest.
For example, for the bubble singularity, solving the Landau equations $F(\alpha) = 0$ and $\partial_\alpha F = 0$ for

$$F(\alpha) = -\alpha(1-\alpha)p^2 + \alpha m_1^2 + (1-\alpha)m_2^2,$$

yields $F(\alpha_0) = -\frac{\Delta(p^2, m_1^2, m_2^2)}{4p^2} = 0$, where $\Delta(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$. This can be factorized by using $\Delta(x, y, z) = (x - (\sqrt{y} - \sqrt{z})^2)(x - (\sqrt{y} + \sqrt{z})^2)$, but in general it can be non-obvious (field extensions to include $\sqrt{y}$ and $\sqrt{z}$).

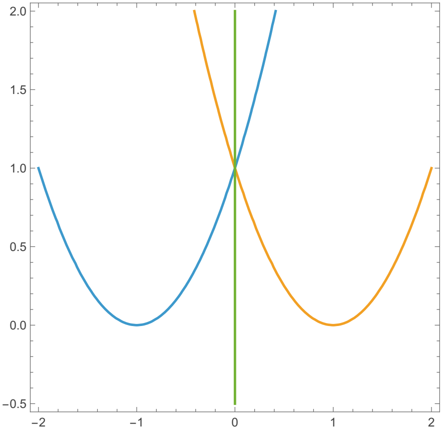
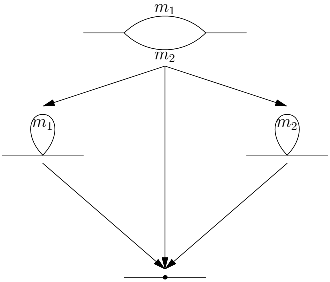


Figure: Threshold and pseudo-threshold for the bubble integral in the (m_1, p^2)-plane for $m_2 = 1$ fixed. For positive mass the threshold (blue) is above the pseudo-threshold (orange). Only the part of the blue curve for $m_1 > 0$ is effective. For $m_1 < 0$ only the part of the orange curve is effective. Which section of the real Landau loci is effective when crossing the green line.

Where do we start the integration path?
No obvious preferred point, but a preferred *set* of points. In the bubble case we can start anywhere in the ineffective part of the bubble Landau locus. This ensures (rather trivially) that the iterated integral does not have a singularity there (before analytic continuation).
There is a singularity in the integrand of the iterated integral at the beginning of the path, but it is integrable ($x^{-\frac{1}{2}}$).
In principle, this determines what have been called "beyond the symbol" terms.

Compatibility



$$(1 - M_{m_1^2=0})(1 - M_{m_2^2=0})I = 0,$$

Pham diagrams

$$
\begin{tikzcd}
\ker \kappa' \arrow[rr, tail, shift left] \arrow[ddr, tail] \arrow[drr, tail] & & \ker \kappa \arrow[ll, two heads, shift left] \arrow[rr, two heads, shift left] \arrow[d, tail] & & \ker \kappa'' \arrow[ll, tail, shift left] \arrow[ddl, tail] \arrow[dll, tail] \\
& & G \arrow[dl, two heads] \arrow[dr, two heads] \arrow[dd, two heads, "\kappa"] & & \\
& G' \arrow[dr, two heads, "\kappa'"] & & G'' \arrow[dl, two heads, "\kappa''"] & \\
& & G_0 & &
\end{tikzcd}
$$

A second type singularity example

Consider the box integral in six dimensions. Unlike its four-dimensional version, it is IR-finite. It can be shown to be proportional to

$$\frac{\log^2 \frac{s}{t} + \pi^2}{u},$$

where $u = -s - t$.

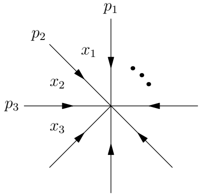
The singularity at $u = 0$ is a second type singularity, which arises at infinity. It is *not* visible in the physical region (since its Landau singularity is at $\sum_e \alpha_e = 0$). It is easy to check that the singularity at $u = 0$ cancels since at $u = 0$ we have $\frac{s}{t} = -1$ and $\log \frac{s}{t} = \pm i\pi$.

Unlike planar $\mathcal{N} = 4$, which has no singularities at infinity, QCD does have them, but they can be understood via renormalization.

Renormalization can be formulated in terms of graph contractions (Connes & Kreimer).

General story?

Generalization from elementary graphs:



Consider a graph contraction

$$\ker \kappa \rightarrowtail G \overset{\kappa}{\twoheadrightarrow} G_0.$$

This induces a projection between on-shell spaces

$$\mathscr{S}(\kappa)\colon \mathscr{S}(G) \to \mathscr{S}(G_0)$$

and

$$d\mathscr{S}(\kappa)_k\colon T_k\mathscr{S}(G) \to T_{\mathscr{S}(\kappa)(k)}\mathscr{S}(G_0).$$

Critical points: $k \in \mathscr{S}(G)$ such that rank $d\mathscr{S}(\kappa)_k$ is not maximal. ([Pham], [Thom], reviewed in [Hannesdóttir, McLeod, CV, Schwartz]).

Given a graph G_1 and a contraction to G_2 ($\kappa\colon G_1 \twoheadrightarrow G_2$), there is a projection $\mathscr{S}(\kappa)\colon \mathscr{S}(G_1) \to \mathscr{S}(G_2)$.
Given a map[2] $f\colon \mathbb{R}^k \to \mathbb{R}^l$ (for $k > l$), we call *critical points* $x \in \mathbb{R}^k$ such that rank $\frac{\partial f_i}{\partial x_j} < l$. If x is a critical point, $f(x)$ is a *critical value*.
The critical values of the map π are connected to *Landau singularities*.
Landau equations: $\sum_{i=1}^{l} \alpha_i \frac{\partial f_i}{\partial x_j} = 0$ for $j = 1, \ldots, k$ and for α_i not all vanishing. Semi-algebraic conditions: $\alpha_i \geq 0$ for singularities in the *physical region*.
Further conditions: positivity[3] of the Hessian

$$\left(\sum_{i=1}^{l} \alpha_i \frac{\partial^2 f_i}{\partial x_p \partial x_q} \right)_{1 \leq p,q \leq l}.$$

Such critical points are called S_1^+ in Thom's classification.

[2] Can also be done in local coordinates for maps between manifolds.
[3] To my knowledge not studied at all in momentum twistor geometry.

Reality conditions

In order to study the signature of the Hessian quadratic form we need to impose some *reality conditions*.
We have *dual momentum twistors* $\bar{Z}_i$ which represent the plane containing the lines ℓ_{i-1} and ℓ_i (can be thought of as $Z_{i-1} \wedge Z_i \wedge Z_{i+1}$). The reality condition is

$$\bar{Z}_i = \epsilon_i \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} Z_i^\dagger,$$

where ϵ_i is the sign of energy for particle i.

Work with Jacob Bourjaily and Matt von Hippel:

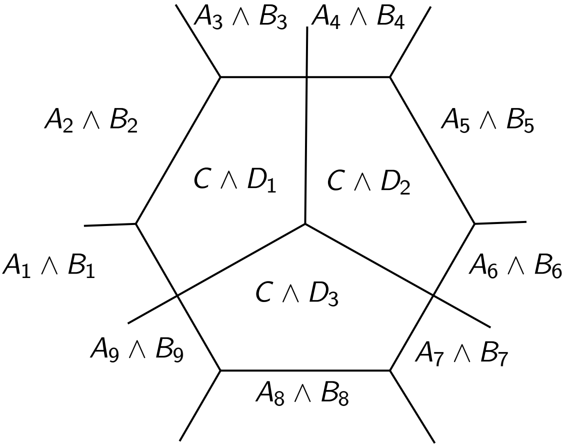


Figure: The dual points are described as lines in momentum twistor space.

Cayley octads

$$(1:0:0:0),\quad (0:1:0:0),\quad (0:0:1:0),\quad (0:0:0:1),$$
$$(1:1:1:1),\qquad (\alpha_6:\beta_6:\gamma_6:\delta_6),\qquad (\alpha_7:\beta_7:\gamma_7:\delta_7).$$

Then, the eighth point has coordinates $(\alpha_8:\beta_8:\gamma_8:\delta_8)$ given by [Plaumann, Sturmfels, Vinzant]

$$\alpha_8 = \frac{-\gamma_6\beta_7+\delta_6\beta_7+\beta_6\gamma_7-\delta_6\gamma_7-\beta_6\delta_7+\gamma_6\delta_7}{-\beta_6\delta_6\beta_7\gamma_7+\gamma_6\delta_6\beta_7\gamma_7+\beta_6\gamma_6\beta_7\delta_7-\gamma_6\delta_6\beta_7\delta_7-\beta_6\gamma_6\gamma_7\delta_7+\beta_6\delta_6\gamma_7\delta_7},$$
$$\beta_8 = \frac{-\gamma_6\alpha_7+\delta_6\alpha_7+\alpha_6\gamma_7-\delta_6\gamma_7-\alpha_6\delta_7+\gamma_6\delta_7}{-\alpha_6\delta_6\alpha_7\gamma_7+\gamma_6\delta_6\alpha_7\gamma_7+\alpha_6\gamma_6\alpha_7\delta_7-\gamma_6\delta_6\alpha_7\delta_7-\alpha_6\gamma_6\gamma_7\delta_7+\alpha_6\delta_6\gamma_7\delta_7},$$
$$\gamma_8 = \frac{-\beta_6\alpha_7+\delta_6\alpha_7+\alpha_6\beta_7-\delta_6\beta_7-\alpha_6\delta_7+\beta_6\delta_7}{-\alpha_6\delta_6\alpha_7\beta_7+\beta_6\delta_6\alpha_7\beta_7+\alpha_6\beta_6\alpha_7\delta_7-\beta_6\delta_6\alpha_7\delta_7-\alpha_6\beta_6\beta_7\delta_7+\alpha_6\delta_6\beta_7\delta_7},$$
$$\delta_8 = \frac{-\beta_6\alpha_7+\gamma_6\alpha_7+\alpha_6\beta_7-\gamma_6\beta_7-\alpha_6\gamma_7+\beta_6\gamma_7}{-\alpha_6\gamma_6\alpha_7\beta_7+\beta_6\gamma_6\alpha_7\beta_7+\alpha_6\beta_6\alpha_7\gamma_7-\beta_6\gamma_6\alpha_7\gamma_7-\alpha_6\beta_6\beta_7\gamma_7+\alpha_6\gamma_6\beta_7\gamma_7}.$$

Configuration of 8 intersection points in $\mathbb{P}^3$ is Gale selfdual (given m points in $\mathbb{P}^n$ for $m > n+2$, they can be put into correspondence with m points in $\mathbb{P}^{m-n-2}$). Used before by [Goncharov] and [Golden, Goncharov, Spradlin, CV, Volovich].

Degenerations

Possible degenerations [Dolgachev & Ortland]

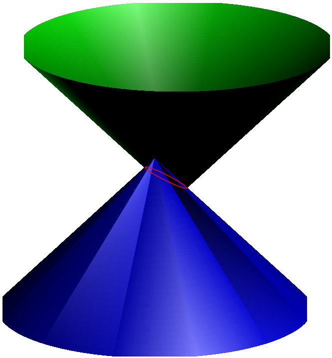
1. two of the eight points become coincident.
2. four points become coplanar (when this happens the other four points also become coplanar as follows from Gale duality).
3. the eight points lie on a twisted cubic. This is a codimension two condition.

Interpretation of some of these singularities from a physics point of view is unclear.

The analysis of these on-shell spaces for the case when they are zero-dimensional has been taken up recently by Hollering, Mazzucchelli, Parisi, Sturmfels.

Complications

On-shell spaces can become singular. Pham suggests using a stratification of the on-shell spaces and applying the usual critical point/value discussion for each stratum.



Two types of limits:

1. soft: vertices of the cones become coincident. The on-shell space at the critical point is a point (vanishing cycle).
2. collinear: vertices of the cones become light-like separated. The on-shell space at the critical point is a line segment (along the common generator of the two cones).

Alternatively, non-uniqueness of internal momenta upon solving the Landau equations.

More complications

1. UV/IR divergences. Dimensional regularization complicates analysis. Analog of tangential basepoint regularization?
2. massless particles: more singular varieties, but less likely to encounter CY at low loop/leg order
3. Connect to cluster algebras?

Thank You!