

# Large deviations on the profile of mutation effects for a model of populations adapting to a changing environment

Aurelien Velleret, E. Pardoux and M. Kopp (PhD advisors), D. Kim (host researcher)  
aurelien.velleret@nsup.org

## Introduction

In this elementary model for the adaptation of a population to changing environmental conditions, the main focus is to gain more insight in the contribution of the various mutation effects.

Averaged over a long time, by looking back at the ancestral line of a surviving population, a specific profile of mutation effects shall arise. So we want to investigate how this profile is constrained.

For instance : how exceptional is it for large mutations to have a larger contribution than expected.

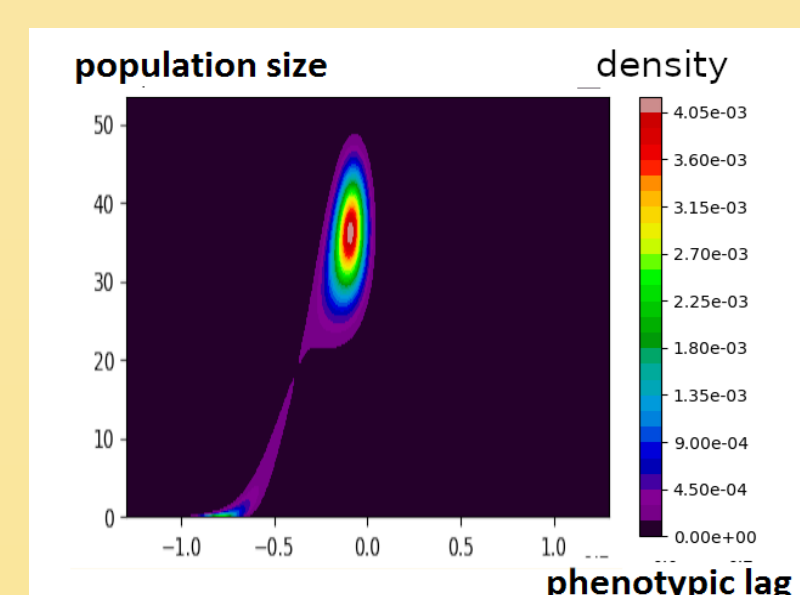
## Ecological aspects

- $(N_t)_{t \geq 0}$  : size of the population
- $(X_t)_{t \geq 0}$  : gap relative to the moving optimum we assume that the growth rate only depends on this adaptation coordinate  $X_t$

Environmental change :

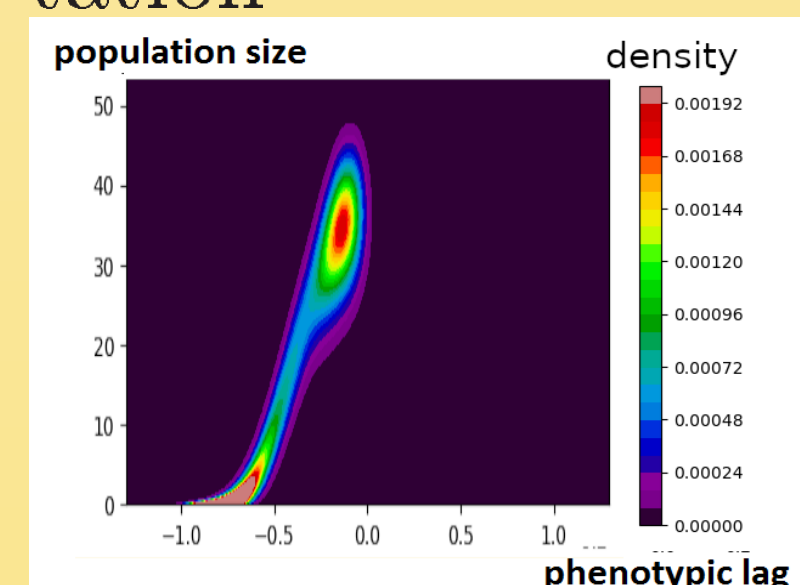
- for simplicity, translation of the growth rate profile at constant speed  $v$
- compensated by the fixation of mutations in the population

## Stability of the adaptation

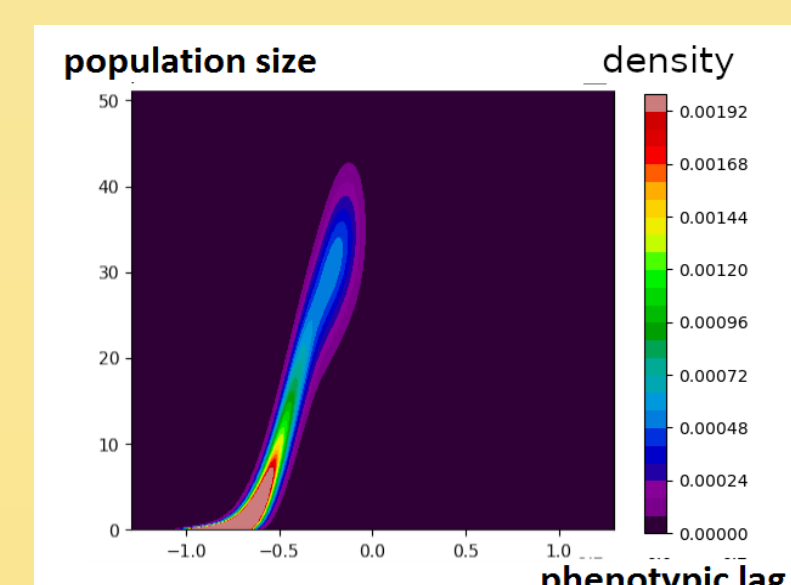


Law of the surviving populations at time  $t$   
 $\mathcal{L}(X_t, N_t | t < \tau_\partial) \sim \alpha$

1. Spontaneous adaptation



2. Intermediate regime



3. Strong survivor effect

## References

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- [2] Kim, D., Kuwae, K., Tawara, Y.: Large deviation principle for generalized Feynman-Kac functionals and its applications. Tohoku Math. J. 68(2), 161-197 (2016)
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- [4] Nassar, E, Pardoux, E; On the large-time behaviour of the solution of a stochastic differential equation driven by a Poisson point process. Advances in Appl. Probab., 49(2), 344-367. (2017)
- [5] Velleret, A.; Unique quasi-stationary distribution, with a possibly stabilizing extinction; preprint on ArXiv : https://arxiv.org/abs/1802.02409 (2018)
- [6] Velleret, A.; Exponential quasi-ergodicity for processes with discontinuous trajectories; preprint on ArXiv : https://arxiv.org/abs/1902.01441 (2019)
- [7] Velleret, A.; Adaptation of a population to a changing environment under the light of quasi-stationarity, preprint available on ArXiv at : https://arxiv.org/abs/1903.10165

## Formalization of the stochastic process

Extinction happens at time  $\tau_\partial$  as soon as  $N_t$  reaches 0 (no more individuals in the population),

$$\text{with } N_t = n + \int_0^t (r(X_s) N_s - c_p (N_s)^2) ds + \sigma \int_0^t \sqrt{N_s} dB_s,$$

where  $B$  is a Brownian Motion and  $r(x) \xrightarrow{|x| \rightarrow \infty} -\infty$  (extreme values of  $X_t$  are not viable)

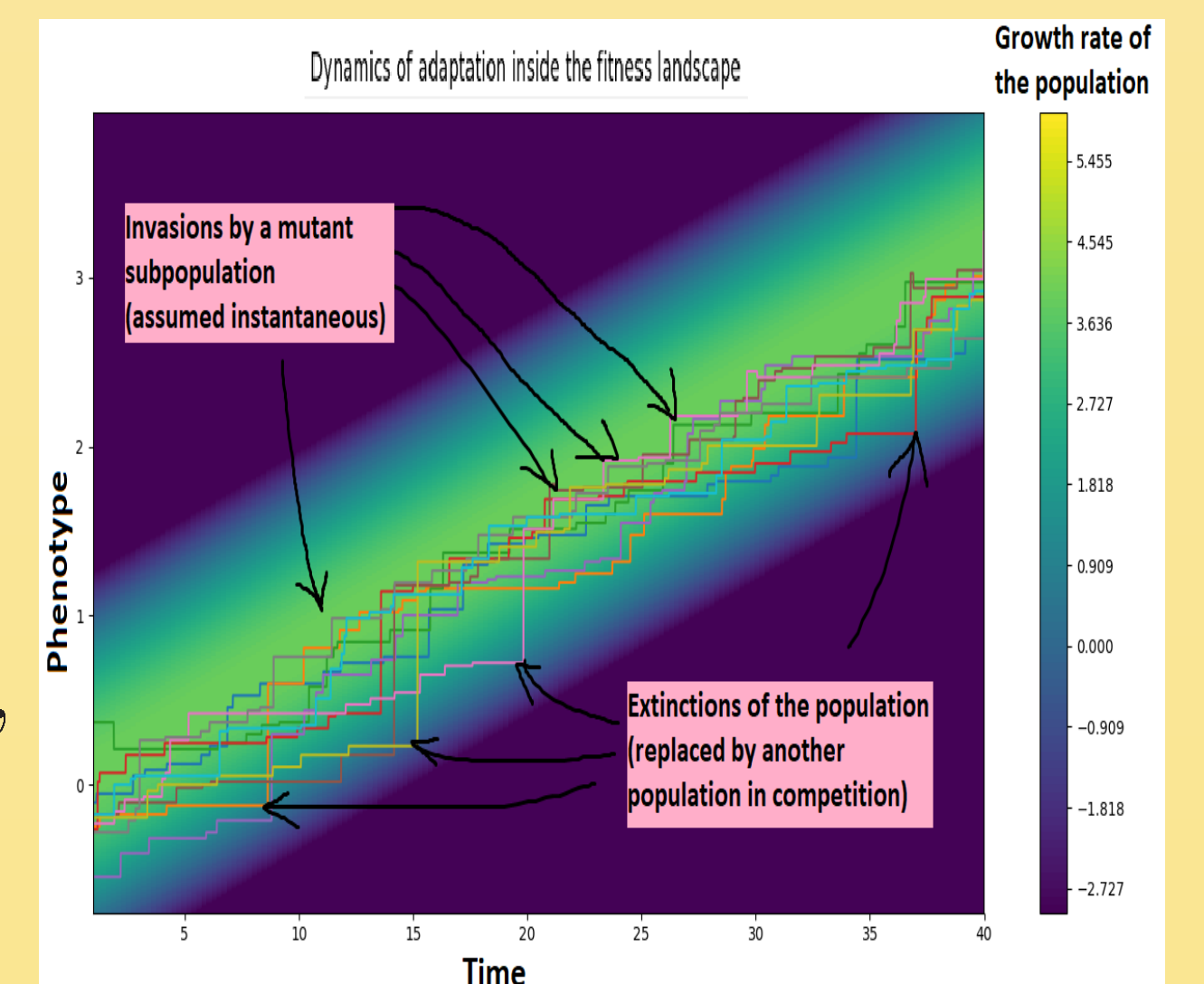
General model for adaptation :

$$X_t = x - vt + \int_{[0,t] \times \mathbb{R}^d \times \mathbb{R}_+} w \mathbf{1}_{\{u \leq f(N_s) g(X_{s-}, w)\}} M(ds, dw, du),$$

where  $M$  is a Poisson Point Process of intensity  $ds \nu(dw) du$ .

Another description of  $X$  when  $f$  is bounded ( $g \leq 1$ ) :  
Let  $M = \{T_i\}_{i \geq 1}$  be a Poisson Point Process with intensity  $\|f\|_\infty$ , ie inter-time are exponential r.v. with average  $1/\|f\|_\infty$ ;  
 $(W_i)_{i \geq 1} \sim \nu(dw)$  be iid rv (effect of the mutations);  
 $(U_i)_{i \geq 1} \sim \mathcal{U}([0, 1])$  be iid rv (filtering of the proposal) :

$$X_t = x - vt + \sum_{T_i \leq t} W_i \mathbf{1}_{\{U_i \leq \frac{1}{\|f\|_\infty} f(N_{T_i}) g(X_{T_i-}, W_i)\}}$$



Population following the environmental change

## An expected profile of mutation effects

Convergence to a unique profile, independent of the initial condition  $\mu \in \mathcal{M}_1(\mathbb{R}^d \times \mathbb{R}_+^*)$  : Whatever the test function  $h : w \rightarrow \mathbb{R}$ , and  $\epsilon > 0$  :

$$\mathbb{P}_\mu [|\Delta_t(h)| > \epsilon | t < \tau_\partial] \xrightarrow{t \rightarrow \infty} 0$$

$$\text{where } \Delta_t(h) := \frac{1}{t} \sum_{T_i \leq t} \mathbf{1}_{\{U_i \leq \frac{1}{\|f\|_\infty} f(N_{T_i}) g(X_{T_i-}, W_i)\}} h(W_i)$$

$$- \int_{\mathbb{R}^d \times \mathbb{R}_+^*} \alpha(dx, dn) \int_{\mathbb{R}^d} \nu(dw) \eta(x+w, n) f(n) g(x, n) h(w)$$

Here, the law  $\alpha$  describes a typical surviving population at a large time  $T$ , while the correction  $\eta(x+w, n)$  precises how likely it is for the population after the fixation to still be surviving until time  $t$ .

## Large deviation results (to be done)

What we'd like to prove is that at least for some test functions  $h$ , we can measure these deviations :

$$\mathbb{P}_\mu [\Delta_t(h) \geq \epsilon | t < \tau_\partial] \sim C(h, \epsilon) \exp[-I(h, \epsilon) t].$$

From the literature on Large Deviation theory, such an estimate of  $I(h, \epsilon)$  is related to the "quasi-stationarity" of  $(X_t, N_t)$  when the natural probability law  $\mathbb{P}_\mu$  is biased to favor larger values of  $h$  (see notably [2]).

It means that for each successful invasion, with effect  $w$ , in a given history, we give to the probability of it happening a bonus  $\exp[\zeta h(w)]$  and renormalize to get a probability law.  $\zeta > 0$  is then the critical value for which  $\Delta_t(h) \geq \epsilon$  is to be expected. Hopefully, the "quasi-stationarity" of the biased process shall rely on the same criteria as for the unbiased process (see [7], based on [5], [6],[1]).

## Unbiased profile

Mutations with predicted proportion with the blue crosses, realized with purple dots (one simulation, about 1500 jumps), with as a reference the profile  $\nu(dw)$  of occurring mutations.

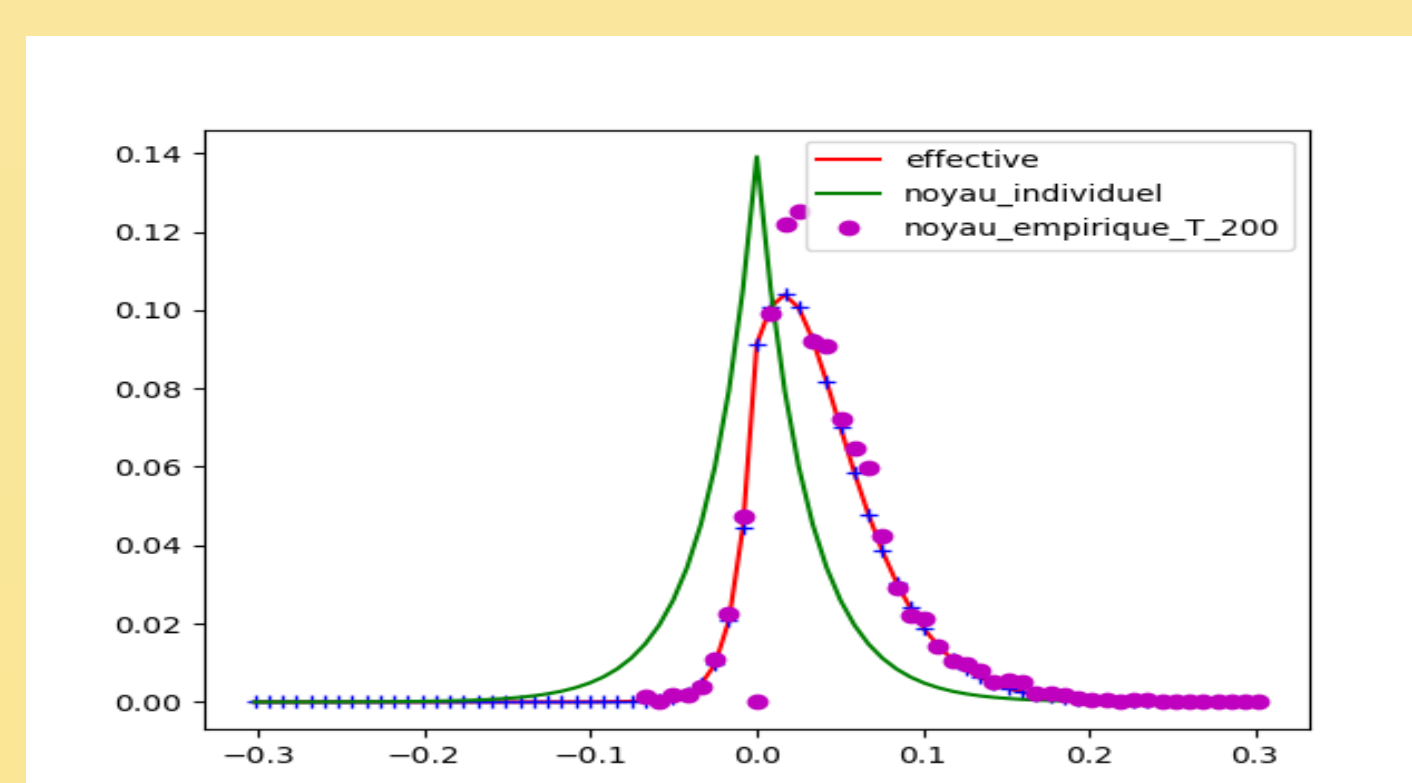
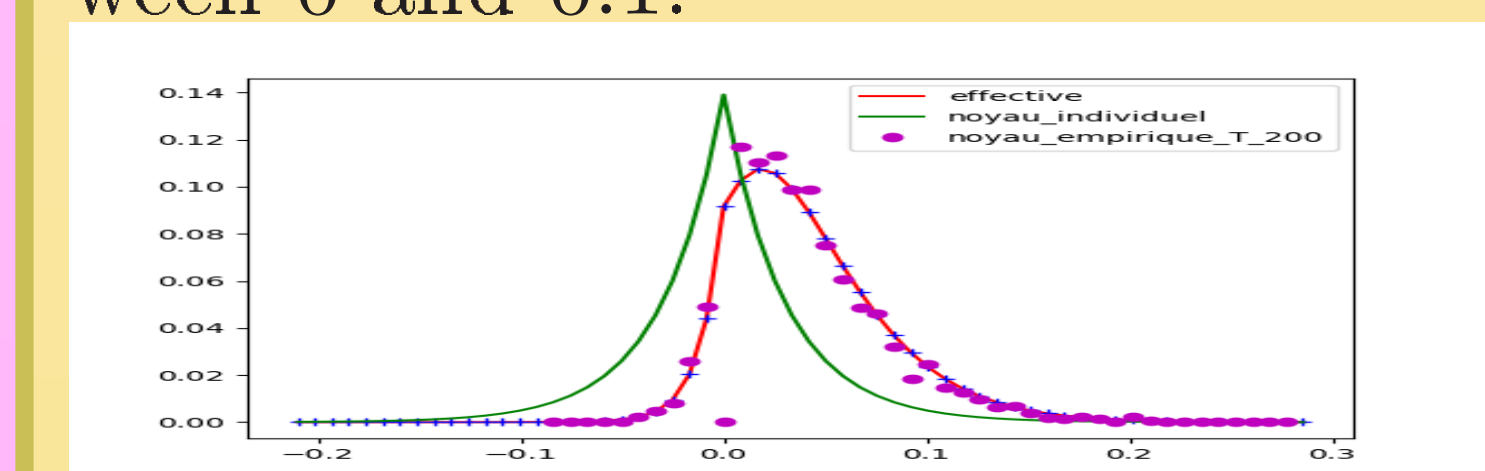


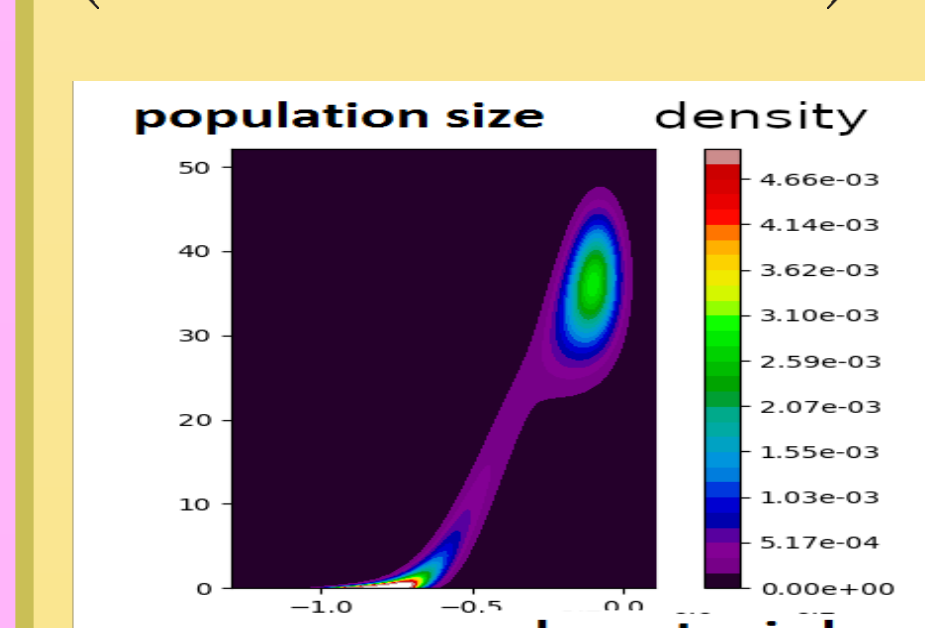
Figure 1: Histogram of the observed mutations Case 1. gives the law of the surviving populations.

## Biased profile

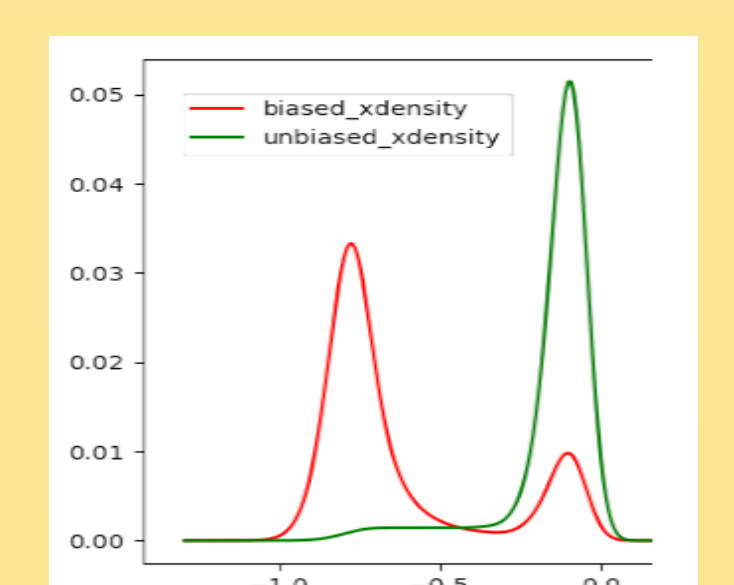
Same profile when the probability gets a multiplicative penalty  $p = 0.7$  for any mutation between 0 and 0.1.



Similar profile of biased mutations ! (but less numerous)



Associated  $\alpha$



Repartition along the phenotypic lag