FINITE DECOMPOSITION COMPLEXITY AND THE INTEGRAL NOVIKOV CONJECTURE FOR HIGHER ALGEBRAIC K–THEORY
(DRAFT)

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ABSTRACT. Decomposition complexity for metric spaces was recently introduced by Guentner, Tessera, and Yu as a natural generalization of asymptotic dimension. We prove a vanishing result for the continuously controlled algebraic K–theory of bounded geometry metric spaces with finite decomposition complexity. This leads to a proof of the integral K–theoretic Novikov conjecture, regarding split injectivity of the K–theoretic assembly map, for groups with finite decomposition complexity and finite CW models for their classifying spaces. By work of Guentner, Tessera, and Yu, this includes all (geometrically finite) linear groups.

1. INTRODUCTION

Decomposition complexity for metric spaces, introduced by Guentner, Tessera, and Yu [12, 13], is a natural inductive generalization of the much-studied notion of asymptotic dimension. Roughly speaking, decomposition complexity measures the difficulty of decomposing a metric space into uniformly bounded pieces that are well-separated from one another. The class of metric spaces with *finite* decomposition complexity (FDC) contains all metric spaces with finite asymptotic dimension [12, Theorem 4.1], as well as all countable linear groups equipped with a proper (left-)invariant metric ([13, Theorem 3.0.1] and [12, Theorems 5.2.2]). In this article, we prove the integral Novikov conjecture for the algebraic K–theory of group rings $R[\Gamma]$, where Γ has FDC and is geometrically finite (i.e. has a finite CW model for its classifying space). This is achieved through a study of the continuously controlled algebraic K–theory of the Rips complexes $P_s(\Gamma)$.

For a discrete group Γ, the classical Novikov conjecture on the homotopy invariance of higher signatures is implied by rational injectivity of the Baum–Connes assembly map [4]. In Yu [29] and Skandalis–Tu–Yu [25], injectivity of the Baum-Connes map was proved for groups coarsely embeddable into Hilbert space. Using this result, Guentner, Higson, and Weinberger [11] proved the Novikov conjecture for linear groups. This inspired the work of Guentner, Tessera, and Yu [13], who proved the *integral* Novikov conjecture (establishing integral injectivity of the L–theoretic assembly map) for

geometrically finite FDC groups, and hence the stable Borel Conjecture for closed aspherical manifolds whose fundamental groups have FDC.

The algebraic K–theory Novikov conjecture claims that Loday's assembly map [17]

$$H_*(\Gamma; \mathbb{K}(R)) \longrightarrow K_*(R[\Gamma]) \tag{1}$$

is (rationally) injective. Here Γ is a finitely generated group and R is an associative, unital ring (not necessarily commutative). The domain of the assembly map is the homology of Γ with coefficients in the (non-connective) K–theory spectrum of R, and the range is the (non-connective) K–theory of the group ring $R[\Gamma]$. For discussions of this conjecture and its relations to geometry, see Hsiang [16] and Farrell–Jones [10]. A great deal is known about the map (1): Bökstedt, Hsiang, and Madsen [5] proved that (1) is *rationally* injective for $R = \mathbb{Z}$ under the assumption that $H_*(\Gamma; \mathbb{Z})$ is finitely generated in each degree. Integral injectivity results were proven for geometrically finite groups with finite asymptotic dimension by Bartels [3] and Carlsson–Goldfarb [8], building on Yu's work [28] (which established injectivity of the Baum–Connes assembly map for groups with finite asymptotic dimension). In Section 7, we prove the following generalization of [3, 8].

Theorem 1.1. *Let Γ be a geometrically finite group with finite decomposition complexity. Then the K–theoretic assembly map (1) is a split injection for all $* \in \mathbb{Z}$.*

Analogous methods yield an integral injectivity result for the assembly map associated to Ranicki's ultimate lower quadratic L–theory $\mathbb{L}^{-\infty}$. We also obtain a large-scale version of the Borel Conjecture for bounded K–theory (Theorem 7.1), analogous to [13, Theorems 4.3.1, 4.4.1].

Guentner, Tessera, and Yu [13] studied the Ranicki–Yamasaki controlled (lower) algebraic K– and L–groups [22, 23] of FDC metric spaces, and established a large-scale vanishing result formulated in terms of Rips complexes. They used this result to study related assembly maps, leading to important geometric rigidity results (in particular, the stable Borel Conjecture). The key technical result in the present paper is a vanishing theorem for *continuously* controlled K–theory, analogous to [13, Theorem 5.1].

Theorem 1.2. *If X is a metric space with bounded geometry and finite decomposition complexity, then $\operatorname{colim}_s K_*^c(P_s X) = 0$ for all $* \in \mathbb{Z}$.*

This theorem is proven in Section 6. Here $K_*^c(Z)$ denotes the continuously controlled K–theory of the metric space Z (see Section 2), and *bounded geometry* means that for each $r > 0$, the exists $N \in \mathbb{N}$ such that each ball of radius r contains at most N elements.

The analogous result from [13] is proven using controlled Mayer–Vietoris sequences for Ranicki and Yamasaki's controlled lower K– and L–groups [22, 23]. In that flavor of controlled algebra, one imposes universal bounds on the propagation of morphisms, and the Mayer–Vietoris sequences are only

exact in a weak sense involving these bounds. While it may be possible to construct quantitative versions of higher algebraic K–groups (analogous to the Ranicki–Yamasaki controlled lower K–groups, and to recent work of Oyono–Oyono and Yu in operator K–theory [18]) such a theory does not currently exist. Instead, we produce analogous (strictly exact) Mayer–Vietoris sequences in *continuously* controlled K–theory. Loosely speaking, this corresponds (in low dimensions) to taking colimits over the propagation bounds in the Ranicki–Yamasaki theory. Our Mayer–Vietoris sequences are produced using the machinery of Karoubi filtrations as developed, for instance, in Cárdenas–Pedersen [6].

In broad strokes, the proof of Theorem 1.2 is similar to the arguments in [13, Section 6]. The starting point is that the theorem holds for *bounded* metric spaces. In [13, Section 6], controlled Mayer–Vietoris sequences were applied to a space X covered by two subspaces, each an r–disjoint union of smaller subspaces. Great care was taken in order to keep r large with respect to the other parameters involved, e.g. the Rips complex parameter and the propagation bound on morphisms. In the present work we consider all at once a *sequence* of such decompositions of X, whose disjointness tends to infinity. For each continuously controlled K–theory class $x \in K_*^c(P_sX)$, we show that at sufficiently high stages in the sequence of decompositions, x can be build from classes supported on the (relative) Rips complexes of the individual factors appearing in the decompositions. An inductive process ensues, in which we further decompose the spaces appearing at each level of the previous sequence of decompositions. Metric spaces with finite decomposition complexity are essentially those for which this process eventually results in (uniformly) bounded pieces. Such considerations lead to the notion of a *decomposed sequence*, introduced in Section 4. Our approach avoids much of the intricate manipulation of various constants in [13, Section 6], but the price we pay is that we must deal with more complicated objects than simply a metric space decomposed as a union of two subspaces.

Our approach to the assembly map makes crucial use both ordinary Rips complexes and the relative Rips complexes introduced in [13]. Given a discrete metric space X and a positive number s, the Rips complex $P_s(X)$ is formed from the vertex set X by laying down a simplex $\langle x_0, \ldots, x_n \rangle$ whenever the pairwise distances $d(x_i, x_j)$ are all at most s. As the parameter s increases, these simplices wipe out any small-scale features of X and expose the large-scale structure of the space. When X is a torsion-free group Γ equipped with the word metric associated to a finite generating set, the Rips complexes also give a sequence of cocompact Γ–spaces approximating the universal free Γ–space $E\Gamma$ (if Γ has torsion, they approximate the universal space for *proper* actions). Theorem 1.1 is deduced from Theorem 1.2 through a comparison between $E\Gamma$ and the Rips complexes, which shows that when Γ is geometrically finite and has FDC, the controlled K–theory of $E\Gamma$ vanishes (Theorem 7.8).

In earlier work on assembly maps in higher algebraic K–theory, nerves of coverings (as in Bartels [3] or Carlsson–Goldfarb [8]) or compactifications of the universal space $E\Gamma$ (as in Carlsson–Pedersen [9] or Rosenthal [24]) played roles similar to the Rips complexes used here. Unlike coverings and compactifications, Rips complexes are built in a canonical way from the underlying metric space. Together with their dual relationships to the large-scale geometry of Γ and to the universal space $E\Gamma$, this makes Rips complexes ideally suited to the study of assembly maps.

Organization: Section 2 reviews notions from geometric algebra. Section 3 establishes algebraic facts about Karoubi filtrations that underly our controlled Mayer–Vietoris sequences. The sequences themselves are constructed in Section 4. This section begins with a general Mayer–Vietoris sequence for proper metric spaces, and then specializes this sequence to Rips complexes and relative Rips complexes. Section 4 also introduces the terminology of *decomposed sequences* used extensively in Section 6. In Section 5, we review the necessary metric properties of Rips complexes and relative Rips complexes. Section 6 reviews the notion of finite decomposition complexity and establishes our vanishing theorem for continuously controlled K–theory. Assembly maps for $K-$ and L–theory are studied in the final section.

2. Geometric modules

Throughout this paper all metrics will be allowed to take on the value ∞, and all categories will be assumed to be small. If X is a metric space and $x \in X$, we set $B_r(x) = \{y \in X : d(x, y) < r\}$ and if $Z \subset X$, we set $N_r(Z) = \{y \in X : d(y, Z) < r\}$. We call a metric space *proper* if the closed ball $\{y \in X : d(x, y) \leqslant r\}$ is compact for every $x \in X$ and every $r > 0$.

Definition 2.1. *Let $\mathcal{A}$ be an additive category (we think of the objects of $\mathcal{A}$ as "modules"). A geometric $\mathcal{A}$–module over a metric space X is a function $M : X \to Ob(\mathcal{A})$. We say that M is* locally finite *if its support $supp(M) = \{x \in X | M(x) \neq 0\}$ is locally finite in X, in the sense that for each $x \in X$ and each compact set $K \subset X$, $supp(M) \cap K$ is finite. (If X proper, this is equivalent to requiring that each $x \in X$ has a neighborhood U_x such that $supp(M) \cap U_x$ is finite.) We will usually abbreviate $M(x)$ by M_x, and for any subspace $Y \subset X$ we define $M(Y)$ to be the geometric module given by*

$$M(Y)_x = \begin{cases} M_x, & x \in Y, \\ 0, & x \notin Y \end{cases}$$

A morphism ϕ from a geometric module M to a geometric module N is collection of morphisms $\phi_{xy} : M_y \to N_x$ for all pairs $(x, y) \in X \times X$, subject

to the condition that for each $x \in X$, the sets

$$\{(x,y) \,|\, \phi_{xy} \neq 0\} \quad and \quad \{(y,x) \,|\, \phi_{yx} \neq 0\}$$

are finite.

One may think of $\phi = \{\phi_{xy}\}$ as a matrix indexed by the points in X, in which each row and each column has only finitely many non-zero entries.

We will deal with a fixed additive category $\mathcal{A}$ throughout the paper, and we will refer to geometric $\mathcal{A}$–modules simply as geometric modules. The main case of interest is when $\mathcal{A}$ is (a skeleton of) the category of finitely generated free R–modules for some associative unital ring R.

Geometric modules and their morphisms form an additive category $\mathcal{A}(X)$, in which composition of morphisms is simply matrix multiplication (which is well-defined due to the row- and column-finiteness of these matrices) and addition of morphisms is defined via entry-wise sum of matrices (using the additive structure of $\mathcal{A}$). The categories we are interested in will impose important additional support conditions on the morphisms ϕ.

Definition 2.2. *We say that a morphism $\phi : M \to N$ of geometric modules over X has* finite propagation *(or is* bounded*) if there exists $R > 0$ such that $\phi_{xy} = 0$ whenever $d(x,y) > R$.*

We may now consider the subcategory of locally finite geometric modules and *bounded* morphisms

$$\mathcal{A}_b(X) \subset \mathcal{A}(X).$$

This is again an additive category, and its K–theory is, by definition, the bounded K–theory of X with coefficients in $\mathcal{A}$.

Remark 2.3. *Throughout this paper, the K–theory of an additive category $\mathcal{C}$ will mean the non-connective K–theory spectrum $\mathbb{K}(\mathcal{C})$ as defined, for example, in* [6, *Section 8*]. *This means we consider $\mathcal{C}$ as a Waldhausen category, in which cofibrations are inclusions of direct summands and weak equivalences are isomorphisms. Since inclusions of direct summands can be characterized in terms of split exact sequences, additive functors $\mathcal{C} \to \mathcal{D}$ always preserve these notions of cofibration and weak equivalence, and hence induce maps $\mathbb{K}(\mathcal{C}) \to \mathbb{K}(\mathcal{D})$. We set $K_*(\mathcal{C}) = \pi_* \mathbb{K}(\mathcal{C})$ for $* \in \mathbb{Z}$.*

Next, we will consider the notion of *continuously controlled morphisms*, which will be the main object of study in this paper. Here and in what follows, we give the half-open interval $[0,1)$ the usual Euclidean metric $d(s,t) = |s - t|$, and for metric spaces (X, d_X) and (Y, d_Y), we give $X \times Y$ the metric $d\left((x,y),(x',y')\right) = d_X(x,x') + d_Y(y,y')$.

Definition 2.4. *A morphism $\phi : M \to N$ of geometric modules over $X \times [0,1)$ is* continuously controlled at 1 *if for each $x \in X$ and each neighborhood U of $(x,1)$ in $X \times [0,1]$, there exists a (necessarily smaller) neighborhood V of $(x,1)$ such that ϕ does not cross $U \setminus V$: that is, if $v \in V$ and $y \notin U$, then $\phi_{yv} = \phi_{vy} = 0$.*

It is an exercise to check that the collection of continuously controlled morphisms in $\mathcal{A}_b(X)$ form a subcategory. Since the control condition only depends on the support of the morphism, this collection of morphisms is also closed under addition and negation, and direct sums in this subcategory agree with direct sums in $\mathcal{A}_b(X)$.

Definition 2.5. *Let X be a proper metric space. The category of locally finite geometric modules over $X \times [0,1)$ and continuously controlled morphisms, denoted $\mathcal{A}_c(X)$, is the subcategory of $\mathcal{A}_b(X \times [0,1))$ containing all objects, but only those morphisms with finite propagation and continuous control at 1. As explained above, this is an additive subcategory of $\mathcal{A}(X)$.*

For $Z \subset X$, we will write $\mathcal{A}_c^X(Z)$ for the category of controlled modules on $Z \times [0,1)$, where Z has the metric inherited from X. (This will be especially relevant when Z and X are simplicial complexes, since then Z has its own intrinsic simplicial metric, giving rise to a different category of controlled modules.) The K–theory of the category $\mathcal{A}_c^X(Z)$ is the continuously controlled algebraic K–theory of Z.

Given a closed subset $Z \subset X$, we define

$$\mathcal{A}_c^{X+}(Z) \subset \mathcal{A}_c(X)$$

to be the full subcategory on those geometric modules $M \in \mathcal{A}_c(X)$ which are supported "near" $Z \times [0,1)$; that is, $M \in \mathcal{A}_c^{X+}(Z)$ if and only if there exists $R > 0$ such that $M_{(x,t)} \neq 0$ implies $d(x, Z) < R$. When X is clear from context, we will simply write $\mathcal{A}_c^+(Z)$ rather than $\mathcal{A}_c^{X+}(Z)$.

Remark 2.6. *In Weiss [26, Section 2], a slightly different support condition for modules is used to define an analogue of our category $\mathcal{A}_c(X)$: namely the support of each module is required to be a discrete, closed subset of $X \times [0,1)$. This condition is equivalent to our local finiteness condition when X is a proper metric space, so that our category $\mathcal{A}_c(X)$ is the same as Weiss's $\mathcal{A}(X \times [0,1], X \times [0,1))$. We will only need to consider proper metric spaces in this article.*

The spaces whose controlled K–theory appears in this paper will all be simplicial complexes, equipped with the simplicial metric (possibly restricted from some larger complex). Given a simplicial complex P, the simplicial metric d_Δ on P is the unique path-length metric which restricts to the standard Euclidean metric on each simplex (we will assume all our simplices have diameter one). Explicitly,

$$d_\Delta(x,y) = \inf \sum_{i=0}^{N} d_\Delta(p_i, p_{i+1})$$

where the infimum is taken over all sequences $x = p_0, p_1, \ldots, p_N = y$ (with N arbitrary) such that p_i and p_{i+1} lie in the same simplex of K, and $d_\Delta(p_i, p_{i+1})$ is the Euclidean metric on a simplex containing both points. When x and y lie in different path components of K, we set $d_\Delta(x,y) = \infty$. Note that locally

finite simplicial complexes are always proper with respect to their simplicial metrics (this follows, for example, from the argument in Lemma 5.1 below, which can be used to show that each ball contains finitely many vertices).

We will need a lemma regarding the functoriality of controlled K–theory for maps between metric spaces. Versions of the following result are stated (without proof) in [2, 3, 26]; an equivariant version is proven in [1, Lemma 3.3]. For completeness, we sketch the argument.

Lemma 2.7. *Let $f : X \to Y$ be a continuous map of proper metric spaces which is proper (that is, $f^{-1}(C)$ is compact in X for all compact sets $C \subset Y$) and metrically coarse (that is, for each $R > 0$ there exists $S > 0$ such that $d_X(x_1, x_2) < R$ implies $d_Y(f(x_1), f(x_2)) < S$). Let $X' \subset X$ and $Y' \subset Y$ be closed subspaces with $f(X') \subset N_t(Y')$ for some $t > 0$. Then f induces functors $f_* \colon \mathcal{A}_c(X) \to \mathcal{A}_c(Y)$ and $f_* \colon \mathcal{A}_c^+(X') \to \mathcal{A}_c^+(Y')$.*

In particular, given a diagram of inclusions of simplicial complexes

$$
\begin{array}{ccccc}
P_0 & \hookrightarrow & P_1 & \hookrightarrow & P_2 \\
\uparrow & & \uparrow & & \uparrow \\
\downarrow & & \downarrow & & \downarrow \\
P_0' & \hookrightarrow & P_1' & \hookrightarrow & P_2'
\end{array}
$$

there is an induced functor $\mathcal{A}_c^{P_1+}(P_0) \to \mathcal{A}_c^{P_1'+}(P_0')$, where P_0 and P_1 are given the subspace metric inherited from the simplicial metric on P_2, while P_0' and P_1' are given the subspace metric inherited from the simplicial metric on P_2'.

Proof. We will construct the functor $f_* \colon \mathcal{A}_c^+(X') \to \mathcal{A}_c^+(Y')$; the other functors are special cases (note that inclusions of simplicial complexes are topological embeddings and decrease distances).

Let M be a geometric module in $\mathcal{A}_c^+(X')$. If f is not injective, one needs to redefine the category $\mathcal{A}_c^+(-)$ so that setting

$$
f_*(M)_{(y,t)} = \bigoplus_{x \in f^{-1}(y)} M_{(x,t)}
$$

is well-defined. See [26] for details. We will ignore this technicality in what follows. Since M is supported on a neighborhood of X', f is metrically coarse, and $f(X') \subset N_t(Y)$, the module $f(M)$ will be supported on a neighborhood of Y'. The behavior of f on morphisms is defined similarly; since f is metrically coarse, we see that $f(\phi)$ has finite propagation for each $\phi \in \mathcal{A}_c^+(X')$. Finally, we must check that for each $\phi \in \mathcal{A}_c^+(X')$, $f(\phi)$ is continuously controlled. Fix $y \in Y$, and consider a neighborhood U of $(y, 1)$ in $Y \times I$. Replacing U with a small ball around $(y, 1)$ if necessary, we may assume that the closure $\overline{U}$ is compact. Let $U' = (f \times \mathrm{Id}_I)^{-1}(U)$. For each $x \in f^{-1}(y)$, U' is a neighborhood of $(x, 1)$, so there exists a smaller neighborhood V_x of $(x, 1)$ such that $\phi_{z',z} = 0$ if $z \in V_x$, $z' \notin U'$ or $z \notin U'$, $z' \in V_x$. Since f is proper, $f^{-1}(y)$ is compact, so we may cover $(f \times \mathrm{Id}_I)^{-1}(y, 1)$

by finitely many of the sets V_x, say to $V_{x_1}, \ldots, V_{x_n}$. Since f is proper and continuous and $\overline{U}$ is compact, it follows that

$$C := (f \times \mathrm{Id}_I)\left((f \times \mathrm{Id}_I)^{-1}(\overline{U}) \setminus \bigcup_i V_{x_i}\right)$$

is compact. Now $V = U \setminus C$ is a neighborhood of $(y, 1)$, and one may now check that ϕ does not cross $U \setminus V = C \cap U$. □

Remark 2.8. *Most uses of Lemma 2.7 below will only require the statement regarding simplicial complexes. Note that by setting $P_0 = P_1$ and $P'_0 = P'_1$, we obtain a statement about the categories $\mathcal{A}_c(-)$.*

3. KAROUBI FILTRATIONS

We will use the notion of a Karoubi filtration to produce various Mayer–Vietoris sequences in controlled K–theory. Algebraically, a Karoubi filtration is a tool for collapsing a full subcategory of an additive category; geometrically it is a method for producing fibrations of K–theory spectra.

By abuse of notation we will write $A \in \mathcal{A}$ to mean that A is an object in $\mathcal{A}$. Furthermore, we will write $A = A_1 \oplus A_2$ to mean that there exist maps $i_j : A_j \to A$ making A the categorical direct sum of A_1 and A_2. We will always implicitly choose particular maps i_j, and we will denote the corresponding projections $A \to A_j$ by π_j.

Definition 3.1. *Let $\mathcal{S} \subset \mathcal{A}$ be a full additive subcategory of a small additive category $\mathcal{A}$. A* Karoubi filtration *on the pair $(\mathcal{A}, \mathcal{S})$ consists of an index set I and for each $A \in \mathcal{A}$ and each $i \in I$, a direct sum decomposition $A = A_i \oplus A'_i$ with $A_i \in \mathcal{S}$. These data must satisfy the following conditions:*

(1) *For each morphism $A \xrightarrow{f} S$ (with $S \in \mathcal{S}$) there exists $i \in I$ such that f factors as*
$$A = A_i \oplus A'_i \xrightarrow{\pi_1} A_i \longrightarrow S$$

(2) *For each morphism $S \xrightarrow{g} A$ (with $S \in \mathcal{S}$) there exists $i \in I$ such that g factors as*
$$S \longrightarrow A_i \xrightarrow{i_1} A_i \oplus A'_i = A$$

(3) *The index set I is a directed poset under the relation $i \leqslant j \iff$ for all $A \in \mathcal{A}$, A_i is a direct summand of A_j and A'_j is a direct summand of A'_i. (Here* directed *means that for each $i, j \in I$, there exists $k \in I$ such that $i, j \leqslant k$.)*

(4) *For each $A, B \in \mathcal{A}$ and each $i \in I$, we have $(A \oplus B)_i = A_i \oplus B_i$ and $(A \oplus B)'_i = A'_i \oplus B'_i$.*

Remark 3.2. *In the literature on Karoubi quotients, the term "filtered" is often used instead of "directed." In category theory, the term "directed" is standard.*

For any full additive subcategory $\mathcal{S} \subset \mathcal{A}$, the Karoubi quotient $\mathcal{A}/\mathcal{S}$ is the category with the same objects as $\mathcal{A}$, but with two morphisms identified if their difference factors through an object of $\mathcal{S}$. The following lemma is surely well-known, but seems not to have been made explicit previously.

Lemma 3.3. *If $\mathcal{S}$ is a full additive subcategory of the additive category $\mathcal{A}$, then $\mathcal{A}/\mathcal{S}$ is an additive category, and if $A_1 \xrightarrow{i_1} A \xleftarrow{i_2} A_2$ is a direct sum diagram in $\mathcal{A}$, then $A_1 \xrightarrow{[i_1]} A \xleftarrow{[i_2]} A_2$ is a direct sum diagram in $\mathcal{A}/\mathcal{S}$.*

Proof. It is elementary to check that $\mathcal{A}/\mathcal{S}$ is a category. The addition on morphisms is given by $[\phi] + [\psi] = [\phi + \psi]$. This is well-defined because if $\phi \colon A \to B$ factors through $S \in \mathcal{S}$ and $\psi \colon A \to B$ factors through $S' \in \mathcal{S}$, then $\phi + \psi$ is the composite

$$A \xrightarrow{\Delta} A \oplus A \xrightarrow{\phi \oplus \psi} B \oplus B \xrightarrow{\pi_1 + \pi_2} B,$$

where Δ is the diagonal map (i_1, i_2), so $\phi + \psi$ factors through $S \oplus S' \in \mathcal{S}$. It is now an elementary exercise to check that $\mathcal{A}/\mathcal{S}$ is additive.

For the comparison of direct sums, consider a diagram

$$
\begin{array}{ccc}
A_1 & \xrightarrow{[i_1]} A \xleftarrow{[i_2]} & A_2 \\
& \searrow_{[f]} \quad \Big\downarrow \quad \swarrow_{[g]} & \\
& C &
\end{array}
$$

in $\mathcal{A}/\mathcal{S}$. We must show that there exists a unique morphism $A \xrightarrow{[\phi]} C$ in $\mathcal{A}/\mathcal{S}$ making the diagram commute. Since $A = A_1 \oplus A_2$ in $\mathcal{A}$, we know there exists a morphism $f \oplus g : A \to C$ in $\mathcal{A}$ making the diagram

$$
\begin{array}{ccc}
A_1 & \xrightarrow{i_1} A \xleftarrow{i_2} & A_2 \\
& \searrow_{f} \quad \Big\downarrow{\scriptstyle f \oplus g} \quad \swarrow_{g} & \\
& C &
\end{array}
$$

commute (in $\mathcal{A}$), so existence is immediate. Now, say we have a second commutative diagram

$$
(2) \qquad\qquad
\begin{array}{ccc}
A_1 & \xrightarrow{[i_1]} A \xleftarrow{[i_2]} & A_2 \\
& \searrow_{[f]} \quad \Big\downarrow{\scriptstyle [\phi]} \quad \swarrow_{[g]} & \\
& C &
\end{array}
$$

in $\mathcal{A}/\mathcal{S}$. We must show that $[\phi] = [f \oplus g]$, i.e. that the morphism

$$f \oplus g - \phi : A \longrightarrow C$$

factors through an object in $\mathcal{S}$.

Note that since $A = A_1 \oplus A_2$ in $\mathcal{A}$, we have $\phi = (\phi \circ i_1) \oplus (\phi \circ i_2)$, i.e. ϕ is the unique map making the diagram

$$
\begin{array}{ccc}
A_1 & \xrightarrow{\;i_1\;} A \xleftarrow{\;i_2\;} & A_2 \\
& \phi\downarrow & \\
\phi i_1 \searrow & & \swarrow \phi i_2 \\
& C &
\end{array}
$$

commute. Similarly, we have

$$(3) \quad f \oplus g - \phi = f \oplus g - (\phi \circ i_1) \oplus (\phi \circ i_2) = (f - \phi \circ i_1) \oplus (g - \phi \circ i_2).$$

Commutativity of (2) means that $f - \phi \circ i_1$ and $g - \phi \circ i_2$ factor through objects S_1 and S_2 in $\mathcal{S}$ (respectively), so we may choose maps α_k and β_k ($k = 1, 2$) in $\mathcal{A}$ such that $f - \phi \circ i_1$ is the composite

$$A_1 \xrightarrow{\;\alpha_1\;} S_1 \xrightarrow{\;\beta_1\;} C$$

and $g - \phi \circ i_2$ is the composite

$$A_2 \xrightarrow{\;\alpha_2\;} S_2 \xrightarrow{\;\beta_2\;} C.$$

We have assumed that direct sums exist in $\mathcal{S}$, so we may choose a direct sum $S_1 \oplus S_2 \in \mathcal{S}$. The map $(f - \phi \circ i_1) \oplus (g - \phi \circ i_2)$ factors through $S_1 \oplus S_2$ because (letting j_1 and j_2 denote the inclusions of the summands into $S_1 \oplus S_2$), the diagram

commutes in $\mathcal{A}$ (by construction). Now (3) shows that $[f \oplus g] = [\phi]$, completing the proof. $\qquad\square$

The utility of Karoubi filtrations comes from the following result due to Pedersen–Weibel [19]; see also Cárdenas–Pedersen [6, Section 8].

Theorem 3.4. *If $\mathcal{S} \subset \mathcal{A}$ is a full additive subcategory of a small additive category $\mathcal{A}$ and $(\mathcal{A}, \mathcal{S})$ admits a Karoubi filtration, then then there is a long exact sequence in non-connective algebraic K–theory*

$$\cdots \longrightarrow K_*\mathcal{S} \longrightarrow K_*\mathcal{A} \longrightarrow K_*\mathcal{A}/\mathcal{S} \xrightarrow{\;\partial\;} K_{*-1}\mathcal{S} \longrightarrow \cdots$$

For our purposes, the key examples of Karoubi filtrations arise from restricting the support of geometric modules.

Definition 3.5. *Given any family of subspaces $\mathcal{Y}$ of a proper metric space X we may consider the full subcategory $\mathcal{A}_c(\mathcal{Y}) \subset \mathcal{A}_c(X)$ on those modules supported on $Y \times [0,1)$ for some $Y \in \mathcal{Y}$. Note that $\mathcal{A}_c^{X+}(Z) = \mathcal{A}_c(\{N_r(Z) : r \in \mathbb{N}\}) \subset \mathcal{A}(X)$. The category $\mathcal{A}_c(\mathcal{Y})$ is unchanged if we enlarge $\mathcal{Y}$ be adding subspaces of elements in $\mathcal{Y}$, so we may always assume that our families are closed under taking subspaces.*

The following lemma is a special case of Bartels and Rosenthal [2, (5.7)].

Lemma 3.6. *Let $\mathcal{Y}$ and $\mathcal{Z}$ be families of subspaces of a proper metric space X, and assume $\mathcal{Y}$ and $\mathcal{Z}$ are closed under finite unions. Then $\mathcal{A}_c(\mathcal{Y}), \mathcal{A}_c(\mathcal{Z}) \subset \mathcal{A}_c(X)$ are additive subcategories, and if for all $Y \in \mathcal{Y}$ there exists $Z \in \mathcal{Z}$ with $Y \subset Z$, then $\mathcal{A}_c(\mathcal{Y})$ is a full (additive) subcategory of $\mathcal{A}_c(\mathcal{Z})$. If, in addition, for each $Y \in \mathcal{Y}$ and each $r \in \mathbb{N}$ there exists $Y' \in \mathcal{Y}$ such that $N_r(Y) \subset Y'$, then the inclusion $\mathcal{A}_c(\mathcal{Y}) \subset \mathcal{A}_c(\mathcal{Z})$ admits a Karoubi filtration.*

In particular, for any subspace $Z \subset X$, the pair $\mathcal{A}_c^+(Z) \subset \mathcal{A}(X)$ admits a Karoubi filtration.

The direct sum decompositions making up these Karoubi filtration come from the following lemma, applied to the subspaces $Y \in \mathcal{Y}$.

Lemma 3.7. *Let $Z \subset X$ be a subspace of the proper metric space X. For any $M \in \mathcal{A}_c(X)$, the inclusions*

$$M(Z \times [0,1)) \hookrightarrow M \text{ and } M\left((X \setminus Z) \times [0,1)\right) \hookrightarrow M$$

yield a direct sum decomposition

$$M = M(Z \times [0,1)) \oplus M\left((X \setminus Z) \times [0,1)\right).$$

Proof. This is immediate, since a morphism $M \to N$ is defined as a family of morphisms $M_{(x,t)} \to N_{(y,s)}$ for all $(x,t), (y,s) \in X \times [0,1)$. $\square$

Definition 3.8. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be full subcategories of an additive category $\mathcal{A}$. Let $\mathcal{A}_1 \cap \mathcal{A}_2$ (the intersection of $\mathcal{A}_1$ and $\mathcal{A}_2$) be the full subcategory generated by those objects lying in both $\mathcal{A}_1$ and $\mathcal{A}_2$.*

Remark 3.9. *Note that if $\mathcal{Y}$ and $\mathcal{Z}$ are families of subspaces of a metric space X, then the intersection category $\mathcal{A}_c(\mathcal{X}) \cap \mathcal{A}_c(\mathcal{Y})$ is simply $\mathcal{A}_c(\{X \cap Y : X \in \mathcal{X}, Y \in \mathcal{Y}\})$. In particular, if $X_1, X_2 \subset X$, then*

$$\mathcal{A}_c^+(X_1) \cap \mathcal{A}_c^+(X_2) = \mathcal{A}_c(\{N_r(X_1) \cap N_s(X_2)\}_{r,s \in \mathbb{N}}),$$

and Lemma 3.6 shows that $\mathcal{A}_c^+(X_1) \cap \mathcal{A}_c^+(X_2) \subset \mathcal{A}_c(X)$ admits a Karoubi filtration. With the exception of the inclusion $\mathcal{A}_c(X)_{<1} \subset \mathcal{A}_c(X)$ discussed in Section 7, all the Karoubi filtrations in this paper follow from Lemma 3.6 by similar arguments.

We need another technical condition for some of our arguments.

Definition 3.10. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be full, additive subcategories of the additive category $\mathcal{A}$. We say that $(\mathcal{A}_1, \mathcal{A}_2)$ is* dispersed *if every morphism $\phi : A_1 \to A_2$ (with $A_i \in \mathcal{A}_i$) factors through an object in $\mathcal{A}_1 \cap \mathcal{A}_2$.*

The dispersion conditions encountered in this paper are all special cases of the following observation.

Lemma 3.11. *Let X be a metric space, and consider a family of subspaces $\{X_i\}_{i \in I}$. Assume that for each $i \in I$ and each $r \in \mathbb{N}$, there exists $j \in I$ such that $N_r(X_i) \subset X_j$. If $\mathcal{S} \subset \mathcal{A}_c(X)$ is a full subcategory that is closed under restriction of modules (meaning that for all $S \in Ob(\mathcal{S})$ and for all $Z \subset X$, $S(Z \times [0, 1)) \in Ob(\mathcal{S})$), then the pair $(\mathcal{A}_c(\{X_i\}), \mathcal{S})$ is dispersed.*

Proof. Consider a morphism $\phi \colon M \to S$ in $\mathcal{A}_c(X)$, with $M \in Ob(\mathcal{A}_c(\{X_i\}))$ and $S \in Ob(\mathcal{S})$. Then $\mathrm{supp}(M) \subset X_i \times [0, 1)$ for some $i \in I$, and

$$\left\{ (z, t) \,:\, \phi_{(z,t),(z',t')} \neq 0 \text{ for some } (z', t') \right\} \subset N_r(X_i) \times [0, 1) \subset X_j \times [0, 1)$$

for some $r > 0$ and some $j \in J$. Now ϕ factors through $S(X_j \times [0, 1))$. $\square$

In Section 4 we will build Mayer–Vietoris sequences in continuously controlled K–theory. These sequences will be applied in Section 6 to spaces of the form $\coprod_{r=1}^{\infty} Z^r$, covered by subspaces $\coprod_{r=1}^{\infty} U^r$ and $\coprod_{r=1}^{\infty} V^r$. These decompositions $Z^r = U^r \cup V^r$ will become finer (in a sense) as r increases, and we will want to ignore the subcategory $\mathcal{A}_c \left(\{\coprod_{r=1}^{R} Z^r\}_{R \in \mathbb{N}} \right)$. This will be done through the use of Karoubi quotients, and in the remainder of this section we discuss the necessary categorical set-up.

Lemma 3.12. *Let $\mathcal{S}, \mathcal{B} \subset \mathcal{A}$ be full additive subcategories of the additive category $\mathcal{A}$. Assume that:*

    (1) *the pairs $(\mathcal{B}, \mathcal{S} \cap \mathcal{B})$, $(\mathcal{A}, \mathcal{S})$, and $(\mathcal{A}, \mathcal{B})$ admit Karoubi filtrations;*
    (2) *$(\mathcal{B}, \mathcal{S})$ is dispersed.*

Then the full subcategory of $\mathcal{A}/\mathcal{S}$ on the objects of $\mathcal{B}$ is precisely $\mathcal{B}/(\mathcal{S} \cap \mathcal{B})$, and the inclusion $\mathcal{B}/(\mathcal{S} \cap \mathcal{B}) \subset \mathcal{A}/\mathcal{S}$ admits a Karoubi filtration.

Proof. We begin by examining the full subcategory of $\mathcal{A}/\mathcal{S}$ on the objects of $\mathcal{B}$. This category is formed by identifying two morphisms $\phi, \psi \colon B_1 \to B_2$ ($B_i \in \mathcal{B}$) if $\phi - \psi$ factors as

$$B_1 \xrightarrow{\alpha} S \xrightarrow{\beta} B_2$$

for some $S \in \mathcal{S}$. Since $(\mathcal{B}, \mathcal{S})$ is dispersed, α factors through an object of $\mathcal{B} \cap \mathcal{S}$, so $\phi \equiv \psi$ (modulo $\mathcal{B} \cap \mathcal{S}$). Hence the full subcategory of $\mathcal{A}/\mathcal{S}$ on the objects of $\mathcal{B}$ is precisely $\mathcal{B}/(\mathcal{B} \cap \mathcal{S})$.

Next, we must show that the inclusion $\mathcal{B}/(\mathcal{B} \cap \mathcal{S}) \subset \mathcal{A}/\mathcal{S}$ admits a Karoubi filtration. The filtration on $\mathcal{B}/(\mathcal{B} \cap \mathcal{S}) \subset \mathcal{A}/\mathcal{S}$ is exactly the same as the filtration on $\mathcal{B} \subset \mathcal{A}$: for $A \in \mathcal{A}$, let I denote the indexing set for the latter filtration. Then for each $i \in I$ we have a decomposition $A = B_i \oplus B_i'$ in $\mathcal{A}$ (with $B_i \in \mathcal{B}$), and this remains a direct sum decomposition in the category $\mathcal{A}/\mathcal{S}$ by Lemma 3.3. It now follows from the definitions that these decompositions give a Karoubi filtration on $\mathcal{B}/(\mathcal{B} \cap \mathcal{S}) \subset \mathcal{A}/\mathcal{S}$. $\square$

We record the universal property of Karoubi quotients, which we will use several times. The proof is an elementary exercise.

Lemma 3.13. *Say $G : \mathcal{A} \to \mathcal{B}$ is a functor between additive categories and $\mathcal{S} \subset \mathcal{A}$ is a full additive subcategory admitting a Karoubi filtration. If $G(\phi) = 0$ whenever $\phi \equiv 0$ (mod $\mathcal{S}$), then there is a unique additive functor*

$$\overline{G}\colon \mathcal{A}/\mathcal{S} \longrightarrow \mathcal{B}$$

such that the composite $\mathcal{A} \to \mathcal{A}/\mathcal{S} \xrightarrow{\overline{G}} \mathcal{B}$ equals G.

Our Mayer–Vietoris sequences will be built using the following version of the Third Isomorphism Theorem from elementary abstract algebra.

Proposition 3.14. *Let $\mathcal{A}$ be a small additive category with full additive subcategories $\mathcal{S}$, $\mathcal{A}_1$ and $\mathcal{A}_2$, and assume that $\mathcal{A}_1$ and $\mathcal{A}_2$ generate $\mathcal{A}$ in the sense that every $A \in \mathcal{A}$ admits a direct sum decomposition $A = A_1 \oplus A_2$ with $A_i \in \mathcal{A}_i$. Set $\mathcal{A}_{12} = \mathcal{A}_1 \cap \mathcal{A}_2$, and similarly set $\mathcal{S}_1 = \mathcal{S} \cap \mathcal{A}_1$, $\mathcal{S}_2 = \mathcal{S} \cap \mathcal{A}_2$, and $\mathcal{S}_{12} = \mathcal{S} \cap \mathcal{A}_{12}$.*

If the triples $\mathcal{S}, \mathcal{A}_1 \subset \mathcal{A}$ and $\mathcal{S}_2, \mathcal{A}_{12} \subset \mathcal{A}_2$ satisfy the conditions of Lemma 3.12 (in other words, if $(\mathcal{A}_1, \mathcal{S})$ and $(\mathcal{A}_{12}, \mathcal{S}_2)$ are dispersed, and all the relevant inclusions admit Karoubi filtrations), and if the pairs $(\mathcal{A}_1, \mathcal{A}_2)$ and $(\mathcal{A}_2, \mathcal{S})$ are dispersed, then the inclusion $\mathcal{A}_2 \hookrightarrow \mathcal{A}$ induces an equivalence of categories between Karoubi quotients as follows:

$$\frac{\mathcal{A}_2/\mathcal{S}_2}{\mathcal{A}_{12}/\mathcal{S}_{12}} \xrightarrow{\simeq} \frac{\mathcal{A}/\mathcal{S}}{\mathcal{A}_1/\mathcal{S}_1}$$

Proof. Lemma 3.12 guarantees that the displayed Karoubi quotients are well-defined. We begin by checking that the composite

$$(4) \qquad\qquad \mathcal{A}_2 \hookrightarrow \mathcal{A} \longrightarrow \mathcal{A}/\mathcal{S} \longrightarrow \frac{\mathcal{A}/\mathcal{S}}{\mathcal{A}_1/\mathcal{S}_1}$$

factors through the identifications in $\frac{\mathcal{A}_2/\mathcal{S}_2}{\mathcal{A}_{12}/\mathcal{S}_{12}}$, so that Lemma 3.13 yields a well-defined functor $F\colon \frac{\mathcal{A}_2/\mathcal{S}_2}{\mathcal{A}_{12}/\mathcal{S}_{12}} \to \frac{\mathcal{A}/\mathcal{S}}{\mathcal{A}_1/\mathcal{S}_1}$.

If $A_2, A_2' \in \mathcal{A}_2$ and $\phi : A_2 \to A_2'$ is a morphism in $\mathcal{A}$ which is equivalent to zero in $\frac{\mathcal{A}_2/\mathcal{S}_2}{\mathcal{A}_{12}/\mathcal{S}_{12}}$, then ϕ must factor through an object in either $\mathcal{S}_2$ or $\mathcal{A}_{12}$. These are subcategories of $\mathcal{S}$ and $\mathcal{A}_1$ (respectively), so such morphisms certainly map to zero under the composite (4). Applying Lemma 3.13 twice yields the desired functor F.

We must show that, up to isomorphism, every object is in the image of F. Every object $A \in \mathcal{A}$ can be written in the form $A = A_1 \oplus A_2$ with $A_i \in \mathcal{A}_i$, and we claim that in the Karoubi quotient $\frac{\mathcal{A}/\mathcal{S}}{\mathcal{A}_1/\mathcal{S}_1}$, the objects A and A_2 are isomorphic. Indeed, the inclusion $i_2 : A_2 \to A$ and the corresponding projection $\pi_2 : A \to A_2$ are inverses in this Karoubi quotient (the composite $\pi_2 i_2$ is the identity on A_2 by definition, and $\mathrm{Id}_A = i_1\pi_1 + i_2\pi_2$, so $\mathrm{Id}_A - i_2\pi_2 = i_1\pi_1$, which factors through A_1). This shows that up to isomorphism, every object is in the image of the map F.

To complete the proof, we must check that F is full and faithful. Fullness follows from the fact that $\mathcal{A}_2$ is a full subcategory of $\mathcal{A}$. Next, if ϕ_1 and ϕ_2

are morphisms in $\mathcal{A}_2$ that are equivalent in $(\mathcal{A}/\mathcal{S})/(\mathcal{A}_1/\mathcal{S}_1)$, then $\phi_1 - \phi_2$ factors through an object in either $\mathcal{S}$ or $\mathcal{A}_1$. Dispersion implies that $\phi_1 - \phi_2$ actually factors through an object of $\mathcal{S}_2$ or $\mathcal{A}_{12}$, so ϕ_1 and ϕ_2 are equivalent in the domain of F. Hence F is faithful. $\square$

4. Mayer–Vietoris sequences in continuously controlled K–theory

In this section we build Mayer–Vietoris sequences in continuously controlled K–theory analogous to the controlled Mayer–Vietoris sequences of Ranicki–Yamasaki [22, 23] (see also [13, Appendix B]). First we produce a general Mayer–Vietoris sequence for metric spaces, and then we specialize the construction to the Rips complexes that will be used in later sections.

4.1. Mayer–Vietoris for continuously controlled K–theory of metric spaces.

Proposition 4.1. *Let X be a proper metric space with subspaces $X_1, X_2 \subset X$ and assume that for some $r > 0$, $N_r(X_1) \cup N_r(X_2) = X$. Consider a family $\{S_i\}_{i \in I}$ of subspaces of X such that for each $t \in \mathbb{N}$ and each $i \in I$, there exists $j \in I$ such that $N_t(S_i) \subset S_j$. Let $\mathcal{S} = \mathcal{A}_c(\{S_i\}_{i \in I})$. We denote the intersection of $\mathcal{S}$ with $\mathcal{A}_c^+(X_i)$ by $\mathcal{S}_i$, and we denote the intersection of $\mathcal{S}$ with $\mathcal{A}_c^+(X_1) \cap \mathcal{A}_c^+(X_2)$ by $\mathcal{S}_{12}$.*

Then the natural maps from $\mathcal{A}_c^+(X_1)/\mathcal{S}_1$ and $\mathcal{A}_c^+(X_2)/\mathcal{S}_2$ to $\mathcal{A}_c(X)/\mathcal{S}$ are isomorphisms onto their images, and the natural map

$$\frac{\mathcal{A}_c^+(X_1) \cap \mathcal{A}_c^+(X_2)}{\mathcal{S}_{12}} \longrightarrow \mathcal{A}_c(X)/\mathcal{S}$$

is an isomorphism onto the intersection of the images of $\mathcal{A}_c^+(X_1)/\mathcal{S}_1$ and $\mathcal{A}_c^+(X_2)/\mathcal{S}_2$. Moreover, there is a long-exact Mayer–Vietoris sequence in non-connective K–theory

$$\cdots \longrightarrow K_{*+1}\left(\mathcal{A}_c(X)/\mathcal{S}\right) \xrightarrow{\partial} K_*\left(\frac{\mathcal{A}_c^+(X_1) \cap \mathcal{A}_c^+(X_2)}{\mathcal{S}_{12}}\right)$$
$$\xrightarrow{((i_1)_*,(i_2)_*)} K_*(\mathcal{A}_c^+(X_1)/\mathcal{S}_1) \oplus K_*(\mathcal{A}_c^+(X_2)/\mathcal{S}_2)$$
$$\xrightarrow{(j_1)_* - (j_2)_*} K_*(\mathcal{A}_c(X)/\mathcal{S}) \xrightarrow{\partial} \cdots,$$

in which $i_1, i_2, j_1,$ and j_2 are induced by the relevant inclusions of categories.

Proof. To construct the Mayer–Vietoris sequence, we will consider the diagram of additive categories

(5)

$$
\begin{array}{ccc}
\frac{\mathcal{A}_c^+(X_1) \cap \mathcal{A}_c^+(X_2, X)}{\mathcal{S}_{12}} & \xrightarrow{\ i_1\ } & \mathcal{A}_c^+(X_1)/\mathcal{S}_1 \\
\downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle j_1} \\
\mathcal{A}_c^+(X_2)/\mathcal{S}_2 & \xrightarrow{\ j_2\ } & \mathcal{A}_c(X)/\mathcal{S} \\
\downarrow{\scriptstyle q_2} & & \downarrow{\scriptstyle q_1} \\
\frac{\mathcal{A}_c^+(X_2)/\mathcal{S}_2}{\left(\mathcal{A}_c^+(X_1) \cap \mathcal{A}_c^+(X_2, X)\right)/\mathcal{S}_{12}} & \xrightarrow[\cong]{\ F\ } & \frac{\mathcal{A}_c(X)/\mathcal{S}}{\mathcal{A}_c^+(X_1)/\mathcal{S}_1},
\end{array}
$$

where the q_i are the Karoubi projections guaranteed by Lemma 3.12 and (as we will check) the induced map F is an equivalence of categories by Proposition 3.14. Applying K–theory produces two vertical long-exact sequences (Theorem 3.4), which can be weaved together using the isomorphism F_* to form the desired Mayer–Vietoris sequence (see, for example, Hatcher [15, Section 2.2, Exercise 38]). The facts that $\mathcal{A}_c^+(X_1)/\mathcal{S}_1$, $\mathcal{A}_c^+(X_2)/\mathcal{S}_2$, and $\frac{\mathcal{A}_c^+(X_1) \cap \mathcal{A}_c^+(X_2)}{\mathcal{S}_{12}}$ are isomorphic to their images in $\mathcal{A}_c^+(X)/\mathcal{S}$ will also follow from Lemma 3.12. The Karoubi filtrations needed to apply Lemma 3.12 come from Lemma 3.6 (see Remark 3.9), while the necessary dispersion conditions can be checked using Lemma 3.11.

To complete the proof, we must check that the conditions of Proposition 3.14 are satisfied, so that we obtain a well-defined equivalence of categories

$$
\frac{\mathcal{A}_c^+(X_2)/\mathcal{S}_2}{\left(\mathcal{A}_c^+(X_1) \cap \mathcal{A}_c^+(X_2, X)\right)/\mathcal{S}_{12}} \xrightarrow{\ F\ } \frac{\mathcal{A}_c(X)/\mathcal{S}}{\mathcal{A}_c^+(X_1)/\mathcal{S}_1}.
$$

The necessary Karoubi filtrations and the dispersion conditions are checked using Lemmas 3.6 and 3.11. To check that $\mathcal{A}_c^+(X_1)$ and $\mathcal{A}_c^+(X_2)$ generate $\mathcal{A}_c^+(X)$, recall that Lemma 3.7 guarantees a direct sum decomposition

$$
M = M\left(N_r(X_1) \times [0, 1)\right) \oplus M\left((X \times [0, 1)) \setminus (N_r(X_1) \times [0, 1))\right).
$$

Since $N_r(X_1) \cup N_r(X_2) = X$, we have

$$
(X \times [0, 1)) \setminus (N_r(X_1) \times [0, 1)) \subset N_r(X_2) \times [0, 1),
$$

and hence $M\left((X \times [0, 1)) \setminus (N_r(X_1) \times [0, 1))\right) \in \mathcal{A}_c^+(X_2)$. $\qquad\square$

4.2. **Decomposed sequences and Rips complexes.** We will apply our general Mayer–Vietoris sequence (Proposition 4.1) to decompositions of Rips complexes arising from decompositions of the underlying metric space. For the proof of our vanishing result for continuously controlled K–theory, it will be necessary to consider an infinite sequence of increasingly refined decompositions of our space. In fact, we will need to consider such sequences all at once by forming an infinite disjoint union of the spaces involved in the decompositions, and we will need to iterate this process (by further

decomposing each space in the initial decomposition). Such considerations lead to the notion of *decomposed sequence* introduced below.

We begin by recalling the construction of the Rips complex.

Definition 4.2. *Given a metric space X and a number $s > 0$, the Rips complex $P_s(X)$ is the simplicial complex with vertex set X and with a simplex $\langle x_0, \ldots, x_n \rangle$ whenever $d(x_i, x_j) \leqslant s$ for all $i, j \in \{0, \ldots, n\}$.*

Note that if X is a metric space with bounded geometry (i.e. if for each $r > 0$ there exists $N > 0$ such that for all $x \in X$, the ball $B_r(x)$ contains at most N points), then the Rips complex $P_s(X)$ is finite dimensional and locally finite. When forming Rips complexes, we will always assume that the underlying metric space has bounded geometry. Note that a finitely generated group, with the word metric arising from a finite generating set, always has bounded geometry. This is our main source of examples.

Definition 4.3. *Let X be a bounded geometry metric space and consider a sequence of subspaces $\mathcal{Z} = \{Z^1, Z^2, \ldots\}$, $Z^i \subset X$, equipped with decompositions*

$$(6) \qquad\qquad Z^r = \bigcup_{\alpha \in A_r} Z^r_\alpha$$

of each Z^r $(r = 1, 2, \ldots)$. We will call this data (the sequence $\mathcal{Z}$ together with the families $\{Z^r_\alpha\}_{\alpha \in A_r}$) a decomposed sequence in X. Note that the Z^r_α need not be disjoint.

Given a decomposed sequence $\mathcal{Z}$ in X, and a sequence of positive real numbers $\mathbf{s}$, the Rips complex $P_{\mathbf{s}}(\mathcal{Z})$ is the simplicial complex

$$P_{\mathbf{s}}(\mathcal{Z}) = \coprod_{r=1}^{\infty} \coprod_{\alpha \in A_r} P_{s_r}(Z^r_\alpha).$$

Note that Z^r_α and Z^s_β may overlap inside of X, so to be precise, points in $P_{\mathbf{s}}(\mathcal{Z})$ have the form (x, r, α) where $\alpha \in A_r$ and $x \in P_{s_r}(Z^r_\alpha)$. Each simplicial complex $P_{s_r}(Z^r_\alpha)$ is equipped with the metric induced by the simplicial metric on $P_{s_r}(X)$ (see Section 2 for a definition of the simplicial metric), and the distance between (x, r, α) and (y, r', β) is set to infinity unless $r = r'$ and $\alpha = \beta$. In other words, we consider $P_{\mathbf{s}}(Z)$ to be a subset of the infinite disjoint union

$$(7) \qquad\qquad \coprod_{r=1}^{\infty} \coprod_{\alpha \in A_r} P_{s_r}(X),$$

with the induced metric.

Remark 4.4. *If the indexing sets are clear from context, we will write X to denote a decomposed sequence in which every piece of the decomposition is the entire space X; then the infinite disjoint union (7) is written $P_{\mathbf{s}}(X)$.*

We will need to consider coverings of one decomposed sequence by two subsequences. In the applications, the subsequences will have lower "decomposition complexity" than the original sequence, in a sense that will be explained in Section 6.

Definition 4.5. *Let $\mathcal{Z} = (Z^1, Z^2, \ldots)$ be a decomposed sequence inside the metric space X, with decompositions $Z^r = \bigcup_{\alpha \in A_r} Z^r_\alpha$. We write $\mathcal{Z} = \mathcal{U} \cup \mathcal{V}$ if $\mathcal{U}$ and $\mathcal{V}$ are decomposed sequences in X whose decompositions are indexed over the same sets A_r and for each r and each α we have*

$$Z^r_\alpha = U^r_\alpha \cup V^r_\alpha.$$

Similarly, we write $\mathcal{U} \subset \mathcal{Z}$ if $\mathcal{U}$ is a decomposed sequence in X with the same indexing sets as $\mathcal{Z}$, and for each r and each α

$$U^r_\alpha \subset Z^r_\alpha.$$

Given a sequence $\mathbf{s}$ of positive real numbers, we define

$$\mathcal{A}^{\mathcal{Z}+}_c(P_\mathbf{s}(\mathcal{U})) := \mathcal{A}^{P_\mathbf{s}(\mathcal{Z})+}_c(P_\mathbf{s}(\mathcal{U}))$$

as in Definition 2.5. Note that both $P_\mathbf{s}(\mathcal{Z})$ and $P_\mathbf{s}(\mathcal{U})$ have the metric induced by the simplicial metric on $P_\mathbf{s}(X)$. We will sometimes drop $\mathcal{Z}$ from the superscript when it is clear from context.

In the proof of our vanishing result for continuously controlled K–theory (Theorem 6.4), it will be important to ignore the initial portion of a decomposed sequence. This is done via the following constructions.

Definition 4.6. *Given proper metric spaces $Y_1, Y_2, \ldots$ and a subcategory*

$$\mathcal{A} \subset \mathcal{A}_c\left(\coprod_{r>0} Y_r\right),$$

we define $\mathcal{S} = \mathcal{S}(\mathcal{A})$ to be the full subcategory of $\mathcal{A}$ consisting of those geometric modules supported on $\coprod_{r \leqslant R} Y_r \times [0, 1)$ for some $R > 0$. Note that

$$\mathcal{S} = \operatorname*{colim}_{R>0}\left(\mathcal{A} \cap \mathcal{A}_c\left(\coprod_{r<R} Y_r\right)\right).$$

We then define $\overline{\mathcal{A}} = \mathcal{A}/\mathcal{S}$.

Given decomposed sequences $\mathcal{U} \subset \mathcal{Z}$ in X, we set $\overline{\mathcal{A}_c}(\mathcal{Z}) = \overline{\mathcal{A}_c(\mathcal{Z})}$ and $\overline{\mathcal{A}_c}^+(\mathcal{U}) = \overline{\mathcal{A}_c}^{\mathcal{Z}+}(\mathcal{U}) = \overline{\mathcal{A}^{\mathcal{Z}+}_c(\mathcal{U})}$.

Remark 4.7. *The constructions $\overline{\mathcal{A}_c}$ and $\overline{\mathcal{A}_c}^+$ enjoy the same sort of functoriality as $\mathcal{A}_c$ and $\mathcal{A}^+_c$. The statements in Lemma 2.7 regarding functoriality of $\mathcal{A}_c$ and $\mathcal{A}^+_c$ for inclusions of simplicial complexes apply to inclusions of Rips complexes associated to inclusions of decomposed sequences, and Lemma 3.13 yields corresponding statements for $\overline{\mathcal{A}_c}$ and $\overline{\mathcal{A}_c}^+$.*

In the sequel, we will simply refer to Lemma 2.7 when constructing functors between categories $\overline{\mathcal{A}_c}(-)$ and $\overline{\mathcal{A}_c}^+(-)$.

4.3. Mayer–Vietoris for Rips complexes.

Theorem 4.8. *Let* $\mathcal{Z}$, $\mathcal{U}$, *and* $\mathcal{V}$ *be decomposed sequences in a bounded geometry metric space* X, *with* $\mathcal{Z} = \mathcal{U} \cup \mathcal{V}$, *and let* $\mathbf{s}$ *be a sequence of positive numbers. Then there is a long exact sequence in non-connective* K–*theory of the form*

$$(8) \qquad \cdots \longrightarrow K_* \left(\mathcal{I}_\mathbf{s}(\mathcal{U}, \mathcal{V}) \right) \xrightarrow{(i_1, i_2)} K_* \overline{\mathcal{A}_c}^{\mathcal{Z}+}(P_\mathbf{s}(\mathcal{U})) \oplus K_* \overline{\mathcal{A}_c}^{\mathcal{Z}+}(P_\mathbf{s}(\mathcal{V}))$$

$$\xrightarrow{(j_1)_* - (j_2)_*} K_* (\overline{\mathcal{A}_c}(P_\mathbf{s}(\mathcal{Z}))) \xrightarrow{\partial} K_{*-1} \left(\mathcal{I}_\mathbf{s}(\mathcal{U}, \mathcal{V}) \right) \longrightarrow \cdots,$$

where $\mathcal{I}_\mathbf{s}(\mathcal{U}, \mathcal{V})$ *denotes the intersection in* $\overline{\mathcal{A}_c}(P_\mathbf{s}(\mathcal{Z}))$ *of* $\overline{\mathcal{A}_c}^{\mathcal{Z}+}(P_\mathbf{s}(\mathcal{U}))$ *and* $\overline{\mathcal{A}_c}^{\mathcal{Z}+} P_\mathbf{s}(\mathcal{V})$. *The maps* i_1 *and* i_2 *are induced by the relevant inclusions of categories and the maps* j_1 *and* j_2 *are the functors associated to the inclusions of simplicial complexes* $P_\mathbf{s}(\mathcal{U}) \hookrightarrow P_\mathbf{s}(\mathcal{Z})$ *and* $P_\mathbf{s}(\mathcal{V}) \hookrightarrow P_\mathbf{s}(\mathcal{Z})$.

Proof. By Proposition 4.1 it suffices to check that

$$N_1 \left(P_\mathbf{s}(\mathcal{U}) \right) \cup N_1 \left(P_\mathbf{s}(\mathcal{V}) \right) = P_\mathbf{s}(\mathcal{Z}).$$

Given a simplex $\sigma = \langle x_0, \ldots, x_n \rangle$ in $P_\mathbf{s}(\mathcal{Z})$, we either have $x_0 \in U_\alpha^r$ for some r and α or we have $x_0 \in V_\alpha^r$ for some r and α. In the former case, $\langle x_0 \rangle$ is a 0–simplex in $P_\mathbf{s}\mathcal{U}$, and $\sigma \subset N_1 \left(\langle x_0 \rangle \right) \subset N_1 \left(P_\mathbf{s}(\mathcal{U}) \right)$. In the latter case, $\sigma \subset N_1 \left(P_\mathbf{s}(\mathcal{V}) \right)$. $\qquad \square$

4.4. Mayer–Vietoris for relative Rips complexes.

We will need another Mayer–Vietoris sequence for the proof of our vanishing theorem (Theorem 6.4), involving the relative Rips complexes introduced by Guentner–Tessera–Yu [13].

Definition 4.9. *Given subspaces* Z, W *of a bounded geometry metric space* X *and real numbers* $0 < s < s'$, *the* relative Rips complex $P_{s,s'}(Z; W)$ *is defined to be the subcomplex of* $P_{s'}(X)$ *consisting of those simplices* $\langle x_0, \ldots, x_n \rangle$ *such that at least one of the following conditions holds:*

    (1) $x_0, \ldots, x_n \in Z$ *and* $d(x_i, x_j) \leqslant s$ *for all* i, j;
    (2) $x_0, \ldots, x_n \in W$ *and* $d(x_i, x_j) \leqslant s'$ *for all* i, j.

We equip $P_{s,s'}(Z, W)$ *with the metric induced by the simplicial metric on* $P_{s,s'}(X, W)$. *(It will be crucial for our arguments that we* do not *use the metric inherited from the simplicial metric on* $P_{s'}(X)$; *see in particular Lemma 5.2.) Note that in this definition, we do not require* $W \subset Z$.

If $\mathcal{Z} = (Z^1, Z^2, \ldots)$ *is a decomposed sequence in* X *with decompositions* $Z^r = \bigcup_{\alpha \in A_r} Z_\alpha^r$, $\mathbb{W} = \{W_\alpha^r\}_{r \geqslant 1, \alpha \in A_r}$ *is a family of subspaces of* X, *and* $\mathbf{s}, \mathbf{s}'$ *are sequences of positive real numbers, we define the relative Rips complex*

$$P_{\mathbf{s}, \mathbf{s}'}(\mathcal{Z}, \mathbb{W}) := \coprod_r \coprod_\alpha P_{s_r, s_r'}(Z_\alpha^r, W_\alpha^r) \subset \coprod_r \coprod_\alpha P_{s_r, s_r'}(X, W_\alpha^r) =: P_{\mathbf{s}, \mathbf{s}'}(X, \mathbb{W}),$$

and we give $P_{\mathbf{s}, \mathbf{s}'}(\mathcal{Z}, \mathbb{W})$ *the metric induced by the simplicial metric on* $P_{\mathbf{s}, \mathbf{s}'}(X, \mathbb{W})$.

Given a covering $\mathcal{Z} = \mathcal{U} \cup \mathcal{V}$ of a decomposed sequence by two sub-sequences, we will need to consider a relative Rips complex in which the "larger" simplices are constrained to lie near both $\mathcal{U}$ and $\mathcal{V}$.

Definition 4.10. *Consider decomposed sequences $\mathcal{Z} = \mathcal{U} \cup \mathcal{V}$ in a metric space X. Given a sequence of positive real numbers $\mathbf{T}$, we define $\mathbb{W}_{\mathbf{T}} = \mathbb{W}_{\mathbf{T}}(\mathcal{U}, \mathcal{V}, \mathcal{Z})$ to be the family*

$$\mathbb{W}_{\mathbf{T}} = \{N_{T_r}(U_\alpha^r) \cap N_{T_r}(V_\alpha^r) \cap Z_\alpha^r\}_{r,\alpha}.$$

We can now form the relative Rips complexes $P_{\mathbf{s},\mathbf{s}'}(\mathcal{Z}, \mathbb{W}_{\mathbf{T}}), P_{\mathbf{s},\mathbf{s}'}(\mathcal{U}, \mathbb{W}_{\mathbf{T}})$ and $P_{\mathbf{s},\mathbf{s}'}(\mathcal{V}, \mathbb{W}_{\mathbf{T}})$ as in Definition 4.9. Following Definition 4.6, we set

$$\overline{\mathcal{A}_c}\left(P_{\mathbf{s},\mathbf{s}'}(\mathcal{Z}, \mathbb{W}_{\mathbf{T}})\right) := \mathcal{A}_c\left(P_{\mathbf{s},\mathbf{s}'}(\mathcal{Z}, \mathbb{W}_{\mathbf{T}})\right)/\mathcal{S}.$$

We define $\overline{\mathcal{A}_c}^{\mathcal{Z}+}\left(P_{\mathbf{s},\mathbf{s}'}(\mathcal{U}, \mathbb{W}_{\mathbf{T}})\right)$ and $\overline{\mathcal{A}_c}^{\mathcal{Z}+}\left(P_{\mathbf{s},\mathbf{s}'}(\mathcal{V}, \mathbb{W}_{\mathbf{T}})\right)$ similarly, by allowing modules supported on neighborhoods of $P_{\mathbf{s},\mathbf{s}'}(\mathcal{U}, \mathbb{W}_{\mathbf{T}})$ (or $P_{\mathbf{s},\mathbf{s}'}(\mathcal{U}, \mathbb{W}_{\mathbf{T}})$) inside $P_{\mathbf{s},\mathbf{s}'}(\mathcal{Z}, \mathbb{W}_{\mathbf{T}})$ (recall that both of these complexes are given the metric inherited from the simplicial metric on $P_{\mathbf{s},\mathbf{s}'}(X, \mathbb{W}_{\mathbf{T}})$).

Theorem 4.11. *Let $\mathcal{Z}$, $\mathcal{U}$, $\mathcal{V}$ and X be as in Theorem 4.8. Then for any sequences $\mathbf{s}, \mathbf{s}'$, and $\mathbf{T}$ of positive real numbers, we can form the sequence $\mathbb{W}_{\mathbf{T}} = \mathbb{W}_{\mathbf{T}}(\mathcal{U}, \mathcal{V}, \mathcal{Z})$ as above, and there is a long exact Mayer–Vietoris sequence in non-connective K–theory of the form*

$$\cdots \longrightarrow K_{*+1}\overline{\mathcal{A}_c}P_{\mathbf{s},\mathbf{s}'}(\mathcal{Z}; \mathbb{W}_{\mathbf{T}}) \xrightarrow{\partial} K_*\mathcal{I}'_{\mathbf{s},\mathbf{s}',\mathbf{T}}(\mathcal{U}, \mathcal{V})$$

$$\longrightarrow K_*\overline{\mathcal{A}_c}^{\mathcal{Z}+}P_{\mathbf{s},\mathbf{s}'}(\mathcal{U}; \mathbb{W}_{\mathbf{T}}) \oplus K_*\overline{\mathcal{A}_c}^{\mathcal{Z}+}P_{\mathbf{s},\mathbf{s}'}(\mathcal{V}; \mathbb{W}_{\mathbf{T}})$$

$$\longrightarrow K_*\overline{\mathcal{A}_c}P_{\mathbf{s},\mathbf{s}'}(\mathcal{Z}; \mathbb{W}_{\mathbf{T}}) \xrightarrow{\partial} \cdots,$$

where $\mathcal{I}'_{\mathbf{s},\mathbf{s}',\mathbf{T}}(\mathcal{U}, \mathcal{V})$ is the intersection of the subcategories $\overline{\mathcal{A}_c}^{\mathcal{Z}+}P_{\mathbf{s},\mathbf{s}'}(\mathcal{U}; \mathbb{W}_{\mathbf{T}})$ and $\overline{\mathcal{A}_c}^{\mathcal{Z}+}P_{\mathbf{s},\mathbf{s}'}(\mathcal{V}; \mathbb{W}_{\mathbf{T}})$ inside $\overline{\mathcal{A}_c}P_{\mathbf{s},\mathbf{s}'}(\mathcal{Z}; \mathbb{W}_{\mathbf{T}})$.

Proof. We apply Proposition 4.1. The conditions are checked just as in the proof of Theorem 4.8: for each r and each α, every point in

$$P_{s_r,s'_r}\left(Z_\alpha^r, N_{T_r}(U_\alpha^r) \cap N_{T_r}(V_\alpha^r)\right)$$

is within distance 1 of either

$$P_{s_r,s'_r}\left(U_\alpha^r, N_{T_r}(U_\alpha^r) \cap N_{T_r}(V_\alpha^r)\right) \text{ or } P_{s_r,s'_r}\left(V_\alpha^r, N_{T_r}(U_\alpha^r) \cap N_{T_r}(V_\alpha^r)\right).$$

$\square$

4.5. **A comparison of Mayer–Vietoris sequences.** For the arguments in Section 6, we will need to compare the absolute and relative Mayer–Vietoris sequences from Sections 4.3 and 4.4.

Theorem 4.12. *Let $\mathcal{Z}, \mathcal{U}, \mathcal{V}$, and X be as in Proposition 4.8.*
    Then there are functors

$$\overline{\mathcal{A}_c}^{\mathcal{Z}+}P_{\mathbf{s}}(\mathcal{U}) \xrightarrow{i_\mathcal{U}} \overline{\mathcal{A}_c}^{\mathcal{Z}+}P_{\mathbf{s},\mathbf{s}'}(\mathcal{U}, \mathbb{W}_{\mathbf{T}}), \quad \overline{\mathcal{A}_c}^{\mathcal{Z}+}P_{\mathbf{s}}(\mathcal{V}) \xrightarrow{i_\mathcal{V}} \overline{\mathcal{A}_c}^{\mathcal{Z}+}P_{\mathbf{s},\mathbf{s}'}(\mathcal{V}, \mathbb{W}_{\mathbf{T}}),$$

$$\overline{\mathcal{A}_c}P_{\mathbf{s}}(\mathcal{Z}) \xrightarrow{\gamma} \overline{\mathcal{A}_c}P_{\mathbf{s},\mathbf{s}'}(\mathcal{Z}, \mathbb{W}_{\mathbf{T}}), \ \ and \ \ \mathcal{I}_{\mathbf{s}}(\mathcal{U}, \mathcal{V}) \xrightarrow{\rho} \mathcal{I}'_{\mathbf{s},\mathbf{s}',\mathbf{T}}(\mathcal{U}, \mathcal{V})$$

such that the diagram of Mayer–Vietoris sequences

(9)

$$
\begin{array}{ccc}
K_*\left(\mathcal{I}_{\mathbf{s}}(\mathcal{U}, \mathcal{V})\right) & \xrightarrow{\ \ \ \ \rho_* \ \ \ \ } & K_*\left(\mathcal{I}'_{\mathbf{s},\mathbf{s}',\mathbf{T}}(\mathcal{U}, \mathcal{V})\right) \\
\downarrow & & \downarrow \\
K_*^+(P_{\mathbf{s}}\mathcal{U}) \oplus K_*^+(P_{\mathbf{s}}\mathcal{V}) & \xrightarrow{(i_{\mathcal{U}})_* \oplus (i_{\mathcal{V}})_*} & K_*^+(P_{\mathbf{s},\mathbf{s}'}(\mathcal{U}, \mathbb{W}_{\mathbf{T}})) \oplus K_*^+(P_{\mathbf{s},\mathbf{s}'}(\mathcal{V}, \mathbb{W}_{\mathbf{T}})) \\
\downarrow & & \downarrow \\
K_*(P_{\mathbf{s}}(\mathcal{Z})) & \xrightarrow{\ \ \ \ \gamma_* \ \ \ \ } & K_*\left(P_{\mathbf{s},\mathbf{s}'}(\mathcal{Z}, \mathbb{W}_{\mathbf{T}})\right) \\
\downarrow \partial & & \downarrow \partial \\
K_{*-1}\left(\mathcal{I}_{\mathbf{s}}(\mathcal{U}, \mathcal{V})\right) & \xrightarrow{\ \ \ \ \rho_* \ \ \ \ } & K_{*-1}\left(\mathcal{I}'_{\mathbf{s},\mathbf{s}',t}(\mathcal{U}, \mathcal{V})\right).
\end{array}
$$

is commutative; note that in the middle two rows we have used K_* and K_*^+ as shorthand for $K_*(\overline{\mathcal{A}_c}(-))$ and $K_*(\overline{\mathcal{A}_c}^{\mathcal{Z}+}(-))$ (respectively).

Proof. The Mayer–Vietoris sequences are produced by Theorem 4.8 and Theorem 4.11. The functors $i_{\mathcal{U}}$, $i_{\mathcal{V}}$, and γ are induced by the relevant inclusions of simplicial complexes, which satisfy the hypotheses of Lemma 2.7. There is then an induced functor $I_{\mathbf{s}}(\mathcal{U}, \mathcal{V}) \xrightarrow{\rho} \mathcal{I}'_{\mathbf{s},\mathbf{s}',\mathbf{T}}$ between the intersection categories. By Lemma 3.13, these functors produce a commutative diagram $\mathcal{D}$ of categories consisting of two diagrams of the form (5), with one mapping to the other. It is an exercise to check that after taking K–theory, we obtain a commutative diagram of Mayer–Vietoris sequences as claimed. (Note that after taking K–theory spectra, the diagram $\mathcal{D}$ consists of a map between two homotopy (co)-cartesian squares of spectra, together with the homotopy cofibers of the vertical maps in these squares. It is a general fact that maps between homotopy (co)-cartesian squares of spectra yield commutative diagrams of Mayer–Vietoris sequences.) $\qquad\square$

5. Metric properties of Rips complexes

In this section we record some basic, well-known geometric results about Rips complexes. Similar statements may be found in Guentner–Tessera–Yu [13, Appendix A]. For completeness, we provide detailed proofs.

Lemma 5.1. *Let X be a metric space with bounded geometry. Then there exists a constant C, depending only on the dimension of $P_s(X)$, such that for all $x, y \in X$,*

$$d(x, y) \leqslant sCd_\Delta(x, y).$$

Proof. We will show that $C = (2\sqrt{2}+1)^{N-1}$ suffices, where N is the dimension of $P_s(X)$. (If $P_s(X)$ is zero-dimensional, we set $C = 1$.) By definition of the simplicial metric, we must show that

$$d(x,y) \leqslant s(2\sqrt{2}+1)^{N-1} l(\gamma)$$

for all sequences $\gamma = (p_0, p_1, \ldots, p_k)$ such that $p_0 = x$, $p_k = y$, and for each i, p_i and p_{i-1} lie in a common simplex σ_i (which we may assume is the smallest simplex containing p_i and p_{i-1}). Let $\dim(\gamma) = \max_i \dim(\sigma_i)$, and note that $\dim(\gamma) \leqslant N$. We will show by induction on $\dim(\gamma)$ that $d(x,y) \leqslant s(2\sqrt{2}+1)^{\dim(\gamma)-1}l(\gamma)$. Note that if $\sigma_i = \sigma_{i+1}$, then we may shorten γ by removing p_{i+1}, so we may assume without loss of generality that $\sigma_i \neq \sigma_{i+1}$, which implies that p_i and p_{i+1} lie in the boundary of σ_i.

If $\dim(\gamma) = 1$, then $p_i \in X$ for each i, and we have

$$d(x,y) \leqslant \sum_{i=0}^{k-1} d(p_i, p_{i+1}) \leqslant sk = s(2\sqrt{2}+1)^0 l(\gamma).$$

Now assume the result for paths of dimension at most $n-1$, and say $\dim(\gamma) = n$. We will replace γ by a nearby path of lower dimension. By assumption, for some i we have $\sigma_i = \langle x_0, \ldots, x_n \rangle$ ($x_i \in X$). Reordering the x_i if necessary, we may further assume that $p_i \in \langle x_0, \ldots, x_{n-1} \rangle$ and $p_{i+1} \in \langle x_1, \ldots, x_n \rangle$. Letting $\overline{p_i}$ and $\overline{p_{i+1}}$ denote the orthogonal projections of these points to the plane spanned by $\langle x_1, \ldots, x_{n-1} \rangle$ (note that these orthogonal projections necessarily lie inside $\langle x_1, \ldots, x_{n-1} \rangle$), we will replace γ by the piecewise geodesic path $\gamma' = (p_0, \ldots, p_i, \overline{p_i}, \overline{p_{i+1}}, p_{i+1}, \ldots, p_k)$.

We claim that $d_\Delta(p_i, \overline{p_i})$ and $d_\Delta(\overline{p_{i+1}}, p_{i+1})$ are at most $\sqrt{2}d_\Delta(p_i, p_{i+1})$. In barycentric coordinates, we may write $p_i = \sum_{i=0}^n a_i x_i$ (with $a_n = 0$) and $p_{i+1} = \sum_{i=0}^n b_i x_i$, (with $b_0 = 0$). Setting $w = (a_0 + a_1)x_1 + \sum_{i=2}^{n-1} a_i x_i$ we have $d_\Delta(p_i, w) = \sqrt{2}a_0$ and $d_\Delta(p_i, p_{i+1}) \geqslant a_0$, so $d_\Delta(p_i, w) \leqslant \sqrt{2}d_\Delta(p_i, p_{i+1})$. Hence

$$d_\Delta(p_i, \overline{p_i}) \leqslant d_\Delta(p_i, w) \leqslant \sqrt{2}d_\Delta(p_i, p_{i+1}),$$

as desired. Similarly, $d_\Delta(\overline{p_{i+1}}, p_{i+1}) \leqslant \sqrt{2}d_\Delta(p_i, p_{i+1})$.

Since orthogonal projections decrease distances, we also have

$$d_\Delta(\overline{p_i}, \overline{p_{i+1}}) \leqslant d_\Delta(p_i, p_{i+1}),$$

and hence the piecewise geodesic path $(p_0, \ldots, p_i, \overline{p_i}, \overline{p_{i+1}}, p_{i+1}, \ldots, p_k)$ has length at most $(2\sqrt{2}+1)d_\Delta(p_i, p_{i+1})$. Repeating this procedure for each n–simplex among the σ_i, we obtain a new path γ' (from x to y) which lies entirely in the $(n-1)$–skeleton of $P_s(X)$ (meaning that $\dim(\gamma') \leqslant n-1$) and satisfies $l(\gamma') \leqslant (2\sqrt{2}+1)l(\gamma)$. By induction, we know that $d(x,y) \leqslant (2\sqrt{2}+1)^{n-2}l(\gamma')$, so $d(x,y) \leqslant (2\sqrt{2}+1)^{n-1}l(\gamma)$, completing the proof. $\square$

It is important to note that no bound exists in the opposite direction: if $d(x,y) > s$, then x and y may lie in different connected components of $P_s(X)$, in which case $d_\Delta(x,y) = \infty$.

The following result will allow us to compare distances in relative Rips complexes. For this result to hold, it is crucial that we give the relative Rips complex $P_{s,s'}(Z,W)$ the metric inherited from the simplicial metric on $P_{s,s'}(X,W)$ rather than $P_{s'}(X)$.

Note that each point x in a simplicial complex K can be written uniquely, in barycentric coordinates, in the form $x = \sum c_{v_i}(x)v_i$ with $c_{v_i}(x) > 0$ for each i. We will refer to the vertices v_i as the *barycentric vertices* of x. Given a vertex $v \in K$, we can extend c_v to a continuous function from K to $[0,1]$ by setting $c_v(x) = 0$ if v is not a barycentric vertex of x.

Lemma 5.2. *Let $W \subset X$ be metric spaces, and assume X has bounded geometry. Given $s' \geqslant s > 0$, let $N_t(P_{s'}(W))$ denote a t–neighborhood of $P_{s'}(W)$ inside $P_{s,s'}(X,W)$. Then for all $x \in X \cap N_t(P_{s'}(W))$ (where X is viewed as the 0–skeleton of $P_{s,s'}(X,W)$), we have*

$$(10) \qquad\qquad d(x,W) \leqslant (t+1)C(s,X)s,$$

where $C(s,X) = (2\sqrt{2} + 1)^{\dim(P_s(X))-1}$.

It follows that inside the simplicial complex $P_{s'}(X)$, we have inclusions

$$(11) \quad N_t(P_{s'}(W)) \subset P_{s,s'}(N_{(t+2)C(s,X)s}(W),W) \subset P_{s'}(N_{(t+2)C(s,X)s}(W)),$$

where on the left, the neighborhood is still taken with respect to the simplicial metric on $P_{s,s'}(X,W)$. Additionally, for any $U \subset X$, we have inclusions

$$(12) \qquad N_t(P_{s,s'}(U,W)) \subset N_t(P_{s'}(U \cup W)) \subset P_{s'}(N_{(t+2)C(s,X)s}(U \cup W)),$$

where the first neighborhood is taken inside $P_{s,s'}(X,W)$ and the second is taken inside $P_{s,s'}(X,U\cup W)$.

Proof. Say $x \in X \cap N_t(P_{s'}(W))$. Then there exists a piecewise geodesic path γ in $P_{s,s'}(X,W)$, starting at x and ending at a point in $P_{s'}(W)$, such that $l(\gamma) < t$, where $l(\gamma)$ is the sum of the lengths of the geodesics making up γ.

Since X has bounded geometry, the path $\gamma\colon [0,1] \to P_s(X)$ meets only finitely many (closed) simplices $\sigma_1,\ldots,\sigma_m$. Let $J \subset \{1,\ldots,m\}$ be the subset of those j such that σ_j has a vertex lying in W; note that $J \neq \emptyset$ since γ ends in $P_{s'}(W)$. Let $r \in [0,1]$ be the minimum element of the compact set $\bigcup_{j\in J} \gamma^{-1}(\sigma_j)$. If $t < r$ then the barycentric vertices of $\gamma(t)$ all lie in $X \setminus W$, so $\gamma(t) \in P_s(X)$. Continuity of the barycentric coordinate functions implies that the barycentric vertices $x_0,\ldots,x_n$ of $\gamma(r)$ all lie in $X \setminus W$ as well. By choice of r, we know that $\gamma(r)$ lies in a simplex σ having a vertex $w \in W$. This simplex must contain $\langle x_0,\ldots,x_n \rangle$, and since $x_i \notin W$ we conclude (from the definition of the relative Rips complex) $\sigma \subset P_s(X)$. Concatenating $\gamma|_{[0,r]}$ with a geodesic in σ connecting $\gamma(r)$ and w yields a piecewise geodesic path, inside $P_s(X)$, of length at most $t+1$. Hence the simplicial distance, in $P_s(X)$, from x to w is at most $t+1$, and Lemma 5.1 tells us that $d(x,w) \leqslant (t+1)C(s,X)s$. This proves (10).

The first containment in (11) follows from the distance estimate (10), since if $z \in N_t(P_{s'}(W))$ lies in a simplex $\langle x_0,\ldots,x_n \rangle \subset P_{s,s'}(X,W)$, then for each

i, the simplicial distance (in $P_{s,s'}(X,W)$) from x_i to $P_{s'}(W)$ is at most $t+1$, and the so the first half of the lemma shows that $x_i \in N_{(t+2)C(s,X)s}(W)$. The second containment in (11) is immediate from the definitions.

The first containment in (12) follows from the fact that the simplicial metric on $P_{s,s'}(X, U \cup W)$ is smaller than the simplicial metric on the subcomplex $P_{s,s'}(X, W)$, while the second follows from (11), with $U \cup W$ playing the role of W. $\qquad \square$

6. Controlled K–theory for spaces of finite decomposition complexity

We now apply the results of Sections 4 and 5 to the continuously controlled K–theory of spaces with *finite decomposition complexity.*

We begin by reviewing some definitions from Guentner–Tessera–Yu [12, 13], where the notion of decomposition complexity was first introduced. A set of metric spaces will be called a *metric family.* Let $\mathfrak{B}$ denote the class of uniformly bounded metric families; that is, a family $\mathcal{F}$ lies in $\mathfrak{B}$ if there exists $R > 0$ such that $\mathrm{diam}(F) < R$ for all $F \in \mathcal{F}$. Given a class $\mathfrak{D}$ of metric families, we say that a metric family $\mathcal{F} = \{F_\alpha\}_{\alpha \in A}$ *decomposes* over $\mathfrak{D}$ if for every $r > 0$ and every $\alpha \in A$ there exists a decomposition $F_\alpha = U_\alpha^r \cup V_\alpha^r$ and r–disjoint decompositions

$$U_\alpha^r = \coprod_{i \in I(r,\alpha)}^{r\text{–disjoint}} U_{\alpha i}^r, \quad V_\alpha^r = \coprod_{j \in J(r,\alpha)}^{r\text{–disjoint}} V_{\alpha j}^r$$

such that the families $\{U_{\alpha i}^r\}_{\alpha, i}$ and $\{V_{\alpha j}^r\}_{\alpha, j}$ lie in $\mathfrak{D}$.[1] Here r–disjoint simply means that $d(U_{\alpha i}^r, U_{\alpha j}^r) > r$ whenever $i \neq j$. We set $\mathfrak{D}_0 = \mathfrak{B}$, and given a successor ordinal $\gamma + 1$ we define $\mathfrak{D}_{\gamma+1}$ to be the class of all metric spaces which decompose over $\mathfrak{D}_\gamma$. If γ is a limit ordinal, we define

$$\mathfrak{D}_\gamma = \bigcup_{\beta < \gamma} \mathfrak{D}_\beta.$$

(This definition will make the limit ordinal cases of all our transfinite induction arguments trivial.)

We will often drop the indexing set from our notation when no confusion is likely; for example we write $\{Z_\alpha\}_\alpha$ rather than $\{Z_\alpha\}_{\alpha \in A}$.

Definition 6.1. *We say that a metric space X has finite decomposition complexity if the single-element family $\{X\}$ lies in $\mathfrak{D}_\gamma$ for some ordinal γ. (We often write $X \in \mathfrak{D}_\gamma$ rather than $\{X\} \in \mathfrak{D}_\gamma$.)*

---

[1]In the original definition in [13, Section 2], one assumes instead that there exists a family $\mathcal{F}' \in \mathfrak{D}$ containing all the spaces $U_{\alpha j}^r$ and $V_{\alpha j}^r$. However, since the collections of families $\mathcal{D}_\gamma$ defined here, and the analogous families defined in [13], are closed under forming finite unions of families and under subfamilies, the two definitions of $\mathcal{D}_\gamma$ agree. (With our definition of D_γ, closure under finite unions is checked by transfinite induction; closure under subfamilies follows, for example, from Lemma 6.3.)

Remark 6.2. *If $X \in \mathfrak{D}_\gamma$ for some ordinal γ, then in fact there exists a countable ordinal γ' such that $X \in \mathfrak{D}_{\gamma'}$. This is proven in Guentner–Tessera–Yu [12, Theorem 2.2.2].*

Lemma 6.3. *Let X be a metric space, and let $\{Z_\alpha\}_{\alpha \in A}$ and $\{Y_\beta\}_{\beta \in B}$ be metric families with $Z_\alpha, Y_\beta \subset X$ for all α, β. Say $\{Z_\alpha\}_{\alpha \in A} \in \mathfrak{D}_\gamma$ for some ordinal γ. Assume further that there exists $t > 0$ such that for all $\beta \in B$, there exists $\alpha \in A$ with $Y_\beta \subset N_t(Z_\alpha)$. Then $\{Y_\beta\}_\beta \in \mathfrak{D}_\gamma$ as well. (Note here that the parameter t is independent of $\beta \in B$.)*

Proof. By transfinite induction. In the base case, we have a uniform bound D on the diameter of the Z_α, and $D + t$ gives a uniform bound on the diameter of the Y_β, so $\{Y_\beta\}_\beta \in \mathfrak{D}_0$. Now say $\gamma = \delta + 1$ is a successor ordinal, and assume the result for $\mathfrak{D}_\delta$. If $\{Z_\alpha\}_\alpha \in \mathfrak{D}_\gamma$, then for each $r > 0$ and each Z_α there exist U_α^r and V_α^r such that $Z_\alpha = U_\alpha^r \cup V_\alpha^r$ and

$$U_\alpha^r = \overset{r-\text{disjoint}}{\coprod_i} U_{\alpha i}^r, \qquad V_\alpha^r = \overset{r-\text{disjoint}}{\coprod_j} V_{\alpha j}^r$$

with $\{U_{\alpha i}^r\}_{\alpha, i}$ and $\{V_{\alpha j}^r\}_{\alpha, j}$ in $\mathfrak{D}_\delta$. Now for each $\beta \in B$, we know there exists $\alpha = \alpha(\beta) \in A$ such that $Y_\beta \subset N_t(Z_\alpha)$. We now have decompositions

$$Y_\beta = (N_t(U_\alpha^r) \cap Y_\beta) \cup (N_t(V_\alpha^r) \cap Y_\beta),$$

and $(r - 2t)$–disjoint decompositions

$$N_t(U_\alpha^r) \cap Y_\beta = \coprod_i N_t(U_{\alpha i}^r) \cap Y_\beta$$

and

$$N_t(V_\alpha^r) \cap Y_\beta = \coprod_j N_t(V_{\alpha j}^r) \cap Y_\beta.$$

By induction we know that the families

$$\{N_t(U_{\alpha i}^r) \cap Y_\beta\}_{\beta, i} \text{ and } \{N_t(V_{\alpha j}^r) \cap Y_\beta\}_{\beta, j}$$

lie in $\mathfrak{D}_\delta$. Since $r - 2t$ tends to infinity with r, we see that the family $\{Y_\beta\}_\beta$ decomposes over $\mathfrak{D}_\delta$, as desired. The case of limit ordinals is trivial. $\square$

We now come to the main result of this section.

Theorem 6.4. *If X is a bounded geometry metric space with finite decomposition complexity, then for each $* \in \mathbb{Z}$ we have*

$$\operatorname*{colim}_{s \to \infty} K_* \left(\mathcal{A}_c(P_s X) \right) = 0,$$

where the colimit is taken with respect to the maps

$$K_* \left(\mathcal{A}_c(P_s X) \right) \xrightarrow{\eta_{s, s'}} K_* \left(\mathcal{A}_c(P_{s'} X) \right)$$

induced by applying Lemma 2.7 to the inclusions

$$P_s X \hookrightarrow P_{s'} X.$$

We will deduce Theorem 6.4 from a closely related vanishing result for the trivial decomposed sequence $\mathcal{X} = \{X, X, X, \ldots\}$ with trivial decomposition (i.e. the decomposition of X into the one-element family $\{X\}$) at each level.

Definition 6.5. *Let* $\underline{\mathbf{Seq}}$ *denote the partially ordered set consisting of all non-decreasing sequences of (non-zero) natural numbers, with the ordering* $(s_1, s_2, \ldots) \leqslant (s'_1, s'_2, \ldots)$ *if* $s_i \leqslant s'_i$ *for all* i. *Note that* $\underline{\mathbf{Seq}}$ *is directed.*
  For each $\mathbf{s} \leqslant \mathbf{s}' \in \underline{\mathbf{Seq}}$, *we define*

$$\eta_{\mathbf{s},\mathbf{s}'} = \eta_{\mathbf{s},\mathbf{s}'}(\mathcal{Z}) \colon K_* \left(\overline{\mathcal{A}_c}(P_\mathbf{s}(\mathcal{Z})) \right) \longrightarrow K_* \left(\overline{\mathcal{A}_c}(P_{\mathbf{s}'}(\mathcal{Z})) \right)$$

to be the map induced by the inclusion $P_\mathbf{s}(\mathcal{Z}) \subset P_{\mathbf{s}'}(\mathcal{Z})$.

Proposition 6.6. *If* X *is a bounded geometry metric space with finite decomposition complexity, then for each non-decreasing sequence* $\mathbf{s} \in \underline{\mathbf{Seq}}$ *and each element* $x \in K_*(\overline{\mathcal{A}_c}(P_\mathbf{s}(\mathcal{X})))$ *there exists* $\mathbf{s}' \in \underline{\mathbf{Seq}}$, *with* $\mathbf{s}' \geqslant \mathbf{s}$, *such that the map*

$$(13) \qquad K_* \left(\overline{\mathcal{A}_c}(P_\mathbf{s}\mathcal{X}) \right) \xrightarrow{\eta_{\mathbf{s},\mathbf{s}'}} K_* \left(\overline{\mathcal{A}_c}(P_{\mathbf{s}'}\mathcal{X}) \right)$$

sends x *to zero.*

  We will see in the proof that $\mathbf{s}'$ may depend on x.

Remark 6.7. *Note that Proposition 6.6 is equivalent to the statement that*

$$\operatorname*{colim}_{\mathbf{s} \in \underline{\mathbf{Seq}}} K_* \left(\overline{\mathcal{A}_c}(P_\mathbf{s}\mathcal{X}) \right) = 0.$$

Proof of Theorem 6.4 assuming Proposition 6.6. We apply Proposition 6.6 with $\mathbf{s} = (s, s, \ldots)$. Given $\mathbf{s}' \geqslant \mathbf{s}$ and $m \geqslant 1$, consider the diagram
(14)

$$\operatorname*{colim}_n \mathcal{A}_c(\textstyle\coprod_{r=1}^n P_{s'_r}(X))$$

Here the maps i, $\bar{i}$, and j_n are induced by inclusions of simplicial complexes, $\pi_\mathbf{s}$ and $\pi_{\mathbf{s}'}$ are the Karoubi projections, the functor μ sends a geometric module M on $P_s(X)$ to the constant sequence $(M, M, \ldots)$ (and similarly for morphisms), and q_m is the functor which restricts a geometric module to the mth term in the sequence $(X, X, \ldots)$.
  Let $x \in K_*\mathcal{A}_c(P_sX)$ be given. In K–theory, $\bar{i}_*$ is the map (13), so Proposition 6.6 implies that we can choose $\mathbf{s}' \geqslant \mathbf{s}$ such that $\bar{i}_* \left(\pi_\mathbf{s} \circ \mu(x) \right) = 0$. For $m > n$ the composite $q_m \circ j_n$ is the constant functor mapping all objects to 0, so $(q_m)_*(j_n)_* = 0$ in K–theory. However, for any m, the composite $q_m \circ i \circ \mu$

is simply the functor η_{s,s'_m} induced by the inclusion $P_s(X) \hookrightarrow P_{s'_m}(X)$. Since the third column of Diagram (14) is a Karoubi sequence, chasing the diagram shows that $i_* \mu_*(x) = (j_m)_*(y)$ for some $m \geqslant 1$ and some $y \in K_* \left(\mathcal{A}_c(\coprod_{r=1}^n P_{s'_r}(X)) \right)$. Now

$$(\eta_{s,s'})_*(x) = (q_{m+1})_* \circ i_* \circ \mu_*(x) = (q_{m+1})_*(j_m)_*(y) = 0.$$

The result now follows, since the colimit in Theorem 6.4 is defined in terms of the maps $\eta_{s,s'}$.   □

To prove the desired vanishing result for the map (13), we will proceed through an induction for decomposed sequences inside X.

Definition 6.8. *Let $\mathcal{Z}$ be a decomposed sequence in X. We say that $\mathcal{Z}$ is a vanishing sequence (or more briefly, $\mathcal{Z}$ is vanishing) if for each $\mathbf{s} \in \underline{\mathbf{Seq}}$ and each $x \in K_*(\overline{\mathcal{A}_c}(P_\mathbf{s} \mathcal{Z}))$, there exists $\mathbf{s'} \geqslant \mathbf{s}$ such that x maps to zero under*

$$K_*(\overline{\mathcal{A}_c}(P_\mathbf{s}(\mathcal{Z}))) \xrightarrow{\eta_{\mathbf{s},\mathbf{s'}}} K_*(\overline{\mathcal{A}_c}(P_{\mathbf{s'}} \mathcal{Z})).$$

For each ordinal γ, let $\mathfrak{D}_\gamma(X)$ denote the set of metric families $\{Z_\alpha\}_{\alpha \in A} \in \mathfrak{D}_\gamma$ with $Z_\alpha \subset X$ for each $\alpha \in A$. If $\mathcal{Z} = \{Z^r\}_r$ is a decomposed sequence in X with decompositions $Z^r = \bigcup_{\alpha \in A_r} Z_\alpha^r$, by abuse of notation we write $\mathcal{Z} \in \mathfrak{D}_\gamma(X)$ if $\{Z_\alpha^r\}_{\alpha \in A_r} \in \mathfrak{D}_\gamma(X)$ for each $r \in \mathbb{N}$.

We say that $\mathfrak{D}_\gamma(X)$ is vanishing if all decomposed sequences $\mathcal{Z} \in \mathfrak{D}_\gamma(X)$ are vanishing.

Definition 6.9. *Let $\mathcal{Z} = \{Z^r\}_r$ be a decomposed sequence in X with decompositions $Z^r = \bigcup_\alpha Z_\alpha^r$, and let $\mathbf{T}$ be a sequence of positive real numbers. We define $N_\mathbf{T}(\mathcal{Z})$ to be the decomposed sequence $\{N_{T_r}(Z^r)\}_r$, with decompositions $N_{T_r}(Z^r) = \bigcup_\alpha N_{T_r}(Z_\alpha^r)$.*

The next lemma is an immediate consequence of Lemma 6.3.

Lemma 6.10. *Let $\mathcal{Z}$ be a decomposed sequence in X, and say $\mathcal{Z} \in \mathfrak{D}_\gamma(X)$ for some ordinal γ. If $\mathcal{Y}$ is another decomposed sequence in X, and $\mathcal{Y} \subset N_\mathbf{T}(\mathcal{Z})$ for some sequence $\mathbf{T}$ of positive real numbers, then $\mathcal{Y} \in \mathfrak{D}_\gamma(X)$ as well.*

We will prove the following generalization of Proposition 6.6.

Proposition 6.11. *If X is a bounded geometry metric space with finite decomposition complexity, then $\mathfrak{D}_\gamma(X)$ is vanishing for every ordinal γ.*

The proof of Proposition 6.11 will be by transfinite induction on the ordinal γ, and will fill the remainder of the section.

For the rest of the section, we fix a bounded geometry metric space X. We first consider the base case of our induction, $\mathcal{Z} \in \mathcal{D}_0(X)$. This means $\mathcal{Z}$ is a decomposed sequence in X for which each family $\{Z_\alpha^r\}_\alpha$ is uniformly bounded. Hence for each r, there exists $N(r)$ such that for

all $\alpha \in A_r$, the diameter of Z_α^r is at most $N(r)$. This means that if $\mathbf{s}' > \mathbf{N} := (N(1), N(2), \ldots)$, the simplicial complex

$$P_{\mathbf{s}'}(\mathcal{Z}) = \coprod_r \coprod_\alpha P_{s'_r}(Z_\alpha^r)$$

is a disjoint union of simplices, one for each pair (r, α). The following lemma will now establish the base case of our induction.

Lemma 6.12. *Say $\mathcal{Z}$ is a decomposed sequence in X such that for some sequence $\mathbf{N} \in \underline{\mathbf{Seq}}$, the diameter of Z_α^r is at most N_r (for all r and all α). Then for $\mathbf{s} \geqslant \mathbf{N}$, we have $K_* \overline{\mathcal{A}_c}(P_{\mathbf{s}}(\mathcal{Z})) = 0$ for all $* \in \mathbb{Z}$.*

Proof. We have already observed that for $\mathbf{s} \geqslant \mathbf{N}$, $P_{\mathbf{s}}(\mathcal{Z})$ is a disjoint union of simplices, all at infinite distance from one another. We claim that the controlled K-theory of such a metric space vanishes. The category $\mathcal{A}_c(\Delta^n)$, where Δ^n is the standard Euclidean n–simplex, is equivalent to the category $\mathcal{A}_c(\{*\})$, where $\{*\}$ is the one-point metric space (this follows from the fact that the identity map on Δ^n is Lipschitz homotopic to a constant map; see Bartels [3, Corollary 3.19]). We now see that $K_* \mathcal{A}_c(P_{\mathbf{s}'} \mathcal{Z})$ is the same as the $K_*(\mathcal{A}_c(\coprod\{*\}))$, where $\coprod\{*\}$ denotes an infinite disjoint union of points at infinite distance from one another. The category $\mathcal{A}_c(\coprod\{*\})$ has trivial K–theory, because it admits an Eilenberg Swindle (this is analogous to Bartels [3, 3.20], which treats the case of a single point). A similar argument shows that the subcategory

$$\mathcal{S} = \operatorname*{colim}_n \mathcal{A}_c \left(\coprod_{r=1}^n \coprod_\alpha P_{s_r}(Z_\alpha^r) \right) \subset \mathcal{A}_c(P_{\mathbf{s}}(\mathcal{Z}))$$

has trivial K–theory. We conclude that $K_* \overline{\mathcal{A}_c} P_{\mathbf{s}}(\mathcal{Z}) = 0$ for all $*$ by examining the long exact sequence in K–theory associated to the Karoubi sequence $\mathcal{S} \hookrightarrow \mathcal{A}_c P_{\mathbf{s}}(\mathcal{Z}) \to \overline{\mathcal{A}_c} P_{\mathbf{s}}(\mathcal{Z})$. $\square$

If γ is a limit ordinal and Proposition 6.11 holds for all $\beta < \gamma$, it follows immediately from the definitions that Proposition 6.11 also holds for γ.

Next, consider a successor ordinal $\gamma = \beta + 1$ and assume that $\mathfrak{D}_\beta(X)$ is vanishing. Consider a decomposed sequence $\mathcal{Z} \in \mathfrak{D}_\gamma(X)$ and a sequence $\mathbf{s} \in \underline{\mathbf{Seq}}$. Let $C_r = (2\sqrt{2}+1)^{\dim(P_{s_r}(X))-1}$ be the sequence of constants from Lemma 5.1, and let $\mathbf{C} = \{C_r\}_r$. Then for each r and each α we may choose decompositions $Z_\alpha^r = U_\alpha^r \cup V_\alpha^r$ and $(C_r \cdot s_r \cdot r)$–disjoint decompositions

$$(15) \qquad U_\alpha^r = \overset{C_r \cdot s_r \cdot r\text{–disjoint}}{\coprod_i} U_{\alpha i}^r \qquad \text{and} \qquad V_\alpha^r = \overset{C_r \cdot s_r \cdot r\text{–disjoint}}{\coprod_j} V_{\alpha j}^r$$

with $\{U_{\alpha i}^r\}_{\alpha,i}$ and $\{V_{\alpha j}^r\}_{\alpha,j}$ in $\mathfrak{D}_\beta$ for each r. Setting $U^r = \bigcup_\alpha U_\alpha^r$ and $V^r = \bigcup_\alpha V_\alpha^r$, we have decomposed sequences $\mathcal{U} = \{U^r\}_r$ and $\mathcal{V} = \{V^r\}_r$, with decompositions $U^r = \bigcup_\alpha U_\alpha^r$ and $V^r = \bigcup_\alpha V_\alpha^r$; note that $\mathcal{Z} = \mathcal{U} \cup \mathcal{V}$.

On the other hand, we can also consider $\mathcal{U}$ and $\mathcal{V}$ as decomposed sequences under the finer decompositions $U^r = \bigcup_{\alpha,i} U^r_{\alpha i}$ and $U^r = \bigcup_{\alpha,j} U^r_{\alpha j}$. We will denote these more finely decomposed sequences by $\mathcal{U}'$ and $\mathcal{V}'$. Note that $\mathcal{U}', \mathcal{V}' \in \mathcal{D}_\beta(X)$. We will use this observation to prove that $\mathcal{U}$ and $\mathcal{V}$ are vanishing sequences (see Lemma 6.17 below).

For any non-decreasing sequences $\mathbf{s} \leqslant \mathbf{s}' \leqslant \mathbf{s}''$, Theorems 4.8, 4.11, and 4.12 imply that there is a commutative diagram as follows, in which the first column comes from the Mayer–Vietoris sequence in Theorem 4.8 and the second column is the colimit, over $t > 0$, of the Mayer–Vietoris sequences from Theorem 4.11 (we write $\bigcup_t$ rather than colim_t to save space):

(16)

$$
\begin{array}{ccc}
\bigcup_t K^+_* \left(P_{\mathbf{s},\mathbf{s}'}(\mathcal{U}, \mathbb{W}_{t\mathcal{C}\mathbf{s}}) \right) & \xrightarrow{\ \mu_{\mathbf{s},\mathbf{s}',\mathbf{s}''}\ } & \bigcup_t K^+_* \left(P_{\mathbf{s}''}(N^{\mathcal{Z}}_{t\mathbf{C}\mathbf{s}}\mathcal{U}) \right) \\
\oplus & & \oplus \\
\bigcup_t K^+_* \left(P_{\mathbf{s},\mathbf{s}'}(\mathcal{V}, \mathbb{W}_{t\mathcal{C}\mathbf{s}}) \right) & & \bigcup_t K^+_* \left(P_{\mathbf{s}''}(N^{\mathcal{Z}}_{t\mathbf{C}\mathbf{s}}\mathcal{V}) \right)
\end{array}
$$

$$
\begin{array}{ccccc}
 & & \downarrow {\scriptstyle i_{\mathcal{U}}+i_{\mathcal{V}}} & & \\
K_*(P_{\mathbf{s}}(\mathcal{Z})) & \xrightarrow{\ \gamma_{\mathbf{s},\mathbf{s}'}\ } & \bigcup_t K_* \left(P_{\mathbf{s},\mathbf{s}'}(\mathcal{Z}, \mathbb{W}_{t\mathcal{C}\mathbf{s}}) \right) & \xrightarrow{\ \zeta_{\mathbf{s},\mathbf{s}',\mathbf{s}''}\ } & K_*(P_{\mathbf{s}''}(\mathcal{Z})) \\
\downarrow {\scriptstyle \partial} & & \downarrow {\scriptstyle \partial} & & \\
K_{*-1}\left(\mathcal{I}_{\mathbf{s}}(\mathcal{U}, \mathcal{V}) \right) & \xrightarrow{\ \rho_{\mathbf{s},\mathbf{s}'}\ } & \bigcup_t K_{*-1}\left(\mathcal{I}'_{\mathbf{s},\mathbf{s}',t}(\mathcal{U}, \mathcal{V}) \right). & &
\end{array}
$$

We now explain the various terms in Diagram (16).

- The functor K_* is shorthand for $K_* \overline{\mathcal{A}_c}$.
- $t\mathcal{C}\mathbf{s}$ is the product sequence with r^{th} term $tC_r s_r$, and $C_r = C_r(s_r, X)$ is the constant from Lemma 5.1.
- The sequence $\mathbb{W}_{t\mathcal{C}\mathbf{s}} = \mathbb{W}_{t\mathcal{C}\mathbf{s}}(\mathcal{U}, \mathcal{V}, \mathcal{Z})$ was defined in Definition 4.10.
- The functor K^+_* is shorthand for $K_* \overline{\mathcal{A}_c}^{\mathcal{Z}+}$ (Definition 4.6 and 4.10).
- The maps $\gamma = \gamma_{\mathbf{s},\mathbf{s}'}$ and $\rho = \rho_{\mathbf{s},\mathbf{s}'}$ are simply the compositions of the maps appearing in Theorem 4.12 (for any chosen $t > 0$) with the natural maps to the colimits. By definition of the colimit, these maps do not depend on a choice of t, and Theorem 4.12 implies that the left-hand square in Diagram (16) commutes.
- In the third column, $N^{\mathcal{Z}}_{t\mathbf{C}\mathbf{s}}\mathcal{U}$ is the decomposed sequence

$$\{ Z^r \cap N_{tC_r s_r} U^r \}_r$$

  with decompositions

$$Z^r \cap N_{tC_r s_r} U^r = \bigcup_\alpha Z^r_\alpha \cap N_{tC_r s_r} U^r_\alpha,$$

  and similarly for $\mathcal{V}$ in place of $\mathcal{U}$.
- The vertical map in the third column arises from the inclusions

$$P_{\mathbf{s}''}\left(N^{\mathcal{Z}}_{t\mathbf{C}\mathbf{s}}\mathcal{U} \right) \subset P_{\mathbf{s}''}(\mathcal{Z}) \text{ and } P_{\mathbf{s}''}\left(N^{\mathcal{Z}}_{t\mathbf{C}\mathbf{s}}\mathcal{V} \right) \subset P_{\mathbf{s}''}(\mathcal{Z}).$$

- To describe the horizontal map $\zeta = \zeta_{\mathbf{s},\mathbf{s}',\mathbf{s}''}$, note that for each $t > 0$, the inclusion of simplicial complexes

$$P_{\mathbf{s},\mathbf{s}'}(\mathcal{Z}, \mathbb{W}_{t\mathcal{C}\mathbf{s}}) \subset P_{\mathbf{s}''}(\mathcal{Z})$$

induces a functor after applying $\overline{\mathcal{A}_c}(-)$ (Lemma 2.7). These maps are compatible as t increases, and ζ is the induced map from the colimit.
- The map $\mu_{\mathbf{s},\mathbf{s}',\mathbf{s}''}$ is the direct sum of maps $\mu_{\mathbf{s},\mathbf{s}',\mathbf{s}''}(\mathcal{U})$ and $\mu_{\mathbf{s},\mathbf{s}',\mathbf{s}''}(\mathcal{V})$ induced by the inclusions

$$P_{\mathbf{s},\mathbf{s}'}(\mathcal{U}, \mathbb{W}_{t\mathcal{C}\mathbf{s}}) \subset P_{\mathbf{s}''}(N_{t\mathbf{C}\mathbf{s}}^{\mathcal{Z}}\mathcal{U}) \quad\text{and}\quad P_{\mathbf{s},\mathbf{s}'}(\mathcal{V}, \mathbb{W}_{t\mathcal{C}\mathbf{s}}) \subset P_{\mathbf{s}''}(N_{t\mathbf{C}\mathbf{s}}^{\mathcal{Z}}\mathcal{V}).$$

Note that the term-wise colimit of a sequence of exact sequences is exact, so the second column of Diagram (16) is exact. Commutativity of the right-hand square in Diagram (16) is immediate from the definitions of the functors inducing the maps.

We will prove the following two lemmas, which will allow us to deduce that $\mathcal{Z}$ is vanishing by chasing Diagram (16).

Lemma 6.13. *For each sequence* $\mathbf{s} \in \underline{\mathbf{Seq}}$ *and each* $x \in K_{*-1}(\mathcal{I}_{\mathbf{s}}(\mathcal{U}, \mathcal{V}))$, *there exists* $\mathbf{s}' \geqslant \mathbf{s}$ *such that* $\rho_{\mathbf{s},\mathbf{s}'}(x) = 0$.

Lemma 6.14. *For each pair of sequences* $\mathbf{s} \leqslant \mathbf{s}'$, *and for each element*

$$x \in \left(\bigcup_t K_*^+ \left(P_{\mathbf{s},\mathbf{s}'}(\mathcal{U}, \mathbb{W}_{t\mathcal{C}\mathbf{s}}) \right) \right) \bigoplus \left(\bigcup_t K_*^+ \left(P_{\mathbf{s},\mathbf{s}'}(\mathcal{V}, \mathbb{W}_{t\mathcal{C}\mathbf{s}}) \right) \right),$$

there exists $\mathbf{s}'' \geqslant \mathbf{s}'$ *such that* $\mu_{\mathbf{s},\mathbf{s}',\mathbf{s}''}(x) = 0$.

Proof of Proposition 6.11 assuming Lemmas 6.13 and 6.14. For simplicity, we drop most subscripts from the maps in Diagram (16). For each $\mathbf{s}$ and each element $y \in K_* \left(\overline{\mathcal{A}_c} \left(P_{\mathbf{s}}(\mathcal{Z}) \right) \right)$, we have $\partial(\gamma x) = \rho(\partial x)$. By Lemma 6.13, we can choose $\mathbf{s}'$ large enough so that $\rho(\partial x) = 0$. Exactness of the second column in Diagram (16) then shows that $\gamma(x) = (i_{\mathcal{U}} + i_{\mathcal{V}})(x_1, x_2)$ for some x_1, x_2. Now Lemma 6.14 tells us that for $\mathbf{s}''$ large enough, we have $\mu(x_1, x_2) = 0$, and it follows from commutativity of the right-hand square of Diagram (16) that $\zeta(\gamma x) = \zeta((i_{\mathcal{U}} + i_{\mathcal{V}})(x_1, x_2)) = 0$. However, the composite $\zeta \circ \gamma$ is simply the natural map

$$\eta_{\mathbf{s},\mathbf{s}'}\colon K_* \left(\overline{\mathcal{A}_c}(P_{\mathbf{s}}(\mathcal{Z})) \right) \xrightarrow{\eta_{\mathbf{s},\mathbf{s}'}} K_* \left(\overline{\mathcal{A}_c}(P_{\mathbf{s}'}(\mathcal{Z})) \right).$$

Hence $\mathcal{Z}$ is a vanishing sequence, and our induction is complete.

To prove Lemma 6.13, we need to compare two versions of the notion of *vanishing sequence*. Consider a pair of decomposed sequences $\mathcal{W} \subset \mathcal{Y}$ in X. For any sequence $\mathbf{s}$, the simplicial complex

$$P_{\mathbf{s}}(\mathcal{W}) = \coprod_r \coprod_\alpha P_{s_r} W_\alpha^r$$

has two metrics: its intrinsic simplicial metric, and the metric it inherits from the intrinsic simplicial metric on the larger simplicial complex

$$P_{\mathbf{s}}(\mathcal{Y}) = \coprod_r \coprod_\alpha P_{s_r} Y_\alpha^r.$$

Correspondingly, there are two different categories of controlled modules on $P_{\mathbf{s}}(\mathcal{W})$, which we denote by

$$\overline{\mathcal{A}_c}^{\mathcal{W}}(P_{\mathbf{s}}(\mathcal{W})) \text{ and } \overline{\mathcal{A}_c}^{\mathcal{Y}}(P_{\mathbf{s}}(\mathcal{W}))$$

respectively.

For the purposes of the next lemma, we will say that $\mathcal{W}$ is *intrinsically vanishing* if for every sequence $\mathbf{s}$ and every $x \in K_*\left(\overline{\mathcal{A}_c}^{\mathcal{W}}(P_{\mathbf{s}}(\mathcal{W}))\right)$, there exists $\mathbf{s}' > \mathbf{s}$ such that the image of x under the natural map

$$K_*\left(\overline{\mathcal{A}_c}^{\mathcal{W}}(P_{\mathbf{s}}(\mathcal{W}))\right) \longrightarrow K_*\left(\overline{\mathcal{A}_c}^{\mathcal{W}}(P_{\mathbf{s}'}(\mathcal{W}))\right)$$

is zero. We will say that $\mathcal{W}$ is *vanishing with respect to* $\mathcal{Y}$ if for every sequence $\mathbf{s}$ and every $x \in K_*\left(\overline{\mathcal{A}_c}^{\mathcal{Y}}(P_{\mathbf{s}}(\mathcal{W}))\right)$, there exists $\mathbf{s}' > \mathbf{s}$ such that the image of x under the natural map

$$K_*\left(\overline{\mathcal{A}_c}^{\mathcal{Y}}(P_{\mathbf{s}}(\mathcal{W}))\right) \longrightarrow K_*\left(\overline{\mathcal{A}_c}^{\mathcal{Y}}(P_{\mathbf{s}'}(\mathcal{W}))\right)$$

is zero.

Lemma 6.15. *Let $\mathcal{W} \subset \mathcal{Y}$ be decomposed sequences in X. Then $\mathcal{W}$ is intrinsically vanishing if and only if $\mathcal{W}$ is vanishing with respect to $\mathcal{Y}$.*

Proof. Since K–theory commutes with directed colimits of additive categories (see Quillen [21, Section 2]), the statement that $\mathcal{W}$ is intrinsically vanishing is equivalent to the statement that the category

$$\operatorname*{colim}_{\mathbf{s} \in \underline{\mathbf{Seq}}} \overline{\mathcal{A}_c}^{\mathcal{W}}(P_{\mathbf{s}}(\mathcal{W}))$$

has trivial K–theory in all degrees. Similarly, the statement that $\mathcal{W}$ is vanishing with respect to $\mathcal{Y}$ is equivalent to the statement that the category

$$\operatorname*{colim}_{\mathbf{s} \in \underline{\mathbf{Seq}}} \overline{\mathcal{A}_c}^{\mathcal{Y}}(P_{\mathbf{s}}(\mathcal{W}))$$

has trivial K–theory in all degrees.

By Lemma 2.7, the identity map on $P_{\mathbf{s}}(\mathcal{W})$ induces a functor

$$\mathcal{I}_{\mathbf{s}} \colon \overline{\mathcal{A}_c}^{\mathcal{W}}(P_{\mathbf{s}}(\mathcal{W})) \longrightarrow \overline{\mathcal{A}_c}^{\mathcal{Y}}(P_{\mathbf{s}}(\mathcal{W}))$$

for each $\mathbf{s} \in \underline{\mathbf{Seq}}$, and we obtain an induced functor

$$\mathcal{I} = \operatorname*{colim}_{\mathbf{s} \in \underline{\mathbf{Seq}}} \mathcal{I}_{\mathbf{s}} \colon \operatorname*{colim}_{\mathbf{s} \in \underline{\mathbf{Seq}}} \overline{\mathcal{A}_c}^{\mathcal{W}}(P_{\mathbf{s}}(\mathcal{W})) \longrightarrow \operatorname*{colim}_{\mathbf{s} \in \underline{\mathbf{Seq}}} \overline{\mathcal{A}_c}^{\mathcal{Y}}(P_{\mathbf{s}}(\mathcal{W})).$$

We claim that there exist functors

$$(17) \qquad \Phi_{\mathbf{s}} \colon \overline{\mathcal{A}_c}^{\mathcal{Y}}(P_{\mathbf{s}}(\mathcal{W})) \longrightarrow \operatorname*{colim}_{\mathbf{s} \in \underline{\mathbf{Seq}}} \overline{\mathcal{A}_c}^{\mathcal{W}}(P_{\mathbf{s}}(\mathcal{W})),$$

which make the diagrams

(18)
$$\mathcal{A}_c^{\mathcal{Y}}(P_\mathbf{s}\mathcal{W}) \xrightarrow{\Phi_\mathbf{s}} \operatorname{colim}_\mathbf{s} \overline{\mathcal{A}_c}^{\mathcal{W}}(P_\mathbf{s}\mathcal{W})$$

$$\mathcal{A}_c^{\mathcal{Y}}(P_{\mathbf{s}'}\mathcal{W}) \quad \xrightarrow{\Phi_{\mathbf{s}'}}$$

and

(19)
$$\overline{\mathcal{A}_c}^{\mathcal{W}}(P_\mathbf{s}(\mathcal{W})) \xrightarrow{i_\mathbf{s}} \operatorname{colim}_{\mathbf{s}\in\underline{\mathbf{Seq}}} \overline{\mathcal{A}_c}^{\mathcal{W}}(P_\mathbf{s}(\mathcal{W}))$$

$$\downarrow{\mathcal{I}_\mathbf{s}} \qquad \xrightarrow{\Phi_\mathbf{s}} \qquad \downarrow{\mathcal{I}}$$

$$\overline{\mathcal{A}_c}^{\mathcal{Y}}(P_\mathbf{s}(\mathcal{W})) \xrightarrow{j_\mathbf{s}} \operatorname{colim}_{\mathbf{s}\in\underline{\mathbf{Seq}}} \overline{\mathcal{A}_c}^{\mathcal{Y}}(P_\mathbf{s}(\mathcal{W}))$$

commute, where $i_\mathbf{s}$ and $j_\mathbf{s}$ are the structure maps for the colimits.

Assuming the existence of functors $\Phi_\mathbf{s}$ with these properties, we complete the proof of the Lemma. Commutativity of (18) implies that there is an induced functor

$$\Phi\colon \operatorname*{colim}_{\mathbf{s}\in\underline{\mathbf{Seq}}} \overline{\mathcal{A}_c}^{\mathcal{Y}}(P_\mathbf{s}(\mathcal{W})) \to \operatorname*{colim}_{\mathbf{s}\in\underline{\mathbf{Seq}}} \overline{\mathcal{A}_c}^{\mathcal{W}}(P_\mathbf{s}(\mathcal{W})).$$

Commutativity of (19) and the universal property of colimits imply that the compositions

$$\operatorname*{colim}_{\mathbf{s}\in\underline{\mathbf{Seq}}} \overline{\mathcal{A}_c}^{\mathcal{W}}(P_\mathbf{s}(\mathcal{W})) \xrightarrow{\mathcal{I}} \operatorname*{colim}_{\mathbf{s}\in\underline{\mathbf{Seq}}} \overline{\mathcal{A}_c}^{\mathcal{Y}}(P_\mathbf{s}(\mathcal{W})) \xrightarrow{\Phi} \operatorname*{colim}_{\mathbf{s}\in\underline{\mathbf{Seq}}} \overline{\mathcal{A}_c}^{\mathcal{W}}(P_\mathbf{s}(\mathcal{W}))$$

and

$$\operatorname*{colim}_{\mathbf{s}\in\underline{\mathbf{Seq}}} \overline{\mathcal{A}_c}^{\mathcal{Y}}(P_\mathbf{s}(\mathcal{W})) \xrightarrow{\Phi} \operatorname*{colim}_{\mathbf{s}\in\underline{\mathbf{Seq}}} \overline{\mathcal{A}_c}^{\mathcal{W}}(P_\mathbf{s}(\mathcal{W})) \xrightarrow{\mathcal{I}} \operatorname*{colim}_{\mathbf{s}\in\underline{\mathbf{Seq}}} \overline{\mathcal{A}_c}^{\mathcal{Y}}(P_\mathbf{s}(\mathcal{W}))$$

are both the identity (in the first case, for example, we have $\Phi \circ \mathcal{I} \circ i_\mathbf{s} = \Phi \circ j_\mathbf{s} \circ \mathcal{I}_\mathbf{s} = \Phi_\mathbf{s} \circ \mathcal{I}_\mathbf{s} = i_\mathbf{s}$), so if one of these colimit categories has vanishing K-theory, so does the other.

To construct the functor $\Phi_\mathbf{s}$, first note that objects in $\overline{\mathcal{A}_c}^{\mathcal{Y}}(P_\mathbf{s}\mathcal{W})$ are also objects in $\overline{\mathcal{A}_c}^{\mathcal{W}}(P_\mathbf{s}\mathcal{W})$ because the change of metrics does not affect which sets are compact, and hence the locally finiteness condition is the same in both cases. We can now define the functor $\Phi_\mathbf{s}$ on objects by setting $\Phi_\mathbf{s}(M) = i_\mathbf{s}(M)$. More care is required to define $\Phi_\mathbf{s}$ on morphisms.

A morphism in $\psi\colon M \to N$ in $\mathcal{A}_c^{\mathcal{Y}}(P_\mathbf{s}\mathcal{W})$ is a bounded map of geometric modules on $P_\mathbf{s}(\mathcal{W}) \times [0,1)$ (with metric induced from $P_\mathbf{s}(\mathcal{Y}) \times [0,1)$) which is controlled at 1. Let $d < \infty$ denote a bound on the propagation of ψ, and let $\mathbf{s}'$ be the sequence with r^{th} term $s_r' = s_r C_r(d+2)$ (where C_r is the constant from Lemma 5.1). We claim that $\psi \in \mathcal{A}_c^{\mathcal{W}}(P_{\mathbf{s}'}(\mathcal{W}))$. First we check that ψ is bounded as a morphism on $P_{\mathbf{s}'}(\mathcal{W}) \times [0,1)$, where $P_{\mathbf{s}'}(\mathcal{W})$ has its intrinsic simplicial metric. Let $(p,r),(q,t) \in P_\mathbf{s}(\mathcal{W}) \times [0,1)$ be points such that $\psi_{(p,r),(q,t)} \neq 0$. Note that this implies that $p,q \in P_{s_r}(W_r)$ for some r.

Choose barycentric vertices $x, y \in W_r$ for p and q (respectively). Lemma 5.1 implies that $d(x, y) \leqslant s_r C_r (d + 2)$, so x and y lie in a common simplex in $P_{s_r'}(W_r)$ (by choice of s_r'). It follows that p and q are at most distance 3 apart in the simplicial metric on $P_{s_r'}(W_r)$, so (p, r) and (q, t) are at most distance 4 apart in the corresponding metric on $P_{s_r'}(W_r) \times [0, 1)$. Hence ψ has propagation at most 4 in this metric.

It remains to check that ψ is controlled as a morphism on $P_{\mathbf{s}'}(\mathcal{W}) \times [0, 1)$. First, note that (continuous) control is a *topological* condition: it does not refer to the metric on the complex in question. We know that ψ is controlled as a morphism on $P_{\mathbf{s}}(\mathcal{W}) \times [0, 1)$. To check control on the larger complex $P_{\mathbf{s}'}(\mathcal{W}) \times [0, 1)$, let $U' \subset P_{\mathbf{s}'}(\mathcal{W}) \times [0, 1]$ be an open neighborhood of some point $(x, 1)$ and let $U = U' \cap (P_{\mathbf{s}}(\mathcal{W}) \times [0, 1])$. Then there exists an open set $V \subset U$ (with $(x, 1) \in V$) such that ψ does not cross $U \setminus V$. Now $C = (P_{\mathbf{s}}(\mathcal{W}) \times [0, 1]) \setminus V$ is closed in $P_{\mathbf{s}}(\mathcal{W}) \times [0, 1]$ and hence also closed in the larger simplicial complex $P_{\mathbf{s}'}(\mathcal{W}) \times [0, 1]$. Letting $V' = (P_{\mathbf{s}'}(\mathcal{W}) \times [0, 1]) \setminus C$, we have $V' \cap (P_{\mathbf{s}}(\mathcal{W}) \times [0, 1]) = V$ and hence ψ does not cross $U' \setminus V'$.

We can now define $\Phi_{\mathbf{s}}([\psi])$ to be the morphism $\Phi_{\mathbf{s}}(M) \to \Phi_{\mathbf{s}}(N)$ represented by ψ. Since $\Phi_{\mathbf{s}}$ does not change the underlying data of either geometric modules or morphisms, it follows that $\Phi_{\mathbf{s}}$ is a functor and that Diagrams (19) and (18) commute. $\square$

The key result behind the proofs of Lemmas 6.13 and 6.14 is a comparison between the categories of controlled modules associated to a decomposed sequence and to a sufficiently good refinement of that sequence. First we record a lemma regarding the construction $\overline{\mathcal{A}_c}(-)$.

Lemma 6.16. *Let $\mathcal{Z} = \{Z^r\}$ be a decomposed sequence in X with decompositions $Z^r = \bigcup_\alpha Z_\alpha^r$. Given a sequence $\mathbf{s} \in \underline{\mathbf{Seq}}$ and an integer $R > 0$, let $\mathbf{s}(R, \infty)$ denote the sequence $(s_R, s_{R+1}, \ldots)$ and let $\mathcal{Z}(R, \infty)$ denote the decomposed sequence $(Z^R, Z^{R+1}, \ldots)$ and with the same decompositions as $\mathcal{Z}$. Then the inclusion of simplicial complexes*

$$P_{\mathbf{s}(R,\infty)}\left(\mathcal{Z}(R, \infty)\right) = \coprod_{r \geqslant R} \coprod_\alpha P_{s_r}(Z_\alpha^r) \hookrightarrow \coprod_{r > 0} \coprod_\alpha P_{s_r}(Z_\alpha^r) = P_{\mathbf{s}}(\mathcal{Z})$$

induces an equivalence of categories

$$i_R \colon \overline{\mathcal{A}_c}\left(P_{\mathbf{s}(R,\infty)}\left(\mathcal{Z}(R, \infty)\right)\right) \xrightarrow{\cong} \overline{\mathcal{A}_c}\left(P_{\mathbf{s}}\left(\mathcal{Z}\right)\right).$$

Proof. The functor i_R exists by Lemma 2.7, and it follows from the definitions that i_R is full and faithful. Each module $M \in \mathcal{A}_c\left(P_{\mathbf{s}}\left(\mathcal{Z}\right)\right)$ is isomorphic, in the Karoubi quotient $\overline{\mathcal{A}_c}\left(P_{\mathbf{s}}\left(\mathcal{Z}\right)\right)$, to its restriction

$$M\left(\coprod_{r \geqslant R} \coprod_\alpha Z_\alpha^r \times [0, 1)\right).$$

This restriction is in the image of i_R, completing the proof. $\square$

Lemma 6.17. *Let $\mathcal{Y} = \{Y^r\}$ be a decomposed sequence in X, with decompositions $Y^r = \bigcup_{\alpha \in A_r} Y_\alpha^r$ and consider sequences $\mathbf{s}$, $\mathbf{f} \in \underline{\mathbf{Seq}}$ satisfying $\lim_{r \to \infty} f_r / C_r s_r = \infty$, where $C_r = C(s_r, X)$ is the constant from Lemma 5.1. Assume that for each r and each $\alpha \in A_r$ we have a decomposition*

$$Y_\alpha^r = \coprod_i^{f_r - disjoint} Y_{\alpha i}^r.$$

Let $\mathcal{Y}'$ be the decomposed sequence $\mathcal{Y}' = \{Y^r\}$ with decompositions

$$Y^r = \bigcup_{\alpha, i} Y_{\alpha i}^r.$$

If $\mathcal{Y}'$ is a vanishing sequence, then so is $\mathcal{Y}$.

Proof. For each $\mathbf{s}' \geqslant \mathbf{s}$, we will construct a commutative diagram

(20)
$$
\begin{array}{ccc}
K_* \left(\overline{\mathcal{A}_c} \left(P_\mathbf{s}(\mathcal{Y}) \right) \right) & \xrightarrow{\eta_{\mathbf{s},\mathbf{s}'}(\mathcal{Y})} & K_* \left(\overline{\mathcal{A}_c} \left(P_{\mathbf{s}'}(\mathcal{Y}) \right) \right) \\
\Psi_\mathbf{s} \uparrow & & \uparrow \Psi_{\mathbf{s}'} \\
K_* \left(\overline{\mathcal{A}_c} \left(P_\mathbf{s}(\mathcal{Y}') \right) \right) & \xrightarrow{\eta_{\mathbf{s},\mathbf{s}'}(\mathcal{Y}')} & K_* \left(\overline{\mathcal{A}_c} \left(P_{\mathbf{s}'}(\mathcal{Y}') \right) \right).
\end{array}
$$

Furthermore, we will show that the map $\Psi_\mathbf{s}$ is an *isomorphism*, which implies that $\eta_{\mathbf{s},\mathbf{s}'}(\mathcal{Y})$ factors through $\eta_{\mathbf{s},\mathbf{s}'}(\mathcal{Y}')$. The lemma will then follow from the definition of a vanishing sequence.

We will define the desired homomorphisms $\Psi_\mathbf{t}$ for each sequence $\mathbf{t} \in \underline{\mathbf{Seq}}$. Our hypotheses imply that there exists $R > 0$ such that if $r \geqslant R$ then $f_r > s_r > 0$. Now if $r \geqslant R$ and $\alpha \in A_r$ we have $Y_{\alpha i}^r \cap Y_{\alpha j}^r = \emptyset$, so there is an injective map of simplicial complexes

$$P_{\mathbf{t}(R,\infty)}(\mathcal{Y}'(R,\infty)) \hookrightarrow P_{\mathbf{t}(R,\infty)}(\mathcal{Y}(R,\infty)).$$

This map decreases distances, so by Lemma 2.7 we have an induced functor

$$\Phi_\mathbf{t} \colon \overline{\mathcal{A}_c} \left(P_{\mathbf{t}(R,\infty)} \left(\mathcal{Y}'(R,\infty) \right) \right) \longrightarrow \overline{\mathcal{A}_c} \left(P_{\mathbf{t}(R,\infty)} \left(\mathcal{Y}(R,\infty) \right) \right).$$

Lemma 6.16 yields a diagram

(21)
$$
\begin{array}{ccc}
\overline{\mathcal{A}_c} \left(P_\mathbf{t}(\mathcal{Y}') \right) & & \overline{\mathcal{A}_c} \left(P_\mathbf{t}(\mathcal{Y}) \right) \\
i' \uparrow \cong & & i \uparrow \cong \\
\overline{\mathcal{A}_c} \left(P_{\mathbf{t}(R,\infty)} \left(\mathcal{Y}'(R,\infty) \right) \right) & \xrightarrow{\Phi_t} & \overline{\mathcal{A}_c} \left(P_{\mathbf{t}(R,\infty)} \left(\mathcal{Y}(R,\infty) \right) \right),
\end{array}
$$

and we define

$$K_* \left(\overline{\mathcal{A}_c} \left(P_\mathbf{t}(\mathcal{Y}') \right) \right) \xrightarrow{\Psi_\mathbf{t}} K_* \left(\overline{\mathcal{A}_c} \left(P_\mathbf{t}(\mathcal{Y}) \right) \right)$$

to be the homomorphism obtained from Diagram 21 by inverting i_*. Given $\mathbf{t}' \geqslant \mathbf{t}$, we obtain a commutative diagram linking the zig-zag (21) to the corresponding zig-zag for $\mathbf{t}'$; this yields commutativity of Diagram (20).

We need to check that $\Psi_\mathbf{s}$ is an isomorphism. We will show that $\Phi_\mathbf{s}$ is an *isomorphism* of categories. Note that if $r \geqslant R$ and $\alpha \in A_r$, then

$d(Y_{\alpha i}^r, Y_{\alpha j}^r) \geqslant f_r > s_r$ $(i \neq j)$, so the simplicial complex $P_{s_r}(Y_\alpha^r)$ is the disjoint union of the subcomplexes $P_{s_r}(Y_{\alpha i}^r)$. This shows that $\Phi_{\mathbf{s}}$ is bijective on objects: each module on $P_{\mathbf{s}}(Y_\alpha^r) \times [0, 1)$ is the direct sum of its restrictions to the disjoint subspaces $P_{\mathbf{s}}(Y_{\alpha,i}^r) \times [0, 1)$. Next, we check that $\Phi_{\mathbf{s}}$ is surjective on morphisms. Given $[\psi] \in \overline{\mathcal{A}_c} P_{\mathbf{s}}(\mathcal{Y})$, let T be a bound on the propagation of ψ. Since $f_r/s_r \to \infty$, there exists R_T such that $f_r > (T + 2)s_r C_r$ for $r \geqslant R_T$. Now $[\psi] = [\psi(R_T, \infty)]$, where $\psi(R_T, \infty)$ denotes the morphism

$$\psi(R_T, \infty)_{a,b} = \begin{cases} \psi_{a,b}, & a, b \in P_{s_r}(Y_\alpha^r) \times [0, 1) \text{ for some } r \geqslant R_T, \\ 0, & else. \end{cases}$$

Lemma 5.1 and our choice of R_T imply that $\psi(R_T, \infty)$ is a direct sum, over $r > R$, α, and i, of morphisms supported on $P_{\mathbf{s}}(Y_{\alpha,i}^r) \times [0, 1)$. Hence $[\psi] = [\psi(R_T, \infty)]$ is in the image of $\Phi_{\mathbf{s}}$. Finally, we must check that Φ is faithful. If $\Phi_{\mathbf{s}}([\psi_1]) = \Phi_{\mathbf{s}}([\psi_2])$, then for sufficiently large R_0, the restrictions of ψ_1 and ψ_2 to $\coprod_{r > R_0, \alpha} P_{\mathbf{s}}(Y_\alpha^r) \times [0, 1)$ are identical, and hence $[\psi_1] = [\psi_2]$. $\square$

Proof of Lemma 6.13. For each $t > 0$ we define the decomposed sequence $\mathcal{W}_t = \mathcal{W}_t(\mathcal{U}, \mathcal{V}, \mathcal{Z})$, whose r^{th} term is

$$(22) \qquad W_t^r = \bigcup_\alpha N_{tC_r s_r}(U_\alpha^r) \cap N_{tC_r s_r}(V_\alpha^r) \cap Z_\alpha^r;$$

the decomposition of W_t^r is exactly that displayed in (22). We claim that $\mathcal{W}_t$ is vanishing. Consider the decomposed sequence $\mathcal{W}_t'$, whose r^{th} term is the same as that of $\mathcal{W}_t$, but with the finer decomposition

$$(23) \qquad W_t^r = \bigcup_{\alpha, i, j} N_{tC_r s_r}(U_{\alpha i}^r) \cap N_{tC_r s_r}(V_{\alpha j}^r) \cap Z_\alpha^r.$$

Since $\{U_{\alpha i}^r\}_{\alpha, i}, \{V_{\alpha j}^r\}_{\alpha, j} \in \mathfrak{D}_\beta$, the induction hypothesis and Lemma 6.10 tell us that $\mathcal{W}_t'$ is a vanishing sequence. For $i \neq i'$ and $j \neq j'$ we have $d(U_{\alpha i}^r, U_{\alpha i'}^r) > C_r s_r r$ and $d(V_{\alpha j}^r, V_{\alpha j'}^r) > C_r s_r r$, so the distance between two sets in the decomposition (23) is at least $C_r s_r r - 2t C_r s_r = C_r s_r (r - 2t)$, and $(C_r s_r (r - 2t))/s_r = C_r(r - 2t)$ tends to infinity with r. By Lemma 6.17, $\mathcal{W}_t$ is vanishing.

We will show that the map $\rho_{\mathbf{s}, \mathbf{s}'}$ factors through a map

$$(24) \qquad \operatorname*{colim}_{t \to \infty} K_{*-1}\left(\overline{\mathcal{A}_c}^X P_{\mathbf{s}}(\mathcal{W}_t)\right) \xrightarrow{(\xi_{\mathbf{s}, \mathbf{s}'})_*} \operatorname*{colim}_{t \to \infty}\left(K_{*-1}\left(\overline{\mathcal{A}_c}^{\text{rel}} P_{\mathbf{s}'}(\mathcal{W}_t)\right)\right),$$

where the superscripts indicate that we give these Rips complexes the subspace metrics induced from the simplicial metrics on $P_{\mathbf{s}} X$ and $P_{\mathbf{s}, \mathbf{s}'}(X, \mathcal{W}_t)$ (this choice of metrics will be important in obtaining the desired factorization of $\rho_{\mathbf{s}, \mathbf{s}'}$). Lemma 2.7 shows that the inclusion of simplicial complexes $P_{\mathbf{s}}(\mathcal{W}_t) \subset P_{\mathbf{s}'}(\mathcal{W}_t)$ induces a functor $\xi_{\mathbf{s}, \mathbf{s}', t}$ on categories of controlled modules (for the chosen metrics); we set $\xi_{\mathbf{s}, \mathbf{s}'} = \operatorname{colim}_t \xi_{\mathbf{s}, \mathbf{s}', t}$, and (24) is the induced map on K–theory.

We will show that for every class on the left-hand side of (24), there exists $\mathbf{s}' \geqslant \mathbf{s}$ such that $(\xi_{\mathbf{s}, \mathbf{s}'})_*(x) = 0$. (The corresponding result for $\rho_{\mathbf{s}, \mathbf{s}'}$ will

follow immediately once we establish the claimed factorization.) It suffices to show that the functor $\xi_{\mathbf{s}} = \mathrm{colim}_{\mathbf{s}' \in \underline{\mathbf{Seq}}} \, \xi_{\mathbf{s},\mathbf{s}'}$ induces the zero-map on K–theory. Recall that in the proof of Lemma 6.15 (see (17), specifically), we constructed functors

$$\Phi_{\mathbf{s},t} \colon \overline{\mathcal{A}_c}^X(P_{\mathbf{s}}(\mathcal{W}_t)) \longrightarrow \mathop{\mathrm{colim}}_{\mathbf{s}' \in \underline{\mathbf{Seq}}} \overline{\mathcal{A}_c}^{\mathcal{W}_t}(P_{\mathbf{s}'}(\mathcal{W}_t)),$$

where again the superscripts indicate the chosen metrics on the Rips complexes. Lemma 2.7 yields functors $\overline{\mathcal{A}_c}^{\mathcal{W}_t}(P_{\mathbf{s}'}(\mathcal{W}_t)) \xrightarrow{\Psi_{\mathbf{s}',t}} \overline{\mathcal{A}_c}^{\mathrm{rel}} P_{\mathbf{s}'}(\mathcal{W}_t)$ for each $\mathbf{s}' \in \underline{\mathbf{Seq}}$ and each $t > 0$, and now $\xi_{\mathbf{s}} = \mathrm{colim}_{\mathbf{s}' \in \underline{\mathbf{Seq}}} \, \xi_{\mathbf{s},\mathbf{s}'}$ factors as

$$\mathop{\mathrm{colim}}_{t \to \infty} \overline{\mathcal{A}_c}^X P_{\mathbf{s}}(\mathcal{W}_t) \xrightarrow{\mathrm{colim}_t \, \Phi_{\mathbf{s},t}} \mathop{\mathrm{colim}}_{t \to \infty} \mathop{\mathrm{colim}}_{\mathbf{s}' \in \underline{\mathbf{Seq}}} \overline{\mathcal{A}_c}^{\mathcal{W}_t}(P_{\mathbf{s}'}(\mathcal{W}_t))$$

$$\xrightarrow{\mathrm{colim}_{t,\mathbf{s}'} \, \Psi_{\mathbf{s}',t}} \mathop{\mathrm{colim}}_{t \to \infty} \mathop{\mathrm{colim}}_{\mathbf{s}' \in \underline{\mathbf{Seq}}} \left(\overline{\mathcal{A}_c}^{\mathrm{rel}} P_{\mathbf{s}'}(\mathcal{W}_t) \right) \cong \mathop{\mathrm{colim}}_{\mathbf{s}' \in \underline{\mathbf{Seq}}} \mathop{\mathrm{colim}}_{t \to \infty} \left(\overline{\mathcal{A}_c}^{\mathrm{rel}} P_{\mathbf{s}'}(\mathcal{W}_t) \right).$$

As established above, $\mathcal{W}_t$ is a vanishing sequence for every t, so Lemma 6.15 implies that the category $\mathrm{colim}_t \, \mathrm{colim}_{\mathbf{s}' \in \underline{\mathbf{Seq}}} \, \overline{\mathcal{A}_c}^{\mathcal{W}_t}(P_{\mathbf{s}'}(\mathcal{W}_t))$ has trivial K–theory. Hence $(\xi_{\mathbf{s}})_* = 0$ as desired.

The desired factorization of $\rho_{\mathbf{s},\mathbf{s}'}$ comes from a sequence of functors

$$\mathcal{I}_{\mathbf{s}}(\mathcal{U}, \mathcal{V}) \xrightarrow{i} \mathop{\mathrm{colim}}_{t \to \infty} \overline{\mathcal{A}_c}^X \left(P_{\mathbf{s}}(\mathcal{Z}) \cap (N_t P_{\mathbf{s}} \mathcal{U}) \cap (N_t P_{\mathbf{s}} \mathcal{V}) \right)$$

(25)
$$\xrightarrow{j} \mathop{\mathrm{colim}}_{t \to \infty} \overline{\mathcal{A}_c}^X \left(P_{\mathbf{s}}(\mathcal{W}_t) \right) \xrightarrow{\mathrm{colim}_t \, \xi_{\mathbf{s},\mathbf{s}',t}} \mathop{\mathrm{colim}}_{t \to \infty} \overline{\mathcal{A}_c}^{\mathrm{rel}} \left(P_{\mathbf{s}'}(\mathcal{W}_t) \right)$$

$$\xrightarrow{l} \mathop{\mathrm{colim}}_{t \to \infty} \mathcal{I}'_{\mathbf{s},\mathbf{s}',t}(\mathcal{U}, \mathcal{V}).$$

The functors i and l are inclusions of categories that exist by the definitions of the intersection terms in the Mayer–Vietoris sequences. (In the case of l, we use the fact that for metric spaces $A \subset X$, a continuously controlled morphism between geometric modules on $A \times [0, 1)$ is also continuously controlled on $X \times [0, 1)$. This was shown in the proof of Lemma 6.15, and also follows from Lemma 2.7.) The functor j exists by Equation (11) in Lemma 5.2 (which may be applied to non-relative Rips complexes simply by setting the two parameters s, s' appearing in the Lemma to be equal), and it is immediate from the definitions that the composite of these functors is the functor inducing $\rho_{\mathbf{s},\mathbf{s}'}$ on K–theory. $\qquad\square$

Proof of Lemma 6.14. We will show that the map $\mu_{\mathbf{s},\mathbf{s}',\mathbf{s}''} = \mu_{\mathbf{s},\mathbf{s}',\mathbf{s}''}(\mathcal{U}) \oplus \mu_{\mathbf{s},\mathbf{s}',\mathbf{s}''}(\mathcal{V})$ factors through the direct sum of the maps

(26) $\quad \mathop{\mathrm{colim}}_{t \to \infty} K_* \overline{\mathcal{A}_c}^X (P_{\mathbf{s}'}(N^{\mathcal{Z}}_{t\mathbf{Cs}} \mathcal{U})) \xrightarrow{\mathrm{colim}_t \, \eta_{\mathbf{s}',\mathbf{s}''}} \mathop{\mathrm{colim}}_{t \to \infty} K_* \overline{\mathcal{A}_c}^X (P_{\mathbf{s}''}(N^{\mathcal{Z}}_{t\mathbf{Cs}} \mathcal{U}))$

and

$$\mathop{\mathrm{colim}}_{t \to \infty} K_* \overline{\mathcal{A}_c}^X (P_{\mathbf{s}'}(N^{\mathcal{Z}}_{t\mathbf{Cs}} \mathcal{V})) \xrightarrow{\mathrm{colim}_t \, \eta_{\mathbf{s}',\mathbf{s}''}} \mathop{\mathrm{colim}}_{t \to \infty} K_* \overline{\mathcal{A}_c}^X (P_{\mathbf{s}''}(N^{\mathcal{Z}}_{t\mathbf{Cs}} \mathcal{V})).$$

As in the proof of Lemma 6.13, the induction hypothesis together with Lemmas 6.10 and 6.17 show that for each $t > 0$, $N^{\mathcal{Z}}_{t\mathbf{Cs}}\mathcal{U}$ and $N^{\mathcal{Z}}_{t\mathbf{Cs}}\mathcal{V}$ are vanishing sequences. The desired result will follow as soon as we establish the claimed factorization.

Let

$$W^r_{t,\alpha} = Z^r_\alpha \cap N_{tC_r s_r}(U^r_\alpha) \cap N_{tC_r s_r}(V^r_\alpha).$$

Given $t' \geqslant 0$, $r \geqslant 1$, and $\alpha \in A_r$, Equation (12) in Lemma 5.2 shows that

$$N_{t'}\left(P_{s_r,s'_r}\left(U^r_\alpha, W^r_{t,\alpha}\right)\right) \cap P_{s_r,s'_r}\left(Z^r_\alpha, W^r_{t,\alpha}\right)$$
$$\subseteq P_{s'_r}\left(N_{(t'+2)C_r s_r}\left(U^r_\alpha \cup W^r_{t,\alpha}\right)\right)$$
$$\subseteq P_{s'_r}\left(Z^r_\alpha \cap N_{(t'+t+2)C_r s_r}(U^r_\alpha)\right),$$

where the first neighborhood is taken inside $P_{s_r,s'_r}(X, W^r_{t,\alpha})$. Lemma 2.7 now yields functors

$$\overline{\mathcal{A}_c}^X\left(P_{\mathbf{s},\mathbf{s}'}\left(\mathcal{Z}, \mathbb{W}_{t\mathbf{Cs}}\right) \cap N_{t'}\left(P_{\mathbf{s},\mathbf{s}'}\left(\mathcal{U}, \mathbb{W}_{t\mathbf{Cs}}\right)\right)\right)$$
$$\xrightarrow{i_{t,t'}} \overline{\mathcal{A}_c}^X\left(P_{\mathbf{s}'}\left(N^{\mathcal{Z}}_{(t+t'+2)\mathbf{Cs}}(\mathcal{U})\right)\right)$$

for each $t, t' > 0$. The colimit, over $t' > 0$, of the categories appearing in the domain of $i_{t,t'}$ is precisely $\overline{\mathcal{A}_c}^{\mathcal{Z}+}\left(P_{\mathbf{s},\mathbf{s}'}\left(\mathcal{U}, \mathbb{W}_{t\mathbf{Cs}}\right)\right)$. Hence these functors combine to yield a functor

$$\operatorname*{colim}_{t\to\infty}\overline{\mathcal{A}_c}^{\mathcal{Z}+}\left(P_{\mathbf{s},\mathbf{s}'}\left(\mathcal{U}, \mathbb{W}_{t\mathbf{Cs}}\right)\right) \xrightarrow{i} \operatorname*{colim}_{t\to\infty}\overline{\mathcal{A}_c}^X\left(P_{\mathbf{s}'}\left(N^{\mathcal{Z}}_{t\mathcal{C}\mathbf{s}}(\mathcal{U})\right)\right).$$

The desired factorization of $\mu_{\mathbf{s},\mathbf{s}',\mathbf{s}''}(\mathcal{U})$ is induced by the functors

$$\operatorname*{colim}_{t\to\infty}\overline{\mathcal{A}_c}^{\mathcal{Z}+}\left(P_{\mathbf{s},\mathbf{s}'}\left(\mathcal{U}, \mathbb{W}_{t\mathbf{Cs}}\right)\right) \xrightarrow{i} \operatorname*{colim}_{t\to\infty}\overline{\mathcal{A}_c}^X\left(P_{\mathbf{s}'}\left(N^{\mathcal{Z}}_{t\mathcal{C}\mathbf{s}}(\mathcal{U})\right)\right)$$
$$\xrightarrow{\eta} \operatorname*{colim}_{t\to\infty}\overline{\mathcal{A}_c}^X\left(P_{\mathbf{s}''}(N^{\mathcal{Z}}_{t\mathcal{C}\mathbf{s}}(\mathcal{U}))\right) \xrightarrow{j} \operatorname*{colim}_{t\to\infty}\overline{\mathcal{A}_c}^{\mathcal{Z}+}\left(P_{\mathbf{s}''}(N^{\mathcal{Z}}_{t\mathcal{C}\mathbf{s}}(\mathcal{U}))\right),$$

where η is the functor inducing (26) and j is induced by the inclusions

$$\mathcal{A}_c^X\left(P_{\mathbf{s}''}(N^{\mathcal{Z}}_{t\mathcal{C}\mathbf{s}}(\mathcal{U}))\right) \subset \mathcal{A}_c^{\mathcal{Z}+}\left(P_{\mathbf{s}''}(N^{\mathcal{Z}}_{t\mathcal{C}\mathbf{s}}(\mathcal{U}))\right).$$

A similar factorization exists for $\mathcal{V}$ in place of $\mathcal{U}$. $\qquad\qquad\square$

7. Assembly for FDC groups

In this section, we apply our vanishing result for continuously controlled K–theory (Theorem 6.4) to study assembly maps. We first prove a large-scale, bounded version of the Borel Conjecture, analogous to Guentner–Tessera–Yu [13, Theorems 4.3.1, 4.4.1], relating the bounded K–theory of the Rips complexes on an FDC metric space to an associated homology theory. Then we study the classical K–theoretic assembly map, using Carlsson's descent argument [7].

Theorem 7.1. *Let X be a bounded geometry metric space with finite decomposition complexity. Then there is an isomorphism*

$$\operatorname*{colim}_{s\to\infty} H_*(P_s(X); \mathbb{K}(\mathcal{A})) \cong \operatorname*{colim}_{s\to\infty} K_*(\mathcal{A}_b(P_s(X))).$$

This result may be thought of as excision statement for bounded K–theory. Before giving the proof, we need some setup. For a proper metric space X, let $\mathcal{A}_c(X)_{<1}$ denote the full additive subcategory of $\mathcal{A}_c(X)$ on those modules M whose support has no limit points at 1; that is,

$$\overline{\operatorname{supp}(M)} \cap (X \times 1) = \emptyset,$$

where the closure $\overline{\operatorname{supp}(M)}$ is taken in $X \times [0,1]$. By an argument similar to the proof of Lemma 3.6, the inclusion of categories

$$\mathcal{A}_c(X)_{<1} \subset \mathcal{A}_c(X)$$

admits a Karoubi filtration.

Definition 7.2. *The Karoubi quotient $\mathcal{A}_c(X)/\mathcal{A}_c(X)_{<1}$ is denoted $\mathcal{A}_\infty(X)$.*

Theorem 3.4 yields a long exact sequence in non-connective K–theory

$$(27) \qquad \cdots \xrightarrow{\partial} K_*\mathcal{A}_c(X)_{<1} \longrightarrow K_*(\mathcal{A}_c(X))$$

$$\longrightarrow K_*\mathcal{A}_\infty(X) \xrightarrow{\partial} K_{*-1}\mathcal{A}_c(X)_{<1} \longrightarrow \cdots .$$

As shown by Weiss [26], $K_*(\mathcal{A}_\infty(-))$ is the (Steenrod) homology theory associated to the non-connective algebraic K–theory spectrum $\mathbb{K}(\mathcal{A})$, with a dimension shift: in particular, if X is a finite CW complex, there are isomorphisms

$$(28) \qquad\qquad K_*(\mathcal{A}_\infty(X)) \cong H_{*-1}(X; \mathbb{K}(\mathcal{A}))$$

for each $* \in \mathbb{Z}$ (the result was first proven, in a slightly different form, in Pedersen–Weibel [20]). The two key components of Weiss's proof are the facts that the functor $X \mapsto K_*(\mathcal{A}_\infty(X))$ is homotopy invariant and satisfies excision. The methods of Weiss and Williams [27] then show that $\mathcal{A}_\infty(X) \simeq X_+ \wedge \mathcal{A}_\infty(*)$ (at least for X an ENR, and in particular for X a finite CW complex). One then identifies the coefficients $\mathcal{A}_\infty(*)$ by observing that $K_*(\mathcal{A}_\infty(\{*\}))$ is isomorphic to $K_{*-1}(\mathcal{A}_c(\{*\})_{<1})$, since the other terms in the long exact sequence (27) vanish when $X = *$ (see, for example Bartels [3, 3.20]), and $K_{*-1}(\mathcal{A}_c(\{*\})_{<1}) \cong K_{*-1}\mathcal{A}$ by Lemma 7.4 below. Details can be found in the above references; see [26, Section 5] in particular.

Remark 7.3. *Weiss [26] uses a somewhat different description of the category $\mathcal{A}_\infty(X)$. He describes the morphisms as "germs" of morphisms in $\mathcal{A}_c(X)$. It is easy to check, however, that Weiss's germ category is the same as the Karoubi quotient $\mathcal{A}_\infty(X)$. Additionally, Weiss works with the idempotent completion his germ category. This does not affect the results though, since the non-connective K–theory spectrum of an additive category $\mathcal{A}$ is weakly equivalent to that for its idempotent completion $\mathcal{A}^\wedge$: this follows from Pedersen–Weibel [19, Lemmas 1.4.2 and 2.3].*

Lemma 7.4. *For every proper metric space X there is an equivalence*

$$\mathcal{A}_c(X)_{<1} \xrightarrow{\cong} \mathcal{A}_b(X).$$

Proof. This equivalence is induced by the inclusion of categories

$$\mathcal{A}_b(X) = \mathcal{A}_b(X \times \{0\}) \subset \mathcal{A}_c(X)_{<1},$$

which is clearly bijective on Hom sets in the domain. We need to check that every object in $\mathcal{A}_c(X)_{<1}$ is isomorphic to an object in $\mathcal{A}_b(X)$. Given a module $M \in \mathcal{A}_c(X)_{<1}$, let $\overline{M} \in \mathcal{A}_b(X)$ be the module

$$\overline{M}_x = \bigoplus_{t \in [0,1)} M_{(x,t)}.$$

Since objects in $\mathcal{A}_b(X \times [0,1))_{<1}$ stay away from 1, $\overline{M}$ is finitely generated at each point, and properness of X implies that $\overline{M}$ is locally finite. We now have an isomorphism $M \to \overline{M}$ sending $M_{(x,t)}$ isomorphically to the corresponding summand of $\overline{M}_x$. This morphism has propagation less than 1, and is continuously controlled due to the support condition on M.   □

Proof of Theorem 7.1 For each s, the isomorphisms given by (28) and Lemma 7.4 show that the long exact sequence (27) has the form

$$\cdots \to K_*(\mathcal{A}_c(P_s(X))) \to H_{*-1}(P_s(X); \mathbb{K}(\mathcal{A})) \xrightarrow{\partial} K_{*-1}\mathcal{A}_b(P_s(X)) \to \cdots.$$

Since colimits preserve exact sequences and the the K–theory of the category $\mathrm{colim}_s \mathcal{A}_c(P_s X)$ vanishes (Theorem 6.4), the colimit (over s) of the boundary maps for this sequence yields the desired isomorphism.   □

We now come to the main result of this paper.

Theorem 7.5. *Let Γ be a group with finite decomposition complexity, and assume there exists a universal principal Γ–bundle $E\Gamma \to B\Gamma$ with $B\Gamma$ a finite CW complex. Then for every ring R, the K–theoretic assembly map*

$$H_*(B\Gamma; \mathbb{K}(R)) \longrightarrow K_*(R[\Gamma])$$

is a split injection for all $* \in \mathbb{Z}$.

In fact, R may be replaced by any additive category $\mathcal{A}$, as in the previous sections of this paper. (Then $K_*(R[\Gamma])$ must be replaced by the K–theory of the category $\mathcal{A}[\Gamma]$, as defined in Bartels [3].)

This result extends the injectivity result of Bartels [3] and Carlsson–Goldfarb [8], which applied to groups with *finite asymptotic dimension*. Recall that by Guentner–Tessera–Yu [13, Theorem 3.0.1] and [12, Theorems 5.2.2], Theorem 7.5 applies to every (geometrically finite) subgroup of $\mathrm{GL}_n(A)$, where A is a commutative ring.

The proof of Theorem 7.5 requires some preliminaries regarding group actions and the "forget-control" description of the assembly map. Let X be a proper metric space with an isometric action of a group Γ. Then Γ acts on $\mathcal{A}_c(X)$ through additive functors (given by translating modules

and morphisms), and this action maps the subcategory $\mathcal{A}_c(X)_{<1}$ into itself. It follows from the definitions that the inclusion of fixed point categories $\mathcal{A}_c(X)_{<1}^{\Gamma} \subset \mathcal{A}_c(X)^{\Gamma}$ admits a Karoubi filtration. We now have a Karoubi sequence

(29) $$\mathcal{A}_c(X)_{<1}^{\Gamma} \subset \mathcal{A}_c(X)^{\Gamma} \longrightarrow \left(\mathcal{A}_c(X)^{\Gamma}\right) / \left(\mathcal{A}_c(X)_{<1}^{\Gamma}\right).$$

Using the fact that Γ acts freely, one may check that there is an equivalence of categories

$$\mathcal{A}_c(X)_{<1}^{\Gamma} \cong \mathcal{A}[\Gamma]_c(X/\Gamma)_{<1}.$$

When $\mathcal{A}$ is the category of finitely generated free R–modules for some ring R, $\mathcal{A}[\Gamma]$ is the category of finitely generated free $R[\Gamma]$–modules. If Γ acts properly discontinuously, there is also an equivalence of categories

$$\left(\mathcal{A}_c(X)^{\Gamma}\right) / \left(\mathcal{A}_c(X)_{<1}^{\Gamma}\right) \cong \mathcal{A}_{\infty}(X/\Gamma)$$

(this is essentially Carlsson–Pedersen [9, 2.8]), and (28) yields

$$K_* \left(\left(\mathcal{A}_c(X)^{\Gamma}\right) / \left(\mathcal{A}_c(X)_{<1}^{\Gamma}\right)\right) \cong H_{*-1}(X/\Gamma; \mathbb{K}\mathcal{A}).$$

The boundary map for the long exact sequence in K–theory associated to (29) now has the form

(30) $$H_*(X/\Gamma; \mathbb{K}\mathcal{A}) \longrightarrow K_* \left(A[\Gamma]_c(X/\Gamma)_{<1}\right).$$

The following lemma identifies the codomain of this map in the case of interest to us.

Lemma 7.6. *If X is a compact metric space with $\mathrm{diam}(X) < \infty$ and $\mathcal{E}$ is an additive category, then there are equivalences of categories*

(31) $$\mathcal{E}_c(X)_{<1} \xrightarrow{\cong} \mathcal{E}_b(X) \cong \mathcal{E}.$$

Proof. The first equivalence is given by Lemma 7.4. Given $x_0 \in X$, the second equivalence is induced by the inclusion of categories

$$\mathcal{E} \cong \mathcal{E}_b(\{x_0\}) \subset \mathcal{E}_b(X).$$

This inclusion is an equivalence because compactness implies that any locally finite module M over X is in fact supported on a finite set $S \subset X$, and is isomorphic to the module $\bigoplus_{x \in S} M_x$ considered as a module over $\{x_0\}$ (this isomorphism has finite propagation because $\mathrm{diam}(X) < \infty$). $\qquad\square$

Under the isomorphism induced by (31), the map (30) agrees with the classical assembly map

$$H_*(X/\Gamma; \mathbb{K}\mathcal{A}) \longrightarrow K_* \left(\mathcal{A}[\Gamma]\right).$$

(For proofs, see [9, 14, 26].) The boundary map for a fibration sequence of spectra can be realized as a map of spectra after looping the base spectrum, so we have a map

$$\Omega\mathbb{K}\mathcal{A}_{\infty}(X) \longrightarrow \mathbb{K}\mathcal{A}_c(X)_{<1}$$

that induces the assembly map after taking fixed-point spectra and then homotopy groups. (We are using the fact that if $\mathcal{C}$ is an additive category with an action of a group G by additive functors, then $\mathbb{K}(\mathcal{C})^G \cong \mathbb{K}(\mathcal{C}^G)$.)

The key ingredient in the proof of Theorem 7.5 will be a variation on Theorem 6.4. First, we need a simple lemma about homotopically finite classifying spaces of groups. Note that up to homotopy, there is no difference between assuming that a group admits a finite CW model for $B\Gamma$ or a finite simplicial complex model, because every finite CW complex is homotopy equivalent to a finite simplicial complex.

Lemma 7.7. *Let X be a contractible, proper metric space with an isometric, properly discontinuous action of a group Γ. If X/Γ is compact, then X is uniformly contractible: that is, for each $R > 0$ there exists $S > R$ such that for all $x \in X$, the inclusion $B_R(x) \hookrightarrow B_S(x)$ is null-homotopic. In particular, if $E\Gamma \to B\Gamma$ is a universal principal bundle with $B\Gamma$ a finite simplicial complex, then $E\Gamma$ is uniformly contractible with respect to the simplicial metric (corresponding to the simplicial structure lifted from $B\Gamma$).*

Proof. If we equip X/Γ with the metric $d([x],[y]) = \inf_{\gamma_1,\gamma_2 \in \Gamma} d(\gamma_1 \cdot x, \gamma_2 \cdot y) = \inf_{\gamma \in \Gamma} d(x, \gamma \cdot y)$ (which is a metric since Γ acts properly discontinuously), then X/Γ has finite diameter $D > 0$. Let $H\colon X \times I \to X$ be a null-homotopy of Id_X, and choose $x_0 \in X$ and $R > 0$. By compactness, there exists $S > D + R$ such that $H(B_{D+R}(x_0) \times I) \subset B_S(x_0)$, and now the inclusion $B_R(x) \hookrightarrow B_{D+S}(x)$ is null-homotopic for every $x \in X$, because (by choice of D) it factors through the inclusion $B_{D+R}(\gamma \cdot x_0) \hookrightarrow B_S(\gamma \cdot x_0)$ for some $\gamma \in \Gamma$. For the application to $E\Gamma$, it is only necessary to check that $E\Gamma$ is proper. Since $B\Gamma$ is a finite simplicial complex, $E\Gamma$ is a locally finite simplicial complex, and hence proper in the simplicial metric (this is similar to the proof of Lemma 5.1). $\qquad\square$

Theorem 7.8. *Let Γ be a group with finite decomposition complexity, and assume that there exists a universal principal Γ–bundle $E\Gamma \to B\Gamma$ with $B\Gamma$ a finite simplicial complex (this implies, in particular, that Γ is finitely generated). Equip $E\Gamma$ with the simplicial metric corresponding to the simplicial structure lifted from $B\Gamma$. Then the category $\mathcal{A}_c(E\Gamma)$ has trivial K–theory.*

Proof. We will construct continuous, proper, metrically coarse maps

$$f_s\colon E\Gamma \to P_s\Gamma, \quad g_s\colon P_s\Gamma \to E\Gamma$$

for all sufficiently large s, having the property that the compositions

$$E\Gamma \xrightarrow{f_s} P_s\Gamma \xhookrightarrow{i} P_{s'}\Gamma \xrightarrow{g_{s'}} E\Gamma$$

always induce the identity map on $K_*\mathcal{A}_c(E\Gamma)$. This will suffice, since given any element $x \in K_*\mathcal{A}_c(E\Gamma)$, Theorem 6.4 guarantees that we can choose s' large enough that $i_*(f_s)_*(x) = 0$ in $K_*\mathcal{A}_c(P_{s'}\Gamma)$; now $(g_{s'})_* i_*(f_s)_*(x) = x$ must be trivial as well.

We recall the constructions of f_s and g_s given in [13, Lemma 4.3.6]. Fix a vertex $x_0 \in E\Gamma$ and consider the embedding $\Gamma \hookrightarrow E\Gamma$, $\gamma \mapsto \gamma \cdot x_0$. The action of Γ on $E\Gamma$ by deck transformations restricts to left multiplication on Γ under this embedding, so the simplicial metric d_Δ on $E\Gamma$ restricts to a proper, left-invariant metric d_Δ on Γ. If we equip Γ with the left-invariant metric d_w associated to a finite generating set, then for each $R > 0$ there exists $S > 0$ such that $d_\Delta(\gamma, \gamma') < R$ implies $d_w(\gamma, \gamma') < S$ (this follows immediately from the fact that the all balls around the identity in the metric d_Δ are finite). In particular, letting D denote the diameter of $B\Gamma = E\Gamma/\Gamma$ as in the proof of Lemma 7.7, there exists $s > 0$ such that $d_\Delta(\gamma, \gamma') < 2(D+1)$ implies $d_w(\gamma, \gamma') < s$. By choice of D, we have an open cover $\{U_\gamma\}_{\gamma \in \Gamma}$ of $E\Gamma$, where

$$U_\gamma = B_{D+1}(\gamma \cdot x_0) \setminus \{\gamma' \cdot x_0 \,:\, \gamma' \neq \gamma\}.$$

If $\{\phi_\gamma\}_{\gamma \in \Gamma}$ is a partition of unity subordinate to this cover, we can define $f_s \colon E\Gamma \to P_s\Gamma$ by the formula

$$f_s(x) = \sum_{\gamma \in \Gamma} \phi_\gamma(x)\gamma,$$

Note that $f_s(x)$ is a well-defined point in $P_s\Gamma$: if $\phi_\gamma(x), \phi_{\gamma'}(x) > 0$, then

$$d_\Delta(\gamma, \gamma') = d_\Delta(\gamma \cdot x_0, \gamma' \cdot x_0) \leqslant d_\Delta(\gamma \cdot x_0, x) + d_\Delta(x, \gamma' \cdot x_0) < 2(D+1)$$

and by choice of s we have $d_w(\gamma, \gamma') < s$. Note that with this choice of cover, we have $\phi_\gamma(\gamma \cdot x_0) = 1$ and hence $f_s(\gamma \cdot x_0) = \langle x_0 \rangle$. (The cover formed from the balls of radius D also works, but the cover $\{U_\gamma\}$ makes the details below easier to check.)

The maps $g_s \colon P_s\Gamma \to E\Gamma$ ($s = 0, 1, \ldots$) are defined by induction over the simplices in $P_s\Gamma$. When $s = 0$, $P_0\Gamma = \Gamma$ and g_0 is just the embedding $\gamma \mapsto \gamma \cdot x_0$. Now assume that g_{s-1} has been defined ($s > 0$). Let $P_s^{(k)}\Gamma$ denote the k–skeleton of $P_s\Gamma$. Viewing $P_{s-1}\Gamma$ as a subcomplex of $P_s\Gamma$, we extend g_{s-1} inductively over the subcomplexes $P_s^{(k)}\Gamma \cup P_{s-1}(\Gamma)$. Assuming g_s has been defined on the $P_s^{(k-1)}\Gamma \cup P_{s-1}(\Gamma)$ for some $k \geqslant 1$, we extend over a k–simplex $\sigma \notin P_{s-1}\Gamma$ as follows. Let $D = \operatorname{diam}(g_s(\partial\sigma))$ and choose $x \in g_s(\partial\sigma)$. By uniform contractibility of $E\Gamma$ (Lemma 7.7) there exists $D' > 0$ (depending only on D) and a nullhomotopy of $g_s|_{\partial\sigma}$ whose image lies inside $B_{D'}(x)$. We now extend g_s over σ using this nullhomotopy.

As shown in [13, Lemma 4.3.6], f_s and g_s are inverse coarse equivalences, so in particular they are each metrically coarse and proper (since $E\Gamma$ and $P_s\Gamma$ are proper).

To show that $g_s \circ f_s$ induces the identity map on continuously controlled K–theory, it suffices to show that this map is Lipschitz homotopic to the identity [3, Proposition 3.17], where a Lipschitz homotopy $H \colon X \times I \to Y$ (with X and Y metric spaces) is simply a continuous, metrically coarse map for which $\{x \in X \,:\, H(x, t) \in C \text{ for some } t \in I\}$ is compact for all compact sets $C \subset Y$. Following Bartels–Rosenthal [2, Lemma 4.4], one constructs a homotopy $H \colon E\Gamma \times I \to E\Gamma$ connecting $g_s \circ f_s$ to $\mathrm{Id}_{E\Gamma}$ by induction over

the skeleta of $E\Gamma \times I$, again using the uniform contractibility of $E\Gamma$. (Here it is most convenient to use the cell structure on $E\Gamma \times I$ in which cells are either of the form $\sigma \times \{0\}$, $\sigma \times \{1\}$, or $\sigma \times I$, with σ a simplex in $E\Gamma$.)

To see that H is metrically coarse, note that its restriction to the zero skeleton of $E\Gamma \times I$ is the disjoint union of $g_s f_s$ and $\mathrm{Id}_{E\Gamma}$, hence is metrically coarse. Assuming H is metrically coarse on the k–skeleton, one checks metric coarseness on the $(k + 1)$–skeleton using the fact that there is a uniform bound $D(k)$ on the diameter of $H(\sigma)$ for σ a k–simplex (note that for 1–simplices, this follows from the fact that $g_s f_s$ is a bounded distance from the identity). For the remaining condition, it suffices to check that

$$\{x \, : \, d(H(x,t), \gamma \cdot x_0) < R \text{ for some } t \in I\}$$

is compact for each $\gamma \in \Gamma$, $R > 0$. This is similar: if x lies in a k–simplex, then $d(H(x,t), g_s f_s(x)) \leqslant D(k)$ and $d(g_s f_s(x), x) \leqslant S$ (for some constant S independent of x), so if $d(H(x,t), \gamma \cdot x_0) < R$, we have $d(x, \gamma \cdot x_0) < S + D(k) + R$, which suffices. $\qquad\square$

Theorem 7.5 can now be proven exactly as in Bartels' proof for groups with finite asymptotic dimension [3, Theorems 5.3 and 6.5]. For convenience of the reader, we recall the argument.

Proof of Theorem 7.5. As explained above, the assembly map

$$H_*\left(B\Gamma; \mathbb{K}(\mathcal{A}[\Gamma])\right) \longrightarrow K_* \mathcal{A}[\Gamma]$$

can be realized as the map of fixed-point spectra

$$(32) \qquad \left(\Omega\mathbb{K}\left(\mathcal{A}_\infty(E\Gamma)\right)\right)^\Gamma \xrightarrow{\partial^\Gamma} \left(\mathbb{K}\left(\mathcal{A}_c(E\Gamma)_{<1}\right)\right)^\Gamma$$

associated to a map of spectra

$$(33) \qquad \Omega\mathbb{K}\left(\mathcal{A}_\infty(E\Gamma)\right) \xrightarrow{\partial} \mathbb{K}\left(\mathcal{A}_c(E\Gamma)_{<1}\right)$$

that induces, on homotopy groups, the K–theoretic boundary map for the Karoubi sequence

$$\mathcal{A}_c(E\Gamma)_{<1} \longrightarrow \mathcal{A}_c(E\Gamma) \longrightarrow \mathcal{A}_\infty(E\Gamma).$$

Given an Ω–spectrum Y with a level-wise action of a group G, let $Y^{h\Gamma}$ denote the homotopy fixed point spectrum; that is, the function spectrum $F^G(EG_+, Y)$ consisting of (unbased) equivariant maps from EG to Y. The map (32) sits in a commutative diagram

$$(34) \qquad \begin{array}{ccc} \left(\Omega\mathbb{K}\left(\mathcal{A}_\infty(E\Gamma)\right)\right)^\Gamma & \xrightarrow{\partial^\Gamma} & \left(\mathbb{K}\left(\mathcal{A}_b(E\Gamma)\right)\right)^\Gamma \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} \\ \left(\Omega\mathbb{K}\left(\mathcal{A}_\infty(E\Gamma)\right)\right)^{h\Gamma} & \xrightarrow{\partial^{h\Gamma}} & \left(\mathbb{K}\left(\mathcal{A}_b(E\Gamma)\right)\right)^{h\Gamma}. \end{array}$$

The fact that $E\Gamma/\Gamma = B\Gamma$ is a finite CW complex implies that i is a weak equivalence of spectra (see, for example, Carlsson–Pedersen [9, Theorem

2.11]). Theorem 7.8, together with the long exact sequence in homotopy associated to the Karoubi sequence

$$\mathcal{A}_c(E\Gamma) \cong \mathcal{A}_c(E\Gamma)_{<1} \hookrightarrow \mathcal{A}_c(E\Gamma) \longrightarrow \mathcal{A}_\infty(E\Gamma),$$

shows that the map (33) is a weak equivalence. It follows that the map $\partial^{h\Gamma}$ in Diagram (34) is also a weak equivalence (every G–equivariant map between Ω–spectra with G–actions that is a weak equivalence, in the usual non-equivariant sense, induces a weak equivalence on homotopy fixed point spectra). Commutativity of (34) implies that the assembly map ∂^Γ in (32) is a split injection on homotopy, with splitting given by $(i_*)^{-1}(\partial_*^{h\Gamma})^{-1}j_*$. $\square$

As is usually the case in this area (see Bartels [3, Section 7], for example), Theorem 7.1 has an analogue for Ranicki's ultimate lower quadratic L–theory spectrum $\mathbb{L}^{-\infty}(\mathcal{A})$ of an additive category $\mathcal{A}$ with involution.

Theorem 7.9. *Let Γ be a group with finite decomposition complexity, and assume there exists a universal principal Γ–bundle $E\Gamma \to B\Gamma$ with $B\Gamma$ a finite CW complex. Let $\mathcal{A}$ be an additive category with involution, and assume that for some $r > 0$ we have $K_*(\mathcal{A}) = 0$ for $* < -r$. Then the assembly map*

$$H_*(B\Gamma; \mathbb{L}^{-\infty}(\mathcal{A})) \longrightarrow K_*(\mathcal{A}[\Gamma]),$$

is a split injection for all $* \in \mathbb{Z}$.

This result may be proven in a manner exactly analogous to the proof of Theorem 7.1. The relevant tools for L–theory are provided in Carlsson–Pedersen [9, Section 4]. The additional condition on $K_*(\mathcal{A})$ is needed in order to apply the L–theoretic analogue of Carlsson–Pedersen [9, Theorem 2.11] (see [9, Theorem 5.5]). We leave further details to the interested reader.

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