On the LAN property for mean-field model of interacting neurons.

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Motivation and setting

In biological neural networks, neurons interact through discrete **spike events**, which are brief and easily identified transitions triggering excitation across the system. In the **mean-field regime**, with a large number of neurons, these interactions can be approximated by effective average fields, simplifying the analysis of large-scale dynamics.

Objective: relate the inference of the spiking rate parameters to the mean-field reduction

Model specification

- N neurons ($N \gg 1$) with one-dimensional internal states, that evolve deterministically between spikes
- A spike triggers uniform mean-field boost on other neurons' states, and possibly a large effect on the spiking neuron

Inference objective

- Full observation: the complete trajectory $X^{(N)}$ over fixed a time-interval $[0,t]$
- The d-dimensional parameter θ^* controlling the spiking intensity is to be estimated by $\hat{\theta}_t^N$, the Maximum Likelihood Estimator (MLE)
- Characterize the performance of the MLE by leveraging asymptotic properties of the likelihood as $N \to \infty$.

Theoretical Guarantees

Consistency

$\hat{\theta}_t^N \to \theta^*$ as $N \to \infty$, under $\mathbb{P}_N^{\theta^*}$ for any $\theta^* \in \mathring{\Theta}$.

Asymptotic normality with minimal variance

$$\sqrt{N}(\hat{\theta}_t^N - \theta^*) \xrightarrow{\mathcal{L}} \mathcal{N}(0, (I_t^{\theta^*})^{-1}),$$

under $\mathbb{P}_N^{\theta^*}$, where $I_t^{\theta^*}$ is the Fisher information matrix.

Efficiency

$\hat{\theta}_t^N$ achieves the Hájek lower bound for regular estimators, with local and global criteria.

References

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Model description

We consider the following model of interacting neurons, where each neuron spikes according to the **jump rate** f_θ that relates to its internal state (representing its membrane potential) through some parameter $\theta \in \mathring{\Theta}$, Θ being a compact set of $\mathbb{R}^d$.

In words, the state of neuron i evolves **between jumps** according to the **flow** driven by b. At a **spike** of neuron i, which happens **at rate** $f_\theta(X_{s-}^{i,N})$ at time s, the state of **neuron** i **is shifted by** $\varphi(X_{s-}^{i,N})$ and of the **all other neurons by** U/N, where U is sampled at each spike with distribution ν. The probability measure ν has finite second moment, i.e. $\int_{\mathbb{R}} u^2 v(\mathrm{d}u) < \infty$.

This is summarized in the following system, where $i \in [\![1,N]\!]$, $s \in [0,t]$:

$$X_s^{i,N} = X_s^{i,N,\theta^*} = \int_0^s b(X_r^{i,N})\mathrm{d}r + \int_{[0,s]\times\mathbb{R}_+\times\mathbb{R}} \varphi(X_{r-}^{i,N})\,\mathbf{1}_{\{z \le f_\theta(X_{s-}^{i,N})\}}\pi^i(\mathrm{d}r,\mathrm{d}z,\mathrm{d}u)$$
$$+ \frac{1}{N}\sum_{j\neq i}^{N} \int_{[0,s]\times\mathbb{R}_+\times\mathbb{R}} u\,\mathbf{1}_{\{z \le f_{\theta^*}(X_{r-}^{N,j})\}}\pi^j(\mathrm{d}r,\mathrm{d}z,\mathrm{d}u)$$

Here, (π^j), $j \in \mathbb{N}^*$ is a family of i.i.d. Poisson Random Measures on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$ with intensity $\mathrm{d}s\,\mathrm{d}z\,\nu(\mathrm{d}u)$. The interaction term is close to $\int_{\mathbb{R}} u\,\nu(\mathrm{d}u) \cdot \int_0^s \mathbb{E}[f_{\theta^*}(X_r^{i,N})]\,\mathrm{d}r$ (mean-field reduction).

You may have in mind $b(x) = -\alpha x$ (exponential leakage), $\phi(x) = -x$ (reset of the neuron state to 0 after the spike), f_θ being a sigmoid function from 0 to a saturation level $\theta_1 > 0$, a threshold level θ_2 on a range of potential specified by θ_3.

Main results

⋆ Under standard regularity and boundedness assumptions, the following **local asymptotic minimax lower-bound** holds for any loss function $w : \mathbb{R}^p \to [0,\infty)$ which are continuous, subconvex and bounded, for any sequence of estimators $(\tilde{\theta}_t^N)$ which estimation errors in θ^* are tight at rate $\sqrt{N}$:

$$\liminf_{C\to\infty}\liminf_{N\to\infty} \sup_{\sqrt{N}|\theta-\theta^*|\le C} \mathbb{E}_{\mathbb{P}_N^\theta}\Big[w\Big((N\,I_t^{\theta^*})^{1/2}\cdot(\tilde{\theta}_t^N - \theta)\Big)\Big] \ge \frac{1}{(\sqrt{2\pi})^d}\int_{\mathbb{R}^d} w(x)\,e^{-\frac{1}{2}\|x\|^2}\mathrm{d}x\,. \qquad (1)$$

On the other hand, the MLE $(\hat{\theta}_t^N)_{N\ge1}$ is **consistent** and **asymptotically normal with minimal variance**. In addition, **local asymptotic minimax optimality** holds in the sense that equality is achieved in (1) with $\tilde{\theta}_t^N = \hat{\theta}_t^N$.

⋆ In the absence of large jumps ($\varphi \equiv 0$), **local asymptotic minimax optimality** holds on larger domains and for a class of possibly unbounded loss functions w, in the following sense:

$$\limsup_{N\to\infty} \sup_{|\theta-\theta^*|\le\delta} \mathbb{E}_{\mathbb{P}^\theta}\Big[w\Big((N\,I_t^{\theta^*})^{1/2}\cdot(\hat{\theta}_t^N - \theta)\Big)\Big] \to \frac{1}{(\sqrt{2\pi})^d}\int_{\mathbb{R}^d} w(x)\,e^{-\frac{1}{2}\|x\|^2}\mathrm{d}x$$

as $\delta \to 0$, where w is continuous, subconvex and polynomial.
Global asymptotic minimax optimality also holds for such loss functions w:

$$\mathcal{R}_w^N(\hat{\theta}_t^N;\Theta_0) = \inf_{\tilde{\theta}_N} \mathcal{R}_w^N(\tilde{\theta}_N;\Theta_0)\,(1+o(1))\,,$$

where the risk is defined as $\mathcal{R}_w^N(\tilde{\theta}_N;\Theta_0) = \sup_{\theta\in\Theta_0}\mathbb{E}_{\mathbb{P}^\theta}\Big[w\Big((N\,I_t^{\theta^*})^{1/2}\cdot(\tilde{\theta}_N - \theta)\Big)\Big]$.

Asymptotic behavior of the likelihood

Limiting distribution

By propagation of chaos results as $N \to \infty$, we identify $\bar{\mu}_t = \mathcal{L}(\bar{X}_t)$ as the limit of the empirical distribution of the $(X_t^{i,N})_{i\in[\![1,N]\!]}$, for any $t > 0$, and

$$\frac{1}{N}\log L_t^{N,\theta\,|\,\theta^*} \to A_t^{\theta^*}[\theta] = \int_0^t \bar{\mu}_s^{\theta^*}\Big[\Big(\log\frac{f_\theta}{f_{\theta^*}} - \frac{f_\theta}{f_{\theta^*}} + 1\Big)\cdot f_{\theta^*}\Big]\,\mathrm{d}s\,,$$

where $A_t^{\theta^*}$ is non-positive and equals 0 only at θ^* under typical identifiability criterion.

LAN property

$$\log L_t^{N,\theta_h^N\,|\,\theta^*} = h^\top\Delta_t^{N,\theta^*} - \frac{1}{2}h^\top I_t^{\theta^*}h + o_{\mathbb{P}}(1)$$

with $\theta_h^N = \theta^* + h/\sqrt{N}$, $\Delta_t^{N,\theta^*} \xrightarrow{\mathcal{L}} \mathcal{N}(0, I_t^{\theta^*})$, $I_t^{N,\theta^*} \xrightarrow{\mathbb{P}} I_t^{\theta^*}$. As the second identifiability criterion, we require that $I_t^{\theta^*} = \int_0^s \bar{\mu}_s^{\theta^*}\Big[\frac{(\dot{f}_{\theta^*})^{\otimes 2}}{f_{\theta^*}}\Big]\,\mathrm{d}s$ is strictly positive definite.
Then, the experiment behaves locally as a Gaussian shift.

Expression of the likelihood

$$L_t^{N,\theta\,|\,\theta^*} = \frac{\mathrm{d}\mathbb{P}^{N,\theta}_{|[0,t]}}{\mathrm{d}\mathbb{P}^{N,\theta^*}_{|[0,t]}}(X^{(N)}) = \prod_{i=1}^N\Big[\prod_{n\ge1:T_n^i\le t}\frac{f_\theta}{f_{\theta^*}}(X_{T_n^i-}^{(i,N)})\Big]\cdot\exp\Big[-\sum_{j=1}^N\int_0^t(f_\theta - f_{\theta^*})(X_s^{(j,N)})\mathrm{d}s\Big],$$

where $(T_n^i)_{n\ge1}$ for $i \in [\![1,N]\!]$ is the ordered sequence of jump times of $X^{(i,N)}$.