Infection dynamics between cities under a lockdown policy

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joint work in progress
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Motivations 1/2

Analysis of local confinement strategies.

We have in mind a unified area with many cities
and a common lockdown strategy.

Three main aspects to validate the efficacy:

- early phase of propagation
- potential eradication of the disease?
  even when supercritical in most cities!
- existence, stability and comparison of equilibrium in the SIRS configuration (not addressed further in the talk)

Motivations 2/2

- Cities as the core of transmissions -> the nodes of the network
- Heavy-tailed distribution of city sizes
  the tail follows a power-law distribution
  see the Communes in France and Gemeinde in Germany
- What reference for the lockdown restrictions?
  Uniform ("variant U") vs Proportional ("variant P")
- Range of the lockdown restrictions
- Rates of transmission potentially not symmetrical
  (regarding their relation to city sizes)

Outline

Outline

The graph of cities

For simplicity, the model is not spatially explicit.

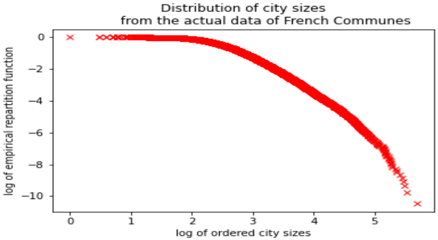
As a reference for the city size distribution β

- the data of around 35,000 French Communes (INSEE)
- the data of around 4400 German Gemeindeverbände (GENESIS)
- power-law distributions like:

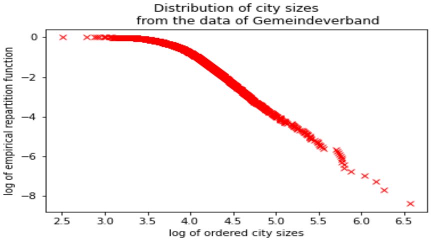
$$\beta(dx) := \frac{1_{\{x > x_L\}}}{Z} x^{-\phi} dx, \text{ for some } x_L > 0 \text{ and } \phi > 1.$$

$\phi > 2$ means finite first moment, $\phi > 3$ means finite variance
We will look at the effect of changing the number of cities, .

The graph of cities



City size distribution of Communes,
$\hat{\phi} \sim 2.08$



and of Gemeindeverbände,
$\hat{\phi} \sim 2.14$

The types of transmission

2 types of transmission between cities:

- An infected citizen from city X generates a new infection while he/she is visiting city Y.
  We say that Y gets infected from inside.
- A citizen from city Y is infected while he/she is visiting the infected city X and comes back propagating the disease in city Y.
  We say that Y gets infected from outside.

The rates of transmission

The kernel of contact interactions is simplified. It depends simply on the city sizes x and y of resp. X and Y as:

$$k(x, y) := k_0\, x^{1+b} y^a + k_0\, x^a y^{1+b}, \text{ for some } a, b \in \mathbb{R}.$$

- a bias towards visiting large cities (neutral is $a = 1$ but $a > 1$ expected)
- b bias towards more travels for citizen of larger cities (neutral $b = 0$, uncertain whether $b > 0$ or $b < 0$ is more valid)

The rates of transmission

An effective regulation must prevent outwards infections with only a few left.
The links to an infected city are simply erased.
Assume there are I_X infected people in city X at the lockdown time.

- Probability that city Y gets infected from inside by citizen of X:
  -> proportional to $I_X \cdot x^b \cdot y^a$
- Probability that city Y gets infected from outside by citizen of X:
  -> proportional to $(I_X/x) \cdot x^a \cdot y^{1+b}$

The lockdown strategy

Complete lockdown assumed. No more transmission afterwards (R state)

In case of a regulation at the scale of single cities:

- Variant "P" for Proportional, most commonly exploited:
  the threshold is set in terms of incidence rate
  -> for some p to be adjusted, $I_X \sim p.x$
- Variant "U" for Uniform, exploited in zero-covid strategies:
  the threshold is set in absolute values
  -> $I_X \sim K$ independent of the infected city X

Natural extension when cities are gathered into larger administrative units like the French departements or the German Landkreise.

Interval duration can be introduced (exposed period, lockdown period).

Comparing lockdown strategy

Which relation between p and K?

- coincide on average when choosing a city at random, i.e.

$$K := \int_{\mathbb{R}_+} p\, x\, \beta(dx).$$

- coincide on average when choosing an individual at random, i.e. :

$$K := \frac{\int_{\mathbb{R}_+} p\, x^2 \beta(dx)}{\int_{\mathbb{R}_+} x\, \beta(dx)}.$$

Outline

Branching process approximation

Cities infected by infectious citizen of city X have size distribution:
$M_A^{(X)}(dy) + M_B^{(X)}(dy)$
Poisson Random Measures with intensity measure
resp. $K_A^{(X)} \cdot \nu_A(dy)$ and $K_B^{(X)} \cdot \nu_B(dy)$ where

$$K_A^{(X)} := k_I \, I_X \, x^b \int_{\mathbb{R}_+} y_2^a \beta(dy_2), \qquad K_B^{(X)} := k_I \, (I_X/x) \cdot x^a \int_{\mathbb{R}_+} y_2^{1+b} \beta(dy_2),$$

$$\nu_A(dy) := \frac{y^a \beta(dy)}{\int_{\mathbb{R}_+} y_2^a \beta(dy_2)}, \qquad \nu_B(dy) := \frac{y^{1+b} \beta(dy)}{\int_{\mathbb{R}_+} y_2^{1+b} \beta(dy_2)}.$$

A two-type reduction of the infection pattern

The system is projected in a two-dimensional subspace:

- ν_A, ν_B city size distributions of cities infected resp. from inside and from outside
- $K_A^{(X)}, K_B^{(X)}$ average number of cities infected resp. from inside and from outside through X
- 2×2-matrix P with entries:

$$P_{AA} = \int \nu_A(dx) K_A^x, \quad P_{AB} = \int \nu_A(dx) K_B^x,$$
$$P_{BA} = \int \nu_B(dx) K_A^x, \quad P_{BB} = \int \nu_B(dx) K_B^x.$$

A two-type reduction of the infection pattern

This reduction to a two-type dynamics extends
when administrative units are considered.
-> administrative units infected from inside vs from outside

Yet, much more debatable how the infected citizen are dispersed
inside the administrative unit.

Comparison of the R_0

For $u > 0$, let $\mathcal{I}_u := \int x^u \beta(dx)$.
Consider the relation $K = p \cdot \mathcal{I}_1$
To show that $R_0^{(U)} \leq R_0^{(P)}$ it is enough to show that $\mathcal{I}_1 r_1^A \leq r_1^B$, where:

$$r_1^A = \mathcal{I}_{a+b} + \sqrt{\mathcal{I}_{2a-1}\mathcal{I}_{2b+1}}, \quad r_1^B = \mathcal{I}_{a+b+1} + \sqrt{\mathcal{I}_{2a}\mathcal{I}_{2b+2}}.$$

This is generally deduced from Hölder's inequality
provided $a + b \geq 1$, $a \geq 1/2$ and $b \geq -1/2$.
The inequality appears to hold more generally, from our simulation results.
Around $a = 1$, $b = 0$, $\phi = 2.1$, $R_0^{(P)}$ many more times larger than $R_0^{(U)}$.

Outline

Method of comparison

Studied essentially in the variant "U", more regular.

Model adjusted to have theoretically $R_0 = 2$

with the empirical distribution (the same for 200 runs)

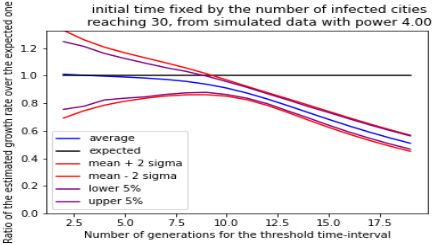
Presented in the case $a = 1.2$, $b = 0$.

Reference time is adjusted by a time-shift between the 200 runs

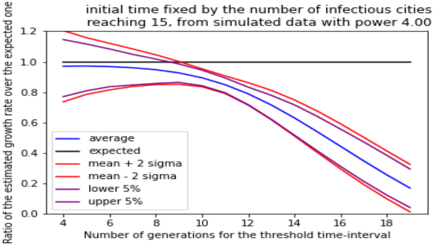
- new time-scale initiated by the number of infected reaching a threshold L
- $L = 30$ for the total number of infected
- $L = 15$ for the number of infectious in one generation

Estimation of the R_0 presented with least square regression on intervals of varying length.

Power law distribution with exponent 4



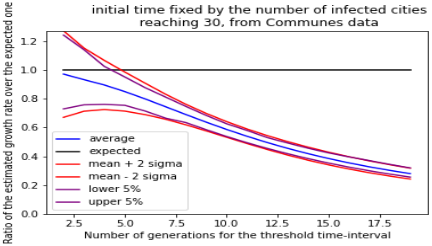
Total number of infected exploited



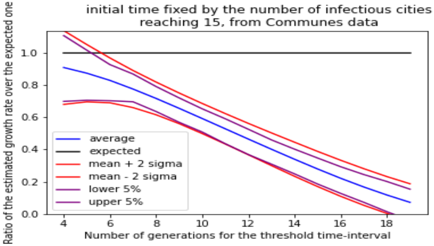
Number of infectious exploited

As a reference for a quite regular situation.

Real case of Communes

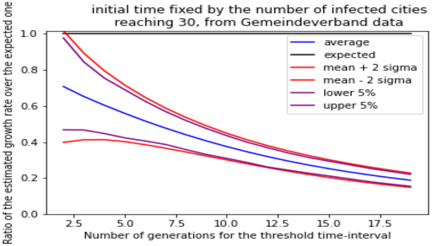


Total number of infected exploited

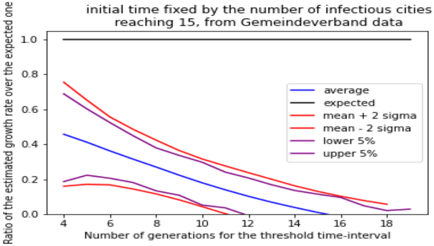


Number of infectious exploited

Real case of Gemeindeverband

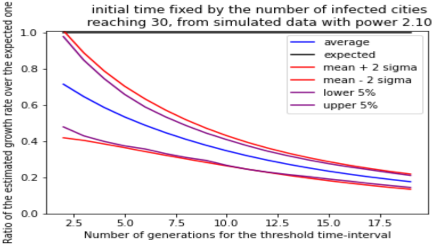


Total number of infected exploited

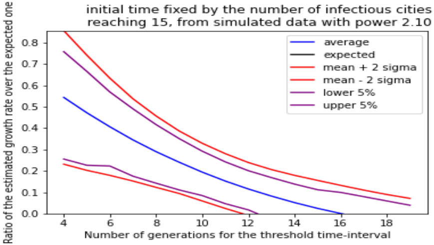


Number of infectious exploited

Power-law at 2.1



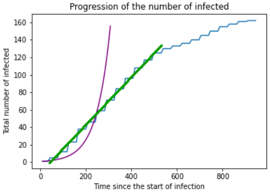
Total number of infected exploited



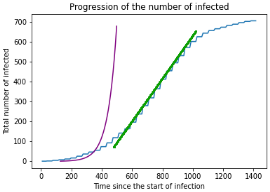
Number of infectious exploited

Power-law at 2.14

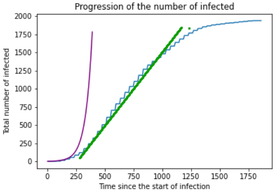
Varying number N_C of cities



$N_C = 100,000$



$N_C = 350,000$

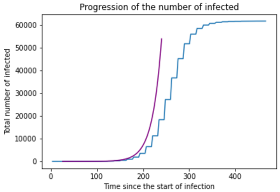


$N_C = 1,000,000$

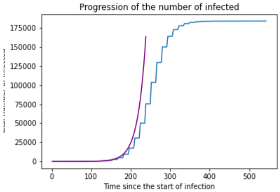
Power-law chosen as for Gemeindeverbände, close to the one of Communes.

Power-law at 3

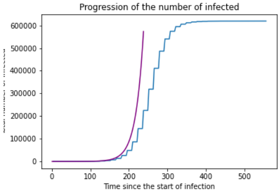
Again varying number N_C of cities.



$N_C = 100,000$



$N_C = 300,000$



$N_C = 1,000,000$

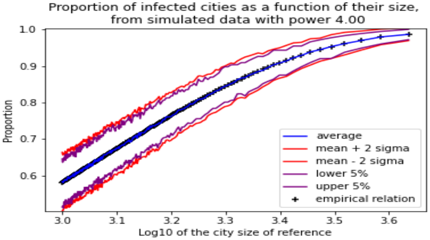
Probability of infection vs city size

$\rho(x)$: probability that a city of size x gets infected.

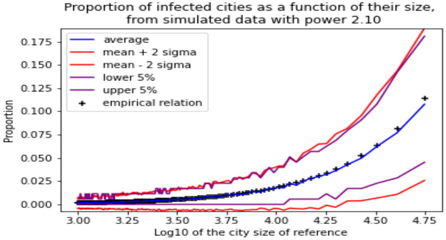
The observation seems totally to agree with the prediction from the formula:

$$\rho_T = 1 - \exp(-\mathcal{T}\rho_T),$$

which characterizes the survival of the backward process.



Exponent 4



Exponent 2.1

Conclusion

The model

The graph of cities

The rates of transmission

The lockdown strategy

Theoretical analysis

Reduction to a 2-dimensional branching dynamics

Comparison of basic reproduction number R_0

First results of simulations

How relevant is R_0?

A linear regime?

The probability of infection vs city size

To keep in mind but beyond our analysis

- uncertainty in the actual number of infected at the time of lockdown
- local transmissions from one city to its neighbors
- heterogeneity of the growth rate in different cities,
  note that relying on its estimation might induce too much of a delay
- temporal fluctuations:
  due to vaccination, climate changes, holidays, special events like the Olympic Games...

Thank you for your attention!