EXERCISES

Stochastic Analysis

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Fall 2023

University of Vienna

Session 1: October 11

Exercise 1. Let X be integrable and $\mathscr{G}, \mathscr{H} \subset \mathscr{F}$ be σ-algebras. Show that if $\sigma(X, \mathscr{G})$ is independent of $\mathscr{H}$, then

$$\mathbb{E}[X|\sigma(\mathscr{G}, \mathscr{H})] = \mathbb{E}[X|\mathscr{G}] \quad a.s.$$

Exercise 2. If $f : \mathbb{R} \to \mathbb{R}$ is convex, and $X, f(X)$ are integrable, prove the conditional Jensen's inequality

$$f(\mathbb{E}[X|\mathscr{G}]) \leq \mathbb{E}[f(X)|\mathscr{G}] \quad a.s.$$

Hint: use the fact that there exists $\mathscr{S} \subset \mathbb{Q}^2$ such that, for all $x \in \mathbb{R}$,

$$f(x) = \sup_{(a,b)\in\mathscr{S}} (ax+b).$$

Exercise 3 (Monotone convergence theorem for conditional expectation)**.** If $(X_n)_{n\geq 0}$ is an increasing sequence of non-negative random variables with $a.s.$ limit X, then

$$\mathbb{E}[X_n|\mathscr{G}] \nearrow \mathbb{E}[X|\mathscr{G}], \quad a.s.$$

Exercise 4. Consider $\mathbb{E}[X|Y]$ where X is integrable. Then there exists a Borel measurable function $h : \mathbb{R} \to \mathbb{R}$ such that $\mathbb{E}[X|Y] = h(Y)\, a.s.$

Exercise 5. Let $X_1, \ldots, X_d$ be real-valued random variables defined on the same probability space. Show that the X_i's are independent if and only if the characteristic function

$$\phi(t_1, \ldots, t_d) := \mathbb{E}\left[e^{i(t_1 X_1 + \ldots + t_d X_d)}\right]$$

can be decomposed as $\phi(t_1, \ldots, t_d) = f_1(t_1) \ldots f_d(t_d)$ for some functions $f_i : \mathbb{R} \to \mathbb{C}$.

Exercise 6. Let $X = (X_1, \ldots, X_d)$ be a centred Gaussian vector composed of independent and identically distributed random variables.

1. Let O be a $d \times d$ orthonormal matrix. Show that OX has the same law as X.
2. Let $a = (a_1, \ldots, a_d)$ and $b = (b_1, \ldots, b_d)$ be two orthonormal vectors of $\mathbb{R}^n$. By considering an orthonormal matrix O whose first two columns coincide with a and b, show that $\sum_{i=1}^d a_i X_i$ and $\sum_{i=1}^d b_i X_i$ are independent.

Exercise 7. Let X and Y be two real-valued random variables defined on the same probability space. We assume that X and Y are independent and that the law of the random vector (X, Y) is invariant under rotations of center $(0, 0)$.

1. Show that $X \sim Y$ and that $X \sim -X$.
2. Show that the characteristic function φ of X satisfies
$$\forall u, v \in \mathbb{R}, \varphi(u)\varphi(v) = \varphi\left(\sqrt{u^2 + v^2}\right).$$
3. Conclude that X is Gaussian.

Session 2: October 25

Exercise 8. Let X and Y be independent, and let $f : \mathbb{R}^2 \to \mathbb{R}$ be bounded measurable. Show that $\mathbb{E}(f(X,Y) \mid Y) = h(Y)$, where $h(t) = \mathbb{E}(f(X,t))$.

Exercise 9. Let $(X,Y) \sim \mathcal{N}(m,\Sigma)$ where $m = (1,0)$ and $\Sigma = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. Compute $\mathbb{E}[X|X-Y]$.

Exercise 10. Let $X_1 \dots X_n$ be i.i.d. standard normal random variables. Explain why the fact that

$$\Big\{(x_1 \dots x_n) \in \mathbb{R}^n : \sum_i x_i = 0\Big\} \text{ and } \{\lambda(1,\dots,1) : \lambda \in \mathbb{R}\}$$

are orthogonal vector subspaces of $\mathbb{R}^n$ implies that

$$\Big(X_1 - \frac{1}{n}\sum_i X_i, \dots, X_n - \frac{1}{n}\sum_i X_i\Big) \text{ and } \sum_i X_i$$

are independent.

Exercise 11 (Central Limit Theorem and random walks)**.** Consider a random walk $S_n = \sum_{i=1}^n X_i$ for $n \geq 0$, where X_i are i.i.d. centered increments with variance $\sigma^2 < \infty$. For $n \geq 0$ and $t \in \mathbb{R}_+$, set $\widetilde{S}_n(t) = \frac{1}{\sigma\sqrt{n}} S_{\lfloor nt \rfloor}$, the rescaled version of S. Now we set $0 \leq t_0 \leq t_1 \leq \dots \leq t_k$ and wish to show convergence in distribution of the random vector $(\widetilde{S}_n(t_0), \dots, \widetilde{S}_n(t_k))$.

1. Show that for every n, the increments $(\widetilde{S}_n(t_i) - \widetilde{S}_n(t_{i-1}))_{1 \leq i \leq k}$ are independent.
2. What is the limit of distribution of each increment? What is the joint limit in distribution of the vectors of the increments?
3. Deduce that the vector $(\widetilde{S}_n(t_0), \dots, \widetilde{S}_n(t_k))$ converges in distribution towards a given centered Gaussian random vector, that we will denote $(B_{t_0}, \dots, B_{t_k})$. What is its covariance matrix?

The next exercise is optional.

Exercise 12 (Time inversion of Brownian motion)**.** Let $(B_t)_{t \geq 0}$ be a Brownian motion starting at zero. Assume that

$$\limsup_{t \to \infty} B_t / t = 0 \qquad \text{a.s.}$$

Show that

$$X_t = \begin{cases} tB_{1/t} & \text{for } t > 0, \\ 0 & \text{for } t = 0, \end{cases}$$

is also a Brownian motion starting at zero.

Session 3: November 8

In the following exercises we will consider a Brownian motion $(B_t)_{t\geq 0}$ starting at zero, i.e. $B_0 = 0$ a.s.

Exercise 13. Let $K > 0$. Show that

$$\mathbb{P}\left(\limsup_{n\to\infty} \sqrt{n}B_{1/n} \geq K\right) > 0.$$

Deduce that this probability is equal to 1 and that almost surely

$$\limsup_{t\searrow 0} \frac{B_t}{\sqrt{t}} = +\infty \text{ and } \liminf_{t\searrow 0} \frac{B_t}{\sqrt{t}} = -\infty.$$

This shows that $(B_t)_{t\geq 0}$ is almost surely not $\frac{1}{2}$-Hölder at 0.

Exercise 14. Show that almost surely for any interval $(s_1, s_2) \subset [0, \infty)$:

$$\sup_{t\in(s_1,s_2)} B_t - B_{s_1} > 0 \text{ and } \inf_{t\in(s_1,s_2)} B_t - B_{s_1} < 0.$$

Deduce that, almost surely, $(B_t)_{t\geq 0}$ is not monotone on (s_1, s_2).

Exercise 15. Prove that

$$\mathbf{P}(t \mapsto B_t \text{ is uniformly continuous on } [0,\infty)) = 0.$$

Exercise 16. The aim of this exercise is to show that almost surely, $B_t/t \to 0$ as $t \to \infty$.

1. Using the law of large numbers, show that

$$\lim_{\substack{n\to\infty \\ n\in\mathbb{N}}} \frac{B_n}{n} = 0 \qquad \text{a.s.}$$

2. Show that

$$\sum_{n\geq 0} \mathbb{P}\left(\sup_{n\leq t<n+1} |B_t - B_n| > \sqrt{n}\right) < \infty.$$

3. Conclude.

Exercise 17. For $a \in \mathbb{R}$, define the stopping time $T_a = \inf\{t > 0 : B_t = a\}$. Let $\tilde{T} = \inf\{t > T_1, B_t = 0\}$. Show the two following equalities in distribution: for all $a \in \mathbb{R}$, $T_a \sim a^2 T_1$ and $\tilde{T} \sim T_2$.

Exercise 18. Let $t > 0$ and for $n \geq 1$, let $0 = t_0^n < t_1^n < \cdots < t_n^n = t$ be such that $\sup_{i=0\ldots n-1}(t_{i+1}^n - t_i^n) \to 0$ as $n \to \infty$. Show the following convergence in L^2:

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} (B_{t_{i+1}^n} - B_{t_i^n})^2 = t.$$

Deduce that the first order variation of Brownian motion over $[0,t]$, i.e.,

$$\sup_{\{(t_0,\cdots,t_n):\,\text{any partition of } [0,t]\}} \sum_{i=0}^{n-1} |B_{t_{i+1}} - B_{t_i}|$$

is infinity almost surely.

Session 4: November 22

Exercise 19. Let $(X_t, t \geq 0)$ be a closed martingale. Prove that $(X_t, t \geq 0)$ is uniformly integrable.

Exercise 20. Suppose $(M_t)_{t\in\mathbb{N}}$ is a discrete time martingale in L^p for some $p > 1$, and let M_n^* be its running maximum at time n. Prove that

$$\mathbb{E}[|M_n^*|^p] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}[|M_n|^p].$$

Hint: $|M_n^*|^p = \int_0^{|M_n^*|} px^{p-1}\, dx = \int_0^\infty px^{p-1} \mathbb{1}_{\{|M_n^*| \geq x\}}\, dx$.

Exercise 21. Let B be a standard Brownian motion, and $\theta > 0$. Prove that

$$\left(e^{\theta B_t - \theta^2/2t}, t \geq 0\right)$$

is a martingale. Compute, for $a \in \mathbb{R}$, $\mathbb{E}\left[e^{-\theta^2 T_a/2}\right]$, where $T_a := \inf\{t \geq 0, B_t = a\}$.

Hint: Use the optional stopping theorem.

Exercise 22. Let B be a standard Brownian motion starting from 1 ($B_0 = 1$ a.s.). For $a > 1$, prove that $\mathbb{P}(T_a < T_0) = 1/a$.

In the next two exercises, let us assume $(M_t)_{t\in\mathbb{N}}$ is a discrete time submartingale with respect to a filtration $(\mathscr{F}_t)_{t\in\mathbb{N}}$. A stochastic process $(F_t)_{t\in\mathbb{N}}$ is said to be predictable if F_0 is $\mathscr{F}_0$-measurable, and F_{t+1} is $\mathscr{F}_t$-measurable for all $t \in \mathbb{N}$. Then one can define a martingale transform

$$X_t = M_0F_0 + (M_1 - M_0)F_1 + \ldots + (M_t - M_{t-1})F_t.$$

Exercise 23. Prove that if $(M_t)_{t\in\mathbb{N}}$ is a martingale and $(F_t)_{t\in\mathbb{N}}$ is bounded predictable, then $(X_t)_{t\in\mathbb{N}}$ is a martingale. If $(M_t)_{t\in\mathbb{N}}$ is a submartingale and $(F_t)_{t\in\mathbb{N}}$ is bounded positive predictable, then $(X_t)_{t\in\mathbb{N}}$ is a submartingale.

Exercise 24. Suppose $(M_t)_{t\in\mathbb{N}}$ is a submartingale. Prove that for any bounded stopping time $S \leq T$, M_S and M_T are integrable and

$$\mathbb{E}[M_T - M_S \,|\, \mathscr{F}_S] \geq 0.$$

Hint: take $F = V\mathbb{1}_{(S,T]}$ where V is bounded, positive, and $\mathscr{F}_S$-measurable.

Exercise 25. Suppose that $a : [0,\infty) \to \mathbb{R}$ is continuously differentiable. Let $V(t)$ be the total variation of a on $[0,t]$. Show that

$$V(t) = \int_0^t |a'(s)|ds.$$

Session 5: December 6

Exercise 26. Let $B = (B_t)_{t\geq 0}$ be a standard Brownian motion, and let

$$\operatorname{sgn}(x) = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Show that $(\operatorname{sgn}(B_t))_{t\geq 0}$ is a previsible process which is neither left nor right continuous.

Exercise 27. Let $X = (X_t)_{t\geq 0}$ be a continuous local martingale. Show that if

$$\mathbb{E}[\sup_{0\leq s\leq t} |X_s|] < \infty$$

for each $t \geq 0$, then X is a martingale.

The rest of the exercises is devoted to a proof of the existence and uniqueness of quadratic variation. We recall this result.

Theorem (Existence and uniqueness of quadratic variation)**.** *Let M be a continuous local martingale ($M \in \mathscr{M}_{c,loc}$). Then there exists a unique continuous, adapted, and nondecreasing process $[M]$ such that $[M]_0 = 0$, and*

$$M^2 - [M] \text{ is a continuous local martingale.} \tag{1}$$

Moreover, if we define

$$[M]^n_t = \sum_{k=0}^{\lceil 2^n t\rceil -1} (M_{(k+1)/2^n} - M_{k/2^n})^2,$$

then $[M]^n \to [M]$ ucp (uniformly in probability on all compacts) as $n \to \infty$.

We assume without loss of generality that, in what follows, $M_0 = 0$ a.s.

Exercise 28. Show the uniqueness of quadratic variation.

Assume for now that $|M|$ is bounded uniformly by some constant $C < \infty$ (and hence a martingale).

Exercise 29. Show that for all $t > 0$,

$$\mathbb{E}\left[([M]^n_t)^2\right] \leq 1000C^4.$$

(The constant 1000 is not optimal.)

Fix $T \in \mathbb{N}$ deterministic, and let

$$H^n_t = M_{2^{-n}\lfloor 2^n t\rfloor} \mathbf{1}_{(0,T]} = \sum_{k=0}^{2^n T-1} M_{k/2^n} \mathbf{1}_{(k/2^n,(k+1)/2^n]}(t),$$

and let

$$X^n_t = (H^n \cdot M)^T_t = \sum_{k=0}^{2^n T-1} M_{k/2^n}(M_{(k+1)/2^n\wedge t} - M_{k/2^n\wedge t}).$$

Exercise 30. Show that $(X^n)_{n\geq 1}$ is Cauchy in $(\mathscr{M}_c^2, \|\cdot\|)$. *Hint:* Show that

$$\|X^n - X^m\|^2 \leq \left(\mathbb{E}\big[\sup_{0\leq t\leq T} |M_{2^{-n}\lfloor 2^n t\rfloor} - M_{2^{-m}\lfloor 2^m t\rfloor}|^4\big]\right)^{1/2} \big(\mathbb{E}\big[([M]_T^n)^2\big]\big)^{1/2}$$

and argue why the right-hand side converges to zero as $n, m \to \infty$.

Recall that $(\mathscr{M}_c^2, \|\cdot\|)$ is complete, and call $X \in \mathscr{M}_c^2$ the limit of X^n whose existence is established by the previous exercise.

Exercise 31. Show that for any $1 \leq k \leq 2^n T$, we have

$$M_{k/2^n}^2 - 2X_{k/2^n}^n = [M]_{k/2^n}^n,$$

and conclude that $M^2 - 2X^n$ is nondecreasing when restricted to the set of times $\{k/2^n : 1 \leq k \leq 2^n T\}$.

The above exercise implies that (after taking the $n \to \infty$ and $T \to \infty$ limits) $M^2 - 2X$ is a nondecreasing process. Set

$$[M] = M^2 - 2X.$$

All in all, $[M]$ is a continuous, nondecreasing process and $M^2 - [M] = 2X$ is a martingale.

Exercise 32. Show that convergence in $(\mathscr{M}_c^2, \|\cdot\|)$ implies ucp convergence. Use this to prove that $[M]^n \to [M]$ ucp as $n \to \infty$.

Now assume that $M \in \mathscr{M}_{c,loc}$ as in the statement of the theorem.

Exercise 33. By using a reducing sequence of stopping times $(T_n)_{n\geq 1}$, and the previous exercises, show part (1) of the theorem in this case.

Session 6: January 10

In the following exercises, $(B_t)_{t\geq 0}$ denotes a Brownian motion starting at the origin.

Exercise 34. Using Itô's formula, show that $B_t^2 - t$ is a local martingale. What is its quadratic variation? Show that it is a true martingale.

Exercise 35. Suppose $f(t,x)$ satisfies the heat equation

$$\frac{\partial f}{\partial t} + \frac{1}{2}\Delta f = 0.$$

Show that $f(t,B_t)$ is a local martingale. Deduce that $X_t = \exp(\lambda^2 t/2)\sin(\lambda B_t)$ is a martingale as well as $Y_t = (B_t + t)\exp(-B_t - t/2)$.

Exercise 36. Show that if X_t is a continuous local martingale, then

$$\mathscr{E}(X)_t = \exp(X_t - \tfrac{1}{2}[X]_t)$$

is a continuous local martingale. $\mathscr{E}(X)_t$ is called the *stochastic exponential*. Conversely, if M_t is a strictly positive continuous martingale, show that M is the stochastic exponential of a local martingale (apply Itô's formula to $\log M_t$).

Exercise 37 (Feynman–Kac formula)**.** Let $f \in C_b^2(\mathbb{R}^d)$ and let $V \in L^\infty(\mathbb{R}^d)$ be a uniformly bounded function. Then show that any solution $u \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$ to the problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\Delta u + Vu & \text{on } \mathbb{R}_+ \times \mathbb{R}^d, \\ u(0,\cdot) = f & \text{on } \mathbb{R}^d, \end{cases}$$

is given by

$$u(t,x) = \mathbb{E}_x\left(f(B_t)\exp\left(\int_0^t V(B_s)ds\right)\right)$$

where $\mathbb{E}_x$ denotes expectation for a Brownian motion starting from x.

[Hint: let $T > 0$ be fixed and set $M_t = u(T-t,B_t)E_t$ for $0 \leq t \leq T$ where $E_t = \exp(\int_0^t V(B_s)ds)$ is of finite variation. Use Itô's formula to show that M is a martingale. Then apply the optional stopping theorem at time T.]

Exercise 38. Let $X \in \mathscr{M}_{c,loc}$. Prove that if for all $t \geq 0$, $\mathbb{E}([X]_t) < \infty$, then X is a true martingale. *Hint:* Use localization and Doob's L^2 inequality.

In the following, let $k \geq 2$ and let B be a k-dimensional Brownian motion. Let $Z_s = \|B_s\|^2$ be the square Euclidean norm of B. Z is called a square Bessel process of dimension k.

Exercise 39. Apply Itô's formula to find the semimartingale decomposition of Z. Show that there is a (one-dimensional) Brownian motion $\tilde{B}$ such that

$$dZ_s = 2\sqrt{Z_s}d\tilde{B}_s + kdt.$$

Exercise 40. Let $R_s = \sqrt{Z_s} = \|B_s\|$, for any $s \geq 0$. Show that

$$dR_t = d\tilde{B}_t + \frac{(k-1)/2}{R_t}dt.$$

Session 7: January 24

This exercise sheet contains different exercises about all parts of the course, and not only the last lectures.

Exercise 41 (Properties of Brownian motion)**.** Let B be a standard Brownian motion. Recall that $\frac{B_t}{t} \to 0$ as $t \to +\infty$, and $\limsup_{t\to 0} \frac{B_t}{\sqrt{t}} = +\infty$.

(i) Show that $0 < \sup_{t\geq 0}(|B_t| - t) < \infty$ a.s., and that $0 < \sup_{t\geq 0} \frac{|B_t|}{1+t} < \infty$ a.s.

(ii) Prove that $\sup_{t\geq 0}(|B_t| - t)$ and $\left(\sup_{t\geq 0} \frac{|B_t|}{1+t}\right)^2$ have the same distribution

(Hint: use the scaling property).

(iii) Show that, for any $p > 0$,

$$\mathbb{E}\left[(\sup_{t\geq 0}((|B_t| - t))^p\right] < \infty.$$

(Hint: prove that $\mathbb{E}\left[\left(\sup_{t\in[0,1]}(|B_t| - t)\right)^p\right] < \infty$ by reflection principle, then use inversion of time)

(iv) Prove that there exists a constant $C > 0$ such that, for any nonnegative random variable, we have $\mathbb{E}[|B_T|] \leq C(\mathbb{E}[T])^{1/2}$.

(Hint: write, for any $a > 0$, $|B_T| = (|B_T| - aT) + aT$, and prove that $\mathbb{E}[|B_T| - aT] \leq \frac{1}{a}\mathbb{E}[\sup_{t\geq 0}(|B_t| - t)]$).

Exercise 42 (A criterion for true martingales)**.** Let $M \in \mathscr{M}_{c,loc}$ such that, for all $t \geq 0$, $\mathbb{E}[\sup_{0\leq s\leq t} |M_s|] < \infty$. Prove that M is a true martingale.

The next exercise was part of the previous exercise sheet but not solved in class. Let $k \geq 2$ and let B be a k-dimensional Brownian motion. Let $Z_s = \|B_s\|^2$ be the square Euclidean norm of B. Z is called a square Bessel process of dimension k.

Exercise 43. Apply Itô's formula to find the semimartingale decomposition of Z. Show that there is a (one-dimensional) Brownian motion $\tilde{B}$ such that

$$dZ_s = 2\sqrt{Z_s}d\tilde{B}_s + kdt.$$

Exercise 44 (More on Bessel processes)**.** Let $R := \sqrt{Z}$ be a Bessel process of dimension $k \geq 3$. Recall that (see Exercise 40):

$$dR_t = d\tilde{B}_t + \frac{(k-1)/2}{R_t}dt.$$

Find α such that R_t^α is a local martingale. Fix $\varepsilon \in (0,1)$. Starting from $B_0 = (1,0,\ldots,0)$, compute the probability that a Brownian motion of dimension k ever hits the ball $B(0,\varepsilon)$.

Exercise 45 (About Girsanov's theorem)**.** Let B be a standard Brownian motion and $\gamma \in \mathbb{R}$. Let

$$T_\gamma = \inf\{t > 0 : |B_t + \gamma t| = 1\}.$$

In questions (i) and (ii) below, we assume that $\gamma = 0$.

(i) Using the symmetry of Brownian motion, show that for any bounded measurable function $f : \mathbb{R} \to \mathbb{R}$,

$$\mathbb{E}\left[\mathbf{1}_{\{B_{T_0}=1\}} f(T_0)\right] = \mathbb{E}\left[f(T_0)\right]/2.$$

(ii) Deduce that B_{T_0} and T_0 are independent.

(iii) More generally, show that for any $\gamma \in \mathbb{R}$, $B_{T_\gamma} + \gamma T_\gamma$ and T_γ are independent.

(Hint: Use Girsanov's theorem to come back to the case $\gamma = 0$).