Quasi-ergodic Theorems for Feynman–Kac Semigroups and Large Deviation for Additive Functionals

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Motivation and setting

With the motivation of characterizing the long-time behavior of symmetric Markov processes under conditioning, through extinction and/or a Feynman-Kac scheme, we study the convergence of associated rescaled additive functionals.

Challenge: How to extend the known quasi-ergodicity results, as in [4], for pure jump effects under the most general conditioning and with unquantified convergence?

Model specification

- Generic space: $(E, \mathfrak{m})$

  Locally compact, separable and metric
- Generic $\mathfrak{m}$-symmmetric strong Markov process, in continuous-time
  $X = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, \mathbb{P}_x, \zeta)$
- μ a signed measure,

  also $\mu = V\mathfrak{m}$, V measurable (particular case)
- $F = F^+ - F^-$, $F^+, F^- \geq 0 \in \mathcal{K}(E\times E)$

  $\mathcal{K}(E\times E)$ the set of Borel measurable functions on $E\times E$ vanishing on the diagonal.
- Additive functional: $A_t^{\mu,F} = A_t^{\mu} + A_t^F$,

  where A_t^{μ} additive functional associated to μ

  $A_t^V = A_t^{V\mathfrak{m}} = \int_0^t V(X_s)\mathrm{d}s$

  $A_t^F := \sum_{0<s\leq t} F(X_{s-}, X_s)$,
- Feynman-Kac (FK) transform:

  $W_t^{\mu,F} = \exp(-A_t^{\mu,F})\mathbf{1}_{\{t<\zeta\}}$

Conditional Measure:

$$\mathbb{P}_{x|t}^{\mu,F}(\Lambda) = \frac{\mathbb{P}_x(\Lambda\cdot W_t^{\mu,F})}{\mathbb{P}_x(W_t^{\mu,F})},\ \Lambda\in\mathcal{F}_t$$

Under (A1), $\mathbb{E}_x[A_t^F, t<\zeta] = \int_0^t p_s(N[F])(x)\,\mathrm{d}s$, for any $x\in E$, $t>0$,
where $N[F](x) = \int_E F(x,y)\,N(x,\mathrm{d}y)$.

Main novelty

Feynman-Kac scheme with extinction ✓
Pure jump effects ✓
Explicit rate function ✓

Main Assumptions

- Kato class conditions: $\mu, N[|F|]\mathfrak{m} \in \mathcal{S}_K^1(X)$
- **(A1)** Lévy system normalization ($\mu_H = \mathfrak{m}$)
- **(A2)** X explosive with (I), (SF), (T):

  (I), Irreducibility:

  $p_t(\mathcal{B}) = \mathcal{B} \Rightarrow [\mathfrak{m}(\mathcal{B}) = 0 \text{ or } \mathfrak{m}(E\setminus\mathcal{B}) = 0]$

  (SF), Strong Feller: $p_t(\mathcal{B}_b(E))\subset C_0(E)$

  (T), Tightness: for any $\epsilon>0$ there exists a compact K s.t.:

  $\sup_{x\in E}\int_0^\infty e^{-t}\mathbb{P}_x(X_t\notin K, t<\zeta) < \epsilon$
- **(A3)** FK semigroup intrinsically ultracontractive (IUC), i.e. for all $t>0$ and $x,y\in E$:

  $$p_t^{\mu,F}(x,y) \leq c_t\phi_0^{\mu,F}(x)\phi_0^{\mu,F}(y),\ c_t>0.$$
- **(A3)$_\theta$**: IUC with $\theta V, \theta F$ for any $\theta\leq 0$ instead of μ, F.

Known Implications

Under $(A1)$ and $(A2)$, by [3, Corollary 1.1]:
Ground state change of measure:

- $\lambda_0^{\mu,F} = \inf\{\mathcal{E}^{\mu,F}(u,u) : \|u\|_2 = 1\}$,

  where there exists a unique minimizer $\phi_0^{\mu,F}$ of

  $$\mathcal{E}^{\mu,F}(u,v) = \mathcal{E}(u,v) + \int_E uv\mathrm{d}\mu$$
  $$+\iint_{E\times E} u(x)v(y)\left(1-e^{-F(x,y)}\right)N(x,\mathrm{d}y)\mathfrak{m}(\mathrm{d}x)$$
- $\phi_0 = \phi_0^{\mu,F}$ strictly positive and continuous ground state
- X^{ϕ_0} conservative and ergodic, so

  $$\lim_{t\to\infty}\phi_0(x)p_t^{\phi_0}\left(\frac{f}{\phi_0}\right)(x) = \phi_0(x)\int_E f\,\phi_0\,\mathrm{d}\mathfrak{m},$$

for any $x\in E$ and $f\in L^1(E;\phi_0\mathfrak{m})$.

Main Results

1. Quasi-Ergodic Theorem:

$$\lim_{t\to\infty}\mathbb{E}_{x|t}^{\mu,F}\left[\frac{1}{t}A_t^{V,G}\right] = \int_E V(x)\nu^{\mu,F}(\mathrm{d}x) + \iint_{E\times E} G(x,y)\mathcal{J}^{\mu,F}(\mathrm{d}x\mathrm{d}y),$$

for any $x\in E$, $V\in L^1(E;\nu^{\mu,F})$ and any $G\in\mathcal{K}(E\times E)$ satisfying $\iint_{E\times E}|G(y,z)|\mathcal{J}^{\mu,F}(\mathrm{d}y\mathrm{d}z)<\infty$,
besides relevant Dynkin conditions: $V, N[|G|], N[|G|^T]\in\mathcal{S}_D^1(X)$,
where $V\in\mathcal{S}_D^1(X)$ if $\sup_{x\in E}\mathbb{E}_x\left[|A_t^V|\right]<\infty$ for some, hence for any $t>0$.

2. Weak Law of Large Numbers:

$$\lim_{t\to\infty}\mathbb{P}_{x|t}^{\mu,F}\left(\left|\frac{1}{t}A_t^{V,G} - \left(\int_E V\mathrm{d}\nu^{\mu,F} + \iint G\mathrm{d}\mathcal{J}^{\mu,F}\right)\right|\geq\varepsilon\right) = 0,$$

for any $\varepsilon>0$, $x\in E$, and V, G as above with in addition $N[G^2], N[(G^2)^T]\in L^1(E;\mathfrak{m})\cap\mathcal{S}_D^1(X)$.

Key quantities:

$\nu^{\mu,F}$ = Quasi-Ergodic Distribution: $\phi_0^2\mathfrak{m}$, $\qquad\mathcal{J}^{\mu,F}$= Quasi-Ergodic Jump Distrib.

$$\mathcal{J}^{\mu,F}(\mathrm{d}x\mathrm{d}y) = \phi_0(x)\phi_0(y)e^{-F(x,y)}N(x,\mathrm{d}y)\mathfrak{m}(\mathrm{d}x) = N^{\phi_0}(x,\mathrm{d}y)\,\phi_0^2(x)\mathfrak{m}(\mathrm{d}x).$$

3. Large Deviation Principle:

Under our main assumptions, $C_{V,F}(\theta) := -\lambda_0^{\theta V,\theta F}$ is strictly convex.
Furthermore, the following holds for any $x\in E$ and any $\gamma\in\Psi_{V,F}(\mathbb{R}_-^1)^o$:

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{P}_x\left(\frac{A_t^{V,F}}{t}\in[\gamma,\infty),\ t<\zeta\right) = C_{V,F}(\theta_\gamma) - \theta_\gamma C'_{V,F}(\theta_\gamma), \tag{1}$$

in which θ_γ is the non-positive real number given by $\theta_\gamma = \Psi_{V,F}^{-1}(\gamma)$ and where we have defined

$$\Psi_{V,F}(\theta) = \int_E V(x)\phi_0^{(\theta)}(x)^2\mathfrak{m}(\mathrm{d}x) + \iint_{E\times E}F(x,y)\mathcal{J}_{\phi_0^{(\theta)}}(\mathrm{d}x\mathrm{d}y),\qquad \phi_0^{(\theta)} = \phi_0^{\theta V,\theta F}.$$

The r.h.s. of (1) is also identified as:

$$-\mathcal{A}\left(\phi_0^{(\theta_\gamma)},\phi_0^{(\theta_\gamma)}\right) = -\mathcal{E}\left(\phi_0^{(\theta_\gamma)},\phi_0^{(\theta_\gamma)}\right)$$
$$-\iint_{E\times E}\phi_0^{(\theta_\gamma)}(x)\phi_0^{(\theta_\gamma)}(y)\left(1-e^{-\theta_\gamma F(x,y)} - \theta_\gamma F(x,y)e^{-\theta_\gamma F(x,y)}\right)N(x,\mathrm{d}y)\mathfrak{m}(\mathrm{d}x).$$

Crucial estimates

$$\Phi_t(x) = e^{\lambda_0 t}\mathbb{E}_x[W_t]\to\phi_0(x)\int\phi_0\mathrm{d}\mathfrak{m}$$

$$\widehat{p}_t^{\mu,F}f(x) = e^{\lambda_0 t}\mathbb{E}_x[W_t f(X_t)]\to\phi_0(x)\int f\phi_0\mathrm{d}\mathfrak{m},$$

$$\Phi_t(x)\mathbb{E}_{x|t}\left[A_t^G\right] = \int_0^t\widehat{p}_s^{\mu,F}\left(N\left[Ge^{-F}\cdot\Phi_{t-s}\right]\right)(x)\mathrm{d}s,$$

$$\Phi_t(x)\mathbb{E}_{x|t}\left[\left(A_t^G\right)^2\right] = \int_0^t\widehat{p}_s^{\mu,F}\left(N\left[G^2e^{-F}\cdot\Phi_{t-s}\right]\right)(x)\mathrm{d}s$$
$$+2\int_0^t\int_0^{t-s}\widehat{p}_s^{\mu,F}\left(N\left[Ge^{-F}\widehat{p}_r^{\mu,F}\left(N\left[Ge^{-F}\Phi_{t-s-r}\right]\right)\right]\right)(x)\,\mathrm{d}r\mathrm{d}s.$$

Applications

⋆ Symmetric Lévy processes with explicit criteria inspired by [2, 3], notably:
∗ Rotationally symmetric α-stable process on $\mathbb{R}^d$.
∗ ⊥ Mixture of two of those ($0<\alpha<\beta<2$).
∗ ⊥ Mixture of α-stable with a Brownian Motion
∗ Symmetric stable-like Lévy processes on $\mathbb{R}^d$
under condition of confining potential

□ Eco-evolutionary models (mutation profiles)?
□ Concentration of parametric estimators for the jump rate?

References

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