Scattering Amplitudes: their structure, their singularities and their geometry

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A royal way to QFT

A perturbative scattering amplitude in a QFT can be written as a sum of diagrams. To each diagram we associate an integral. The result of integration is often a very interesting quantity: MZV, polylogarithms, more general *periods*.
Minkowski space M is $\mathbb{A}^D$ with a constant[1] (independent on position) symmetric nondegenerate quadratic form[2] g with signature $(1, D-1)$. Often one uses the notation $M = \mathbb{R}^{1,D-1}$.
Minkowski space unifies space and time into a new space called *space-time* (one can "rotate" between space and time "directions"). Orthogonal transformations $O(1, D-1)$ which preserve the quadratic form g are called *Lorentz transformations*.

[1]In cosmological applications one deals with de Sitter or FRW spaces, which have nontrivial curvature.

[2]Usually one chooses coordinates such that $g = \mathrm{diag}(1, -1, -1, -1)$ but other choices of coordinates are sometimes useful.

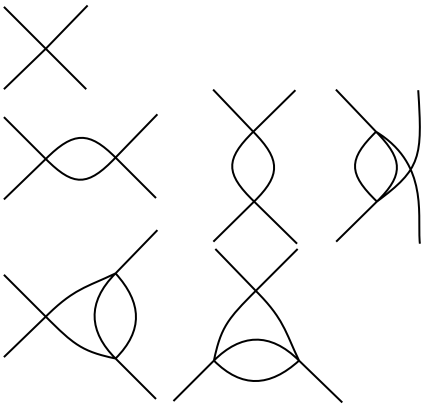
A royal way to QFT (2)

On M one has *local fields*. $\phi\colon M \to \mathbb{R}$ (or $\phi\colon M \to \mathbb{C}$) is a real (complex) *scalar* field. We usually do a Fourier transform $\hat{\phi}(p) = \frac{1}{(2\pi)^D} \int_M d^D x e^{i\langle p,x\rangle} \phi(x)$. x is called *position* and p is called *momentum*. Free fields satisfy equations $(\Box_g + m^2)\phi(x) = 0$ where $\Box_g$ is the Laplacian with respect to the "metric" g and m is called the *mass* of the particle. This implies that $(p^2 - m^2)\hat{\phi}(p) = 0$ (where $p^2 = g^{-1}(p,p)$). A particle with momentum p such that $p^2 = m^2$ is called *on-shell*. For $m > 0$ this is geometrically a hyperboloid with two sheets (one corresponding to *positive energy* and the other to *negative energy*). For $m = 0$ it is a cone, called *light-cone*.

A royal way to QFT (3)

Other types of fields are *vector bosons* $A\colon M \to T_* M$ and *spinors* (fermions) $\Psi\colon M \to S_\pm$.
A *Feynman diagram* is a (multi-)graph, with each vertex being associated to an interaction and each edge with a particle. A QFT is characterized by a set of rules, called *Feynman rules*, which associate to each vertex some rational function of momenta of the incident particles and to each edge a rational function called *propagator*. Given a graph, we can write an integral associated to it by taking the product of the terms at all the vertices and all the edges and integrating[3] over the undetermined momenta.

[3] And summing over all other particle characteristics like species, color, etc.



For the bubble integral we have:

$$\int \frac{d^D q_1}{(q_1^2 - m_1^2)(q_2^2 - m_2^2)}$$

for $p = q_1 + q_2$ which for $D = 2$ integrates to (up to some numerical prefactors)

$$\frac{1}{\sqrt{r_+}\sqrt{r_-}}\Big(\log(\sqrt{r_+} + \sqrt{r_-}) - \log(\sqrt{r_+} - \sqrt{r_-})\Big), \quad (1)$$

for $r_\pm = p^2 - (m_1 \pm m_2)^2$.
Here $\frac{1}{\sqrt{r_+}\sqrt{r_-}}$ is an example of leading singularity.

In practice, many (but not all!) integrals we encounter can be put in an iterated integral form.
This is a good idea for serveral reasons

1. Fewer integrals to do (a Feynman integral with L loops in D dimensions has LD integrals, but its iterated integral form is conjectured to have at most $\lfloor \frac{LD}{2} \rfloor$ integrals).
2. The singularities and algebraic relations between iterated integrals are much more obvious than for Feynman integrals.
3. One can build an ansatz for the final answer and constrain it by imposing various physical constraints (bypassing the difficult computation of Feynman integrals).

Iterated integrals

Consider a complex manifold X and a piecewise continuous path $\gamma\colon [0,1] \to X$ with $\gamma(1) = z$. Take $\omega_1, \ldots, \omega_n$ meromorphic differential one-forms such that $d\omega_i = 0$ outside their polar locus. Then, define the *iterated integral* along γ:

$$I_\gamma(z) = \int_\gamma \omega_1 \circ \cdots \circ \omega_n = \int_\Delta (\gamma^*\omega_1)(t_1) \cdots (\gamma^*\omega_n)(t_n), \qquad (2)$$

where $\Delta = \{(t_1, \ldots, t_n) \in [0,1]^n \mid t_1 \leq t_2 \cdots \leq t_n\}$.
In general we have linear combinations
$\sum_{i_1,\ldots,i_n} c_{i_1\cdots i_n}(z) \int_\gamma \omega_{i_1} \circ \cdots \circ \omega_{i_n}$, where c's are algebraic functions of z. If $c_{i_1\cdots i_n}(z) = n_{i_1\cdots i_n} c(z)$ with $n_{i_1\cdots i_n} \in \mathbb{Z}$, then we say that $c(z)$ is a *leading singularity*.
The length of the iterated integral is called *transcendental weight*.[4]

[4]Only well-defined conjecturally.

Example

For $X = \mathbb{C}$, $\omega_0 = \frac{dz}{z}$, $\omega_1 = \frac{dz}{1-z}$ and γ a straight path from 0 to z_0, we have

$$\mathrm{Li}_n(z_0) = \int_\gamma \omega_1 \circ \underbrace{\omega_0 \circ \ldots \circ \omega_0}_{n-1}.$$

The forms ω_0, ω_1 are *d-log* forms $\omega_0 = d \log z$, $\omega_1 = -d \log(z-1)$. One can encode the essential information about the integrand by a *symbol*

$$-(z-1) \otimes z \ldots \otimes z,$$

or

$$-[z-1|z|\cdots|z].$$

Since $d \log(fg) = d \log f + d \log g$ we have that

$$\cdots \otimes (fg) \otimes \cdots = \cdots \otimes f \otimes \cdots + \cdots \otimes g \otimes \cdots .$$

By the previous property we have that $\cdots \otimes 1 \otimes \cdots = 0$. Similarly, if ζ is an N-th root of unity $\zeta^N = 1$, then we have that $N(\cdots \otimes \zeta \otimes \cdots) = 0$, so $\cdots \otimes \zeta \otimes \cdots$ is N-torsion. In practice we just discard these terms and we take

$$\cdots \otimes (-f) \otimes \cdots = \cdots \otimes f \otimes \cdots .$$

The set of entries in the symbol associated to a Feynman integral or to a physical process is called the *symbol alphabet* (an individual entry is called a *letter*). Letters encode singularities in z of the function $I_\gamma(z)$, but not in one-to-one way (a single letter can contain several singularities).

Integrability

We want to impose the constraint that the function $I_\gamma(z)$ does not depend on small deformations of the path which preserve the endpoints and such that during the deformation γ doesn't cross any polar loci of the differential forms ω_i. This imposes *integrability conditions*:

$$\sum_{i_1,\dots,i_n} c_{i_1\cdots i_n}(z)\,\omega_{i_1}\otimes\cdots\otimes(\omega_{i_k}\wedge\omega_{i_{k+1}})\otimes\cdots\otimes\omega_{i_n}=0,$$

for $k=1,\dots,n-1$.

This turns out to be a surprisingly strong constraint in practice. The *bootstrap* approach consists in finding a symbol alphabet, writing a general integrable symbol at the required weight (for an L-loop quantity in four dimensions the weight is $2L$) and imposing various physical constraints. In many cases this suffices to determine the symbol uniquely!

Nontrivial integrability example

For $X = \mathbb{C}^2$ and γ a path from $(0,0)$ to (x,y) in $\mathbb{C}^2$ minus the locations of the poles ($x=1,\ y=0,\ y=1,\ xy=1$), consider the iterated integral

$$\mathrm{Li}_{1,1}(x,y) = \int_\gamma \Big[\frac{dy}{1-y}\circ\frac{dx}{1-x} + \big(\frac{dx}{1-x}+\frac{dy}{y(y-1)}\big)\circ\frac{d(xy)}{1-xy}\Big]. \quad (3)$$

The symbol is

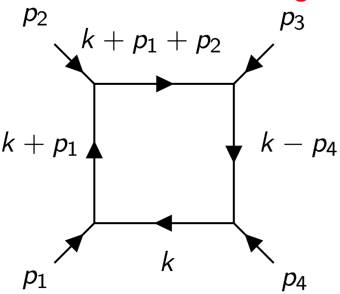
$$(1-y)\otimes(1-x) + (1-x)\otimes(1-xy) + \frac{y}{1-y}\otimes(1-xy) \quad (4)$$

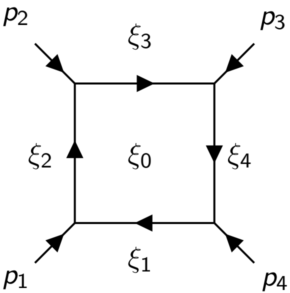
and is integrable

$$\frac{dy}{1-y}\wedge\frac{dx}{1-x} + \big(\frac{dx}{1-x}+\frac{dy}{y(y-1)}\big)\wedge\frac{d(xy)}{1-xy} = 0. \quad (5)$$

Leading singularities from residues

Besides Feynman integrals, sometimes we are interested in computing integrals of various residues of the Feynman integrands. These are called *cut integrals*.





Computing the cuts/residues is equivalent to choosing some special homology cycles which are different from the physical integration contour.

$$\Omega = \frac{d^4 k}{k^2(k+p_1)^2(k+p_1+p_2)^2(k-p_4)^2} = \frac{d^4\xi_0}{(\xi_0-\xi_1)^2\cdots(\xi_0-\xi_4)^2}.$$

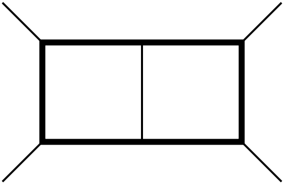
Take a contour defined by $|(\xi_0-\xi_1)^2| = \epsilon$ for $i = 1,\ldots,4$. Then the integral reduces to evaluating

$$\frac{d^4\xi_0}{d(\xi_0-\xi_1)^2\wedge\cdots\wedge d(\xi_0-\xi_4)^2} = \left.\frac{1}{2^4\det(\xi_0-\xi_i)}\right|_*,$$

where $*$ means one of the two roots of the system $(\xi_i-\xi_0)^2 = 0$. The square of the determinant is easy to compute $\big(\det(\xi_{0,*}-\xi_i)\big)^2 = \det(\xi_{0,*}-\xi_i)(\xi_{0,*}-\xi_j) = 2^{-4}\det\xi_{ij}^2$. The leading singularity is then

$$\pm\frac{1}{2^2\sqrt{\det(\xi_i-\xi_j)^2}}.$$

A more complicated case



Impose seven conditions on eight variables we obtain a curve (genus one). If the curve is smooth we get elliptic integrals. But if it is singular, we can make some progress. It can be shown that this curve has a double point singularity $y^2 = x^2(x+1)$, $\omega = \frac{dx}{y}$, after blow-up $y = xu$, the singular point $(x, y) = (0, 0)$ is blown to two points $(x, u) = (0, \pm 1)$ and $\omega = \frac{dx}{xu}$ with $\operatorname{res} \omega = \pm 1$.
Other cases, take seven residues on the propagators, which produce a Jacobian factor which can itself be singular.
Singularities help calculations!

Kinematics

For a four-point scattering process, we have four momenta $p_1, \ldots, p_4 \in \mathbb{R}^{1,D}$ which we take on-shell $p_i^2 = m_i^2$. They satisfy momentum conservation $p_1 + \cdots + p_4 = 0$. We need to consider momenta satisfying these conditions, modulo Lorentz transformations. It is customary to use *Mandelstam invariants* $s = (p_1 + p_2)^2 = (p_3 + p_4)^2$, $t = (p_2 + p_3)^2 = (p_1 + p_4)^2$ (and $u = (p_1 + p_3)^2 = (p_2 + p_4)^2$). The variable u is not independent since $s + t + u = m_1^2 + \cdots + m_4^2$.

The variables s, t, u have dimensions of energy squared. Nontrivial functions can not depend on variables with dimension, so in general we can parametrize the kinematics by ratios of dimensionful quantities $x = \frac{s}{t}$.

Using variables s and t or $\frac{s}{t}$ is convenient but not innocent. Indeed, the change of variables from the space of external momenta $p_1, \ldots, p_4$ such that $p_i^2 = m_i^2$ and modulo the action of the Lorentz group $O(1,3)$ to (s,t) is not one-to-one. Indeed, if $m_1 = m_2$ and $m_3 = m_4$

$$\frac{\partial(s,t)}{\partial(p_1,p_2,p_3)} = 2\begin{pmatrix} p_1+p_2 & 0 \\ p_1+p_2 & p_2+p_3 \\ 0 & p_2+p_3 \end{pmatrix} \tag{6}$$

the Jacobian is not full rank for $p_1 = -p_2$ (and $p_3 = -p_4$) so one can not apply the implicit function theorem.
This doesn't seem to matter much in practice, but perhaps for profound reasons.

Complexification and compactification

Four-dimensional Minkowski space M can be complexified $M(\mathbb{C})$ and that has a nice compactification to $G(2,4)$. It can be constructed as follows: take $x = (x_0, x_1, x_2, x_3)$ and build a matrix

$$X = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix}$$

such that $\det X = g(x,x) = x_0^2 - x_1^2 - x_2^2 - x_3^2$. This 2×2 matrix can be thought of as a two-plane in $\mathbb{C}^4$, via the map $\mathbb{C}^2 \ni \lambda \mapsto (\lambda, \lambda X) \in \mathbb{C}^4$. So we have an embedding $M(\mathbb{C}) \to G(2,4)$. Its image is a cell in $G(2,4)$ where the first two columns form a rank two matrix (in the representation of $G(2,4)$ as a rank two 2×4 matrix modulo the action of $\mathrm{GL}(2)$ on the left).

Momentum twistors

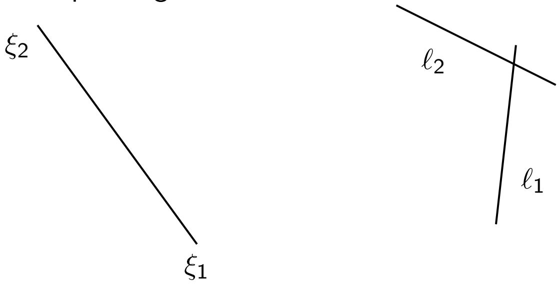
Momentum space and position space are different in a few ways: position space is an affine space while momentum space is a vector space. There is a momentum conservation condition $\sum_i p_i = 0$ which follows from the translation invariance (no preferred origin) in position space. Momentum conservation can be solved by introducing $p_i = \xi_i - \xi_{i+1}$ where now the ξ_i are called *dual coordinates* (sometimes confusingly called x_i in the physics literature).
The dual coordinates are similar to position space coordinates; they belong to an affine space and are not constrained. We can define the compactification of their complexification like before:

$$\Xi = \begin{pmatrix} \xi_0 + \xi_3 & \xi_1 - i\xi_2 \\ \xi_1 + i\xi_2 & \xi_0 - \xi_3 \end{pmatrix}$$

and $\mathbb{C}^2 \ni \lambda \mapsto (\lambda, \lambda\Xi) \in \mathbb{C}^4$.

Momentum twistors

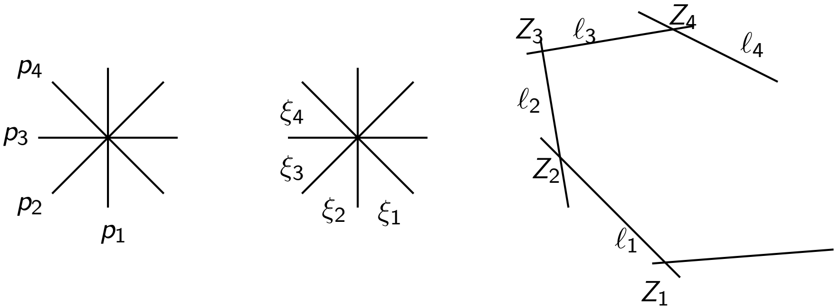
The construction on the previous slide can be projectivized: we can associate to a *dual point* a projective line $\mathbb{P}^1$ inside $\mathbb{P}^3$. This $\mathbb{P}^3$ is called *momentum twistor space*. If two points ξ_1 and ξ_2 are light-like separated $g^{-1}(\xi_1 - \xi_2, \xi_1 - \xi_2) = 0$, then their corresponding lines intersect.



There is a natural action of PSL(4) on $\mathbb{P}^3$. This is isomorphic to Spin$(6, \mathbb{C})$ group which is bigger than the (complexified) Lorentz group SO$(4, \mathbb{C})$. The extra symmetries form *dual conformal group*.[5]

[5] In fact, this is part of a larger symmetry called *dual superconformal* group which is itself part of an even larger symmetry called a *Yangian*.

Momentum twistors



The complexified kinematic space of n massless particles can be rationally parametrized by configurations of n cyclicly[6] ordered points in $\mathbb{P}^3$.

One can now introduce Plücker coordinates $\langle ijkl \rangle$, do cluster mutations, etc.

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[6]Cyclic ordering follows from *color ordering* in gauge theory, which I didn't discuss.

A caveat

The map between

$$\{(p_1, \ldots, p_n) \in (\mathbb{C}^4)^n \mid g^{-1}(p_i, p_i) = 0, \sum p_i = 0\} / \operatorname{SO}(4, \mathbb{C})$$

and

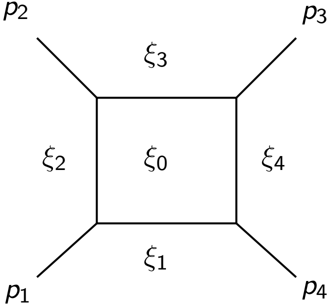
$$\{(Z_1, \ldots, Z_n) \in (\mathbb{P}^3)^n\} / \operatorname{PSL}(4)$$

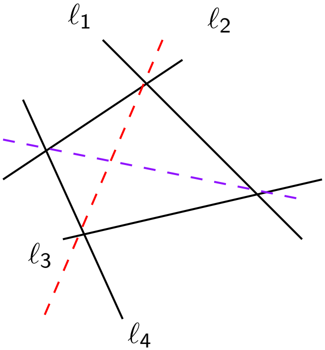
is not quite one-to-one.

Problems arise when some neighboring p_i are collinear. Transition to momentum twistor space looks like a blow-up.

On-shell spaces

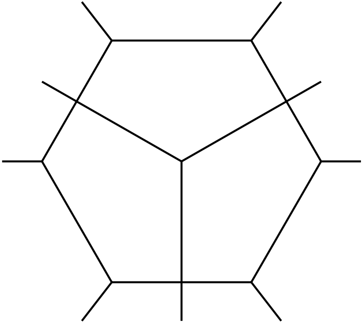
Following Pham, define the on-shell space $\mathcal{S}(G)$ of a planar embedding of a graph G: configurations of lines (points) in $\mathbb{P}^3$ whose intersections are prescribed by the edges of G.





Cayley octads

Sometimes we get some very interesting spaces:



The on-shell space of this graph consists of eight points and can be parametrized rationally (see Plaumann, Sturmfels & Vinzant).

Questions/conjectures

When can the on-shell space be parametrized rationally?
Show that when the on-shell spaces of the contractions of a graph are rational varieties, then the Feynman integral can be written as an iterated integral (with weight at most half of the original integral).

Given a graph G_1 and a contraction to G_2 ($\kappa \colon G_1 \twoheadrightarrow G_2$), there is a projection $\mathcal{S}(\kappa) \colon \mathcal{S}(G_1) \to \mathcal{S}(G_2)$.
Given a map[7] $f : \mathbb{R}^k \to \mathbb{R}^l$ (for $k > l$), we call *critical points* $x \in \mathbb{R}^k$ such that rank $\frac{\partial f_i}{\partial x_j} < l$. If x is a critical point, $f(x)$ is a *critical value*.
The critical values of the map π are connected to *Landau singularities*.
Landau equations: $\sum_{i=1}^{l} \alpha_i \frac{\partial f_i}{\partial x_j} = 0$ for $j = 1, \dots, k$ and for α_i not all vanishing. Semi-algebraic conditions: $\alpha_i \geq 0$ for singularities in the *physical region*.
Further conditions: positivity[8] of the Hessian

$$\left(\sum_{i=1}^{l} \alpha_i \frac{\partial^2 f_i}{\partial x_p \partial x_q} \right)_{1 \leq p,q \leq l} .$$

Such critical points are called S_1^+ in Thom's classification.

[7] Can also be done in local coordinates for maps between manifolds.
[8] To my knowledge not studied at all in momentum twistor geometry.

Reality conditions

In order to study the signature of the Hessian quadratic form we need to impose some *reality conditions*.
We have *dual momentum twistors* $\bar{Z}_i$ which represent the plane containing the lines ℓ_{i-1} and ℓ_i (can be thought of as $Z_{i-1} \wedge Z_i \wedge Z_{i+1}$). The reality condition is

$$\bar{Z}_i = \epsilon_i \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right) Z_i^\dagger,$$

where ϵ_i is the sign of energy for particle i.

Volumes of geodesic simplices in hyperbolic space

Let X be the space of $(n+1)\times(n+1)$ complex symmetric matrices A and consider the diagonal minors $D(I)$ $(I\subset\{1,\ldots,n+1\})$ and non-diagonal minors $D\binom{I}{J}$ with $|I|=|J|$. Given I with $|I|=p$ and J with $|J|=p+2$ and $I\subset J$, consider K_1, K_2 such that $I\subset K_i\subset J$, $|K_i|=p+1$ and define

$$\omega\binom{I}{J}=\frac{1}{2i}d\log\left(\frac{-D\binom{K_1}{K_2}+i\sqrt{D(I)D(J)}}{-D\binom{K_1}{K_2}-i\sqrt{D(I)D(J)}}\right). \qquad (7)$$

Jacobi identity: $D(I)D(J)+D\binom{K_1}{K_2}^2=D(K_1)D(K_2)$ so $\omega\binom{I}{J}$ has poles along $D(K_1)=0$ and $D(K_2)=0$.

Theorem (Aomoto 1977): for odd n the volume of a geodesic simplex is[9]

$$V = \sum_{\substack{\emptyset = I_0 \subset I_1 \subset \ldots \subset I_\nu \\ |I_0|=0,\ldots,|I_\nu|=2\nu}} \sum_{\sigma=0}^{\nu} \frac{\mathrm{Vol}(S^{n-2\sigma})}{2^{n+1-2\sigma}} \int_E^A \omega\binom{I_0}{I_1} \circ \cdots \circ \omega\binom{I_{\sigma-1}}{I_\sigma},$$

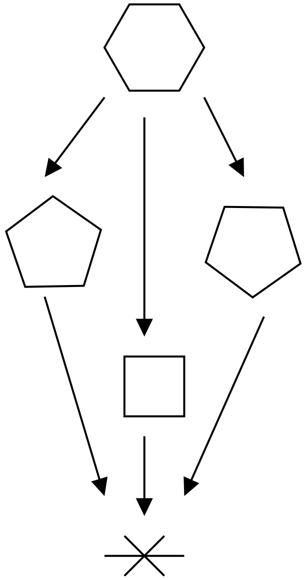
where the path of integration is from the identity matrix E to A. Integrability condition

$$\sum_{\substack{I \subset K \subset J \\ |K|=|I|+2, |J|=|I|+4}} \omega\binom{I}{K} \wedge \omega\binom{K}{J} = 0.$$

[9] $\mathrm{Vol}(S^{-1}) = 1$.

Remarks:

1. Up to some adjustments the same result for D-polygonal one-loop integrals in D dimensions.
2. The structure of the answer is $\pi^{\lceil \frac{n}{2} \rceil} F_{\lfloor \frac{n}{2} \rfloor}$, where F_w is a function of weight w.
3. Even and odd dimensions are very different. For odd dimensions one can apply a Gauss-Bonnet theorem (for manifolds with boundaries and corners).
4. There are square root singularities as well. Aomoto considers a 2^{2^n-1}-covering $\pi \colon \widehat{X} \to X$.



$$V = \sum_{\substack{\emptyset = I_0 \subset I_1 \subset \ldots \subset I_\nu \\ |I_0|=0,\ldots,|I_\nu|=2\nu}} \sum_{\sigma=0}^{\nu} \frac{\mathrm{Vol}(S^{n-2\sigma})}{2^{n+1-2\sigma}} \times \int_E^A \omega\binom{I_0}{I_1} \circ \cdots \circ \omega\binom{I_{\sigma-1}}{I_\sigma}, \quad (8)$$

Thank You!