Intuitionism and *LJ* Sequent Calculus

Third Lecture

19th August 2004

Brouwer

All began with Brouwer who rejected the excluded-middle principle.

Why?
A view of mathematics centered on the mathematician so that the formula A is understood as "I know that A" or more precisely: "I have a proof of A". With this in mind, the logical connectives and the logical rules must be reconsidered.

In particular, the disjunction $A \vee B$ means "I have a proof of A or I have a proof of B" ... and the excluded middle is no more a suitable logical principle since $A \vee \neg A$ means that we always have a proof of a formula or of its negation... which is a very strong requirement.

Constructivism: a proof must provide a way to build an object that represents the property we proved.

What Disjunction?

In the previous lectures, two kinds of rules for disjunction on the right:

$$\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}$$

and

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \vee B, \Delta} \qquad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \vee B, \Delta}$$

Which one shall we choose for intuitionistic logic?

Informal Heyting's Semantics of *Intuitionistic Proofs*

- A proof of $A \vee B$ is a pair (i, π) with $i \in \{1; 2\}$ and if $i = 1$ then π is a proof of A, else it is a proof of B;
- A proof of $A \wedge B$ is a pair (π, π') of a proof of A and a proof of B;
- A proof of $A \Rightarrow B$ is a *function* which maps the proofs of A into the proofs of B (it is a transformation of proofs);
- a proof of $\exists x A$ is a pair (t, π) with t a term and π a proof of $A[t/x]$;
- a proof of $\forall x A$ is a function which maps each term t to a proof of $A[t/x]$;
- a proof of $\neg A$ is a function mapping the proofs of A to proofs of $F \wedge \neg F$.

$\Longrightarrow$ ***Natural Deduction***

Identity Rules

$$\frac{}{A \vdash A}\ \textit{axiom} \qquad \frac{\Gamma_1 \vdash A \quad \Gamma_2, A \vdash \Xi}{\Gamma_1, \Gamma_2 \vdash \Xi}\ \textit{cut}$$

Structural Rules

$$\frac{\Gamma_1, B, A, \Gamma_2 \vdash \Xi}{\Gamma_1, A, B, \Gamma_2 \vdash \Xi}\ \textit{LEx} \qquad \frac{\Gamma \vdash \Xi}{\Gamma, A \vdash \Xi}\ \textit{LW} \qquad \frac{\Gamma \vdash}{\Gamma \vdash A}\ \textit{RW} \qquad \frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}\ \textit{LC}$$

Logical Rules

$$\frac{\Gamma \vdash A}{\Gamma, \neg A \vdash}\ L\neg \qquad \frac{\Gamma, A \vdash}{\Gamma \vdash \neg A}\ R\neg$$

$$\frac{\Gamma_1 \vdash A \quad \Gamma_2, B \vdash \Xi}{\Gamma_1, \Gamma_2, A \Rightarrow B \vdash \Xi}\ L \Rightarrow \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B}\ R \Rightarrow$$

LJ Rules (2)

$$\frac{\Gamma, A \vdash \Xi}{\Gamma, A \wedge B \vdash \Xi}\ L\wedge 1 \qquad \frac{\Gamma, B \vdash \Xi}{\Gamma, A \wedge B \vdash \Xi}\ L\wedge 2 \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ R\wedge$$

$$\frac{\Gamma, A \vdash \Xi \quad \Gamma, B \vdash \Xi}{\Gamma, A \vee B \vdash \Xi}\ L\vee \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\ R\vee 1 \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\ R\vee 2$$

$$\frac{\Gamma, A[t/x] \vdash \Xi}{\Gamma, \forall x A \vdash \Xi}\ L\forall \qquad \frac{\Gamma \vdash A}{\Gamma \vdash \forall x A}\ R\forall \quad (*)$$

$$\frac{\Gamma, A \vdash \Xi}{\Gamma, \exists x A \vdash \Xi}\ L\exists \quad (**) \qquad \frac{\Gamma \vdash A[t/x]}{\Gamma \vdash \exists x A}\ R\exists$$

(*) For this rule, $x \notin FV(\Gamma)$.
(**) For this rule, $x \notin FV(\Gamma, \Xi)$.

Correspondence between classical and intuitionistic provability (1)

LJ is clearly weaker than *LK*: $\Gamma \vdash_{LJ} A$ implies $\Gamma \vdash_{LK} A$

Can we make more precise the relation between the two notions of provability?

We will see that *LJ* can be considered not to be weaker than *LK* but finer!

Correspondence between classical and intuitionistic provability (2)

Remember that in LJ, contraction is not available on the right of $\vdash$ but it is freely available on the left.

$A \vee \neg A$ is not provable in LJ but $\neg\neg(A \vee \neg A)$ is:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{A \vdash A}\;Axiom}{\vdash A, \neg A}\;R\neg}{\vdash A, A \vee \neg A}\;R\vee}{\vdash A \vee \neg A, A \vee \neg A}\;R\vee}{\vdash A \vee \neg A}\;RC$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{A \vdash A}\;Axiom}{A \vdash A \vee \neg A}\;R\vee}{\neg(A \vee \neg A), A \vdash}\;L\neg}{\neg(A \vee \neg A) \vdash \neg A}\;R\neg}{\neg(A \vee \neg A) \vdash A \vee \neg A}\;R\vee}{\neg(A \vee \neg A), \neg(A \vee \neg A) \vdash}\;L\neg}{\neg(A \vee \neg A) \vdash}\;LC}{\vdash \neg\neg(A \vee \neg A)}\;R\neg$$

Correspondence between classical and intuitionistic provability (3)

Gödel Translation

The idea of the intuitionistic proof of $\neg\neg(A \vee \neg A)$ is to send the formula to the left so that it is possible to use left contraction. The occurrence of the double negation $\neg\neg$ precisely allows to cross twice the $\vdash$ and to use left contraction.

Definition: Gödel Translation

- $A^\star = \neg\neg A$ *for* A *atomic;*
- $(A \wedge B)^\star = A^\star \wedge B^\star$;
- $(\forall x A)^\star = \forall x A^\star$;
- $(\neg A)^\star = \neg A^\star$;
- $(A \Rightarrow B)^\star = A^\star \Rightarrow B^\star$;
- $(A \vee B)^\star = \neg\neg(A^\star \vee B^\star)$;
- $(\exists x A)^\star = \neg\neg\exists x A^\star$.

Correspondence between classical and intuitionistic provability (4)

Theorem

$\Gamma \vdash_{LK} A$ *iff* $\Gamma^\star \vdash_{LJ} A^\star$

Lemma

$\vdash_{LK} A \Leftrightarrow A^\star$

Definition

A is said to be stable when $\vdash_{LJ} \neg\neg A \Rightarrow A$.

Lemma

For all formula A, $A^\star$ is stable.

Lemma

If $\Gamma \vdash_{LK} \Delta$ then $\Gamma^\star, \neg\Delta^\star \vdash_{LJ}$.

Correspondence between classical and intuitionistic provability (5)

$$A \equiv_{LK} B \supseteq A \equiv_{LJ} B$$

$$A \equiv B?$$

$$\Gamma \vdash A \quad \textit{iff} \quad \Gamma \vdash B \qquad \textit{OR} \qquad \vdash A \Leftrightarrow B \qquad ?$$

It is the same!

Correspondence between classical and intuitionistic provability (6)

Why is LJ finer than LK?

An intuitionistic logician cannot necessarily prove a formula A that a classical mathematician can prove, BUT he can find another formula ($A^\star$) that the classical mathematician cannot distinguish from the previous one.

In particular, in intuitionnistic, the use of excluded middle (or contraction on the right) shall be explicitly mentioned in the formula thanks to the use of double negation (in Linear Logic, we shall have the same kind of things but for all structural rules, not only RC).